402.

ON A SINGULARITY OF SURFACES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 332—338.]

A SURFACE having a nodal line has in general on this nodal line points where the two tangent planes coincide, or as I propose to term them "pinch-points." Thus, if the nodal line be the curve of complete intersection of any two surfaces $P=0$, $Q=0$, then the equation of the general surface having this curve for a nodal line is $(a, b, c \between P, Q)^2=0$ (where a, b, c are any functions of the coordinates), and the pinch-points are given as the intersections of the nodal line $P=0$, $Q=0$ with the surface $ac-b^2=0$. Consider the case where the nodal curve is a curve of partial intersection represented by the equations $\left\|\begin{matrix} P, & Q, & R \\ P', & Q', & R' \end{matrix}\right\|=0$, or say by the equations $p=0$, $q=0$, $r=0$ (viz. p, q, r denote the functions $QR'-Q'R$, $RP'-R'P$, $PQ'-P'Q$ respectively), and consequently we have identically

$$(P, Q, R \between p, q, r)=0,$$
$$(P', Q', R' \between p, q, r)=0,$$

or what is the same thing, (λ, μ) being arbitrary,

$$(\lambda P+\mu P', \lambda Q+\mu Q', \lambda R+\mu R' \between p, q, r)=0.$$

The general surface having the curve in question for its nodal line is represented by the equation

$$(a, b, c, f, g, h \between p, q, r)^2=0,$$

(where (a, b, c, f, g, h) are any functions of the coordinates), and it is easy to see that the condition for a pinch-point is the same as that which (considering p, q, r as coordinates and all the other quantities as constants), expresses that the line

$$(\lambda P+\mu P', \lambda Q+\mu Q', \lambda R+\mu R' \between p, q, r)=0,$$

touches the conic

$$(a,\ b,\ c,\ f,\ g,\ h)(p,\ q,\ r)^2=0,$$

viz. A, B, C, F, G, H being the inverse coefficients, $A=bc-f^2$, &c., this condition is

$$(A,\ B,\ C,\ F,\ G,\ H)(\lambda P+\mu P',\ \lambda Q+\mu Q',\ \lambda R+\mu R')^2=0,$$

or what is the same thing, the pinch-points are given as the common intersections of the nodal line $p=0$, $q=0$, $r=0$ with each of the three surfaces

$$\begin{aligned}(A,\ B,\ C,\ F,\ G,\ H)(P,\ Q,\ R)^2 &= 0,\\ (A,\ B,\ C,\ F,\ G,\ H)(P,\ Q,\ R)\ (P',\ Q',\ R') &= 0,\\ (A,\ B,\ C,\ F,\ G,\ H)\qquad (P',\ Q',\ R')^2 &= 0,\end{aligned}$$

these last three equations in fact, adding only a single relation to the relations expressed by the equations

$$p=0,\ q=0,\ r=0.$$

If the functions P, Q, R, P', Q', R' are linear functions of the coordinates, then the curve $(p=0,\ q=0,\ r=0)$ is a cubic curve in space, or skew cubic; and if moreover $(a,\ b,\ c,\ f,\ g,\ h)$ are constants, then the equation

$$(a,\ b,\ c,\ f,\ g,\ h)(p,\ q,\ r)^2=0,$$

belongs to a quartic surface having the skew cubic for a nodal line: this surface is (it may be observed) a ruled surface, or scroll. With a view to ulterior investigations, I propose to study the theory of the pinch-points in regard to this particular surface; and to simplify as much as possible, I fix the coordinates as follows:

Considering the skew cubic as given, let any point O on the cubic be taken for the origin; let $x=0$ be the equation of the osculating plane at O; $y=0$ that of any other plane through the tangent line at O; $z=0$, that of any other plane through O, not passing through the tangent line; and $w=0$ that of a fourth plane; then the equation of the cubic will be

$$\left\|\begin{matrix} x, & y, & z \\ y, & z, & w \end{matrix}\right\|=0,$$

or what is the same thing, the values of p, q, r are $yw-z^2$, $zy-xw$, and $xz-y^2$ respectively. And conversely, the cubic being thus represented, the point $(x=0,\ y=0,\ z=0)$ may be considered as standing for any point whatever on the skew cubic; the osculating plane at this point being $x=0$, and the tangent line being $x=0$, $y=0$. For the purpose of the present investigations, we may without loss of generality write $w=1$; and for convenience I shall do this; the values of p, q, r thus become $y-z^2$, $yz-x$, $xz-y^2$, and the equation of the surface is

$$(a,\ b,\ c,\ f,\ g,\ h)(y-z^2,\ yz-x,\ xz-y^2)^2=0.$$

At a pinch-point, we have

$$(A, B, C, F, G, H)(x, y, z)^2 = 0,$$

$$(A, B, C, F, G, H)(x, y, z)(y, z, 1) = 0,$$

$$(A, B, C, F, G, H)(y, z, 1)^2 = 0,$$

and hence the origin will be a pinch-point if $C = 0$, that is, if $ab - h^2 = 0$. This however appears more readily by remarking, that the equation of the pair of tangent planes at the origin is

$$(a, b, c, f, g, h \between y, -x, 0)^2 = 0,$$

or what is the same thing,

$$(a, h, b \between y, -x)^2 = 0;$$

the two tangent planes therefore coincide, or there is a pinch-point, if only $ab - h^2 = 0$.

By what precedes, it appears that if we wish to study the form of the quartic surface, 1°, in the neighbourhood of an arbitrary point on the nodal line; 2°, in the neighbourhood of a pinch-point; it is sufficient in the first case to consider the general surface

$$(a, b, c, f, g, h \between y - z^2, yz - x, xz - y^2)^2 = 0,$$

in the neighbourhood of the origin; and in the second case, to study the special surface for which $ab - h^2 = 0$, or writing for convenience $a = 1$, and therefore $b = h^2$, the surface

$$(1, h^2, c, f, g, h \between y - z^2, yz - x, xz - y^2)^2 = 0,$$

in the neighbourhood of the origin.

Consider first the surface

$$(a, b, c, f, g, h)(y - z^2, yz - x, xz - y^2)^2 = 0.$$

A plane through the origin is either a plane not passing through the tangent line ($x = 0$, $y = 0$), and the equation $z = 0$ will serve to represent any such plane; or if it pass through the tangent line, then it is either a non-special plane, which may be represented by the equation $y = 0$; or it is a special plane: viz. either the osculating plane $x = 0$ of the nodal line, or else one or the other of the two tangent planes $(a, h, b \between y, -x)^2 = 0$ of the surface. I consider therefore the sections of the surface by these planes $z = 0$, $y = 0$, $x = 0$, $(a, h, b \between y, -x)^2 = 0$ respectively.

Section by the non-special plane $z = 0$.

The equation is

$$(a, b, c, f, g, h \between y, -x, -y^2)^2 = 0,$$

which represents a curve having at the origin an ordinary node, the equations of the two tangents being $(a, h, b \between y, -x)^2 = 0$, viz. these are the intersections of the two tangent planes by the plane $z = 0$.

Section by the non-special plane through the tangent line, viz. the plane $y=0$.

The equation is

$$(a,\ b,\ c,\ f,\ g,\ h \between -z^2,\ -x,\ xz)^2=0,$$

or what is the same thing,

$$bx^2-2fx^2z+2hxz^2+cx^2z^2-2yxz^3+az^4=0.$$

Writing as usual $x=Az^\mu+\&c.$ we have

$$\mu=2,\quad bA^2+2hA+a=0,$$

and since $ab-h^2$ is by hypothesis not $=0$, A has two unequal values; we have at the origin two branches $x=A_1z^2+B_1z^3+\&c.$, $x=A_2z^2+B_2z^3+\&c.$, having the common tangent $x=0$ (viz. this is the tangent $x=0$, $y=0$ of the nodal curve), and with a two-pointic intersection of the two branches, that is, the point at the origin is an ordinary tacnode.

Section by the osculating plane $x=0$.

The equation is

$$(a,\ b,\ c,\ f,\ g,\ h \between y-z^2,\ yz,\ -y^2)^2=0.$$

We may write $y=z^2+Az^\mu+\&c.$, we at once find $\mu=3$, and then

$$(a,\ b,\ c,\ f,\ g,\ h \between Az^3+\&c.,\ z^3+\&c.,\ -z^4+\&c.)^2=0,$$

that is

$$(a,\ h,\ b \between A,\ 1)^2=0.$$

A has two unequal values, and the branches through the origin are

$$y=z^2+A_1z^3+B_1z^4+\&c.,\quad y=z^2+A_2z^3+B_2z^4+\&c.\ \ldots,$$

viz. the branches have the common tangent line $y=0$ (the tangent $x=0$, $y=0$ of the nodal curve), but in the present case a three-pointic intersection.

Section by one of the tangent planes $(a,\ h,\ b \between y,\ -x)^2=0$.

Writing $y=-mx$, and therefore $(a,\ h,\ b \between m,\ -1)^2=0$, the equation is

$$(a,\ b,\ c,\ f,\ g,\ h \between -mx-z^2,\ -x-mzx,\ xz-m^2x^2)^2=0,$$

which represents of course the projection of the section on the plane $z=0$, $x=0$, but which (since there is no alteration in the singularities) may be considered as representing the section itself. Developing, the coefficient of x^2 is $am^2+2hm+b$, which is $=0$, and the equation becomes

$$\begin{array}{llll}
 & 2m^2(gm+f) & x^3 + & cm^4 \quad x^4 \\
+2\,\{hm^2+(b-g)\,m-f\} & & x^2z + & 2m^2(fm-c) \quad x^3z \\
+ & 2\,(am+h) & xz^2 + & (b+2g)\,m^2-2fm+c \quad x^2z^2 \\
 & & + & 2\,(hm-g) \quad xz^3 \\
 & & + & a \quad z^4=0,
\end{array}$$

so that the curve has at the origin a triple point, the tangent to one branch being the line $x=0$ (the tangent $x=0$, $y=0$ of the nodal curve).

Consider next the surface

$$(1,\ h^2,\ c,\ f,\ g,\ h\between y-z^2,\ yz-x,\ xz-y^2)^2=0,$$

being as already remarked, the general surface referred to a pinch-point as origin.

Section by the non-special plane $z=0$.

The equation is

$$(1,\ h^2,\ c,\ f,\ g,\ h\between y,\ -x,\ -y^2)^2=0,$$

where, attending only to the terms of the lowest order, we find $(1,\ h,\ h^2\between y,\ -x)^2=0$, that is $(y-hx)^2=0$, showing that the origin is a cusp.

Section by the non-special plane through the tangent line, viz. the plane $y=0$.

The equation is

$$(1,\ h^2,\ c,\ f,\ g,\ h\between -z^2,\ -x,\ xz)^2=0,$$

or what is the same thing,

$$h^2x^2+2hxz^2-2fx^2z+cx^2z^2-2gxz^3+z^4=0,$$

that is

$$(hx+z^2)^2-2fx^2z+cx^2z^2-2gxz^3=0,$$

writing $hx=-z^2+Ax^\mu$, we find at once $\mu=\frac{5}{2}$, and then $A^2=\dfrac{2f}{h^2}-\dfrac{2g}{h}$, so that the branches are $hx=-z^2\pm Ax^{\frac{5}{2}}$; whence we have at the origin a cusp of the second order or node cusp.

Section by the osculating plane $x=0$.

The equation is

$$(1,\ h^2,\ c,\ f,\ g,\ h\between y-z^2,\ yz,\ -y^2)^2=0;$$

writing $y=z^2-hz^3+Az^\mu$, we easily find $\mu=\frac{7}{2}$, and then

$$(1,\ h^2,\ c,\ f,\ g,\ h\between -hz^3+Az^{\frac{7}{2}},\ z^3-hz^4,\ -z^4)^2=0,$$

where the terms in z^6, and $z^{6+\frac{1}{2}}$ disappear of themselves, the terms in z^7 give $A^2+2gh=0$, and the branches are

$$y=z^2-hz^3\pm Az^{\frac{7}{2}}\ \&c.,$$

viz. there is a cusp of a superior order.

Section by the tangent plane $y=hx$.

The equation is

$$(1,\ h^2,\ c,\ f,\ g,\ h\between hx-z^2,\ -x+hzx,\ xz-h^2x^2)^2=0,$$

representing the projection on the plane of zx. Developing, the equation is

$$\begin{array}{lll} 2(f-gh)\,x^2(h^2x-z) & +\,x^4ch^4 & \\ & +\,x^3z\;-2h^2(hf+c) & \\ & +\,x^2z^2\;\;(h^4+2g^2h^2+2fh+c) & \\ & +\,xz^3\;-2(h^2+g) & \\ & +\,z^4\quad\;\; 1 & =0, \end{array}$$

and there is at the origin a triple point (= cusp + 2 nodes) arising from the passage of an ordinary branch through a cusp; the tangent at the cusp being it will be noticed the line $x=0$, that is the tangent $x=0$, $y=0$ to the nodal curve at the pinch-point.

The results of the investigation may be presented in a tabular form as follows:

Plane of Section.	Nature of Section. Origin, an ordinary point.	Nature of Section. Origin, a Pinch-point.
Non-special.	Node.	Cusp.
Ditto, through tangent line of nodal curve.	Tacnode = 2 nodes.	Node-cusp, = node + cusp.
Osculating plane of nodal curve.	$y = z^2 + \left.\begin{matrix}A_1\\A_2\end{matrix}\right\} z^3 + \&c.$	$y = z^2 + hz^3 \pm Az^{\frac{7}{2}}$ &c.
Either of the two tangent planes.	Triple point, one branch touching the tangent of nodal line.	
The single tangent plane.		Triple point, = cusp + 2 nodes; the cuspidal branch touching the tangent of the nodal line.

I have not considered the special cases where one of the two tangent planes, or (as the case may be) the single tangent plane of the surface coincides with the osculating plane of the nodal curve.