401.

A NOTATION OF THE POINTS AND LINES IN PASCAL'S THEOREM.

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TAKING six points 1, 2, 3, 4, 5, 6 on a conic; let $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O$ denote each a combination of three lines, thus

$12.34.56 = A$	$12.35.64 = F$	$12.36.45 = K$
$13.45.62 = B$	$13.46.25 = G$	$13.42.56 = L$
$14.56.23 = C$	$14.52.36 = H$	$14.53.62 = M$
$15.62.34 = D$	$15.63.42 = I$	$15.64.23 = N$
$16.23.45 = E$	$16.24.53 = J$	$16.25.34 = O$

then any hexagon formed with the six points may be represented by a combination of some two of the letters A, B, &c., viz. the three alternate sides are the lines represented by one letter, and the other three alternate sides the lines represented by the other letter: for example, the hexagon 123456 is AE; and so for the other hexagons. Any duad AE thus representing a hexagon may be termed a hexagonal duad; the number of such duads is sixty. Each Pascalian line may be denoted by the symbol of the hexagon to which it belongs; thus, the line which belongs to the hexagon AE, is the line AE.

I form the following combinations:

$IMO.DHJ$	each involving all the duads 12, &c.	except those of		123.456,
$DEG.BNO$	"	"	"	124.356,
$ELM.BCJ$	"	"	"	125.346,
$HLN.CGI$	"	"	"	126.345,
$EFI.JKN$	"	"	"	134.256,
$AEH.CKO$	"	"	"	135.246,
$AMN.CDF$	"	"	"	136.245,
$AGJ.ELO$	"	"	"	145.236,
$ABI.DKL$	"	"	"	146.235,
$GKM.BFH$	"	"	"	156.234,

and also the combinations:

AEGMI	involving all the duads 12, 13, &c.,	
ABHJN	,,	,,
BCFIO	,,	,,
CDGJK	,,	,,
DEFHL	,,	,,
KLMNO	,,	,,

which I call respectively the ten-partite and six-partite arrangements. It is to be remarked that (considering *IMO.DHJ* as standing for the six duads *IM*, *IO*, *MO* *DH*, *DJ*, *HJ*, and so for the others) the ten-partite arrangement contains all the sixty hexagonal duads: and in like manner, (considering *AEGMI* as standing for the ten duads *AE*, *AG*, *AM*, *AI*, *EG*, *EM*, *EI*, *GM*, *GI*, *MI*, and so for the others) the six-partite arrangement contains all the sixty hexagonal duads.

The 60 Pascalian lines intersect by 4's in the 45 Pascalian points p, by 3's in 20 points g and in 60 points h, and by 2's in 90 points m, 360 points r, 360 points t, 360 points z, and 9 points w.

The intersections of the Pascalian lines thus are

45 p	counting as	270
20 g	,, ,,	60
60 h	,, ,,	180
90 m	,, ,,	90
360 r	,, ,,	360
360 t	,, ,,	360
360 z	,, ,,	360
90 w	,, ,,	90
		$1770 = \frac{1}{2}60\,.\,59$,

and the intersections on each Pascalian line are

3 p	counting as	9
1 g	,, ,,	2
3 h	,, ,,	6
3 m	,, ,,	3
12 r	,, ,,	12
12 t	,, ,,	12
12 z	,, ,,	12
3 w	,, ,,	3
		59.

For the ten-partite arrangement, any double triad such as $ABI\,.\,DKL$ gives 15 intersections; $10 \times 15 = 150$; and any pair of double triads such as $ABI\,.\,DKL$ and $AEH\,.\,CKO$ gives 36 intersections; $45 \times 36 = 1620$; and these are

$$
\begin{array}{rll}
10 \times \left\{ \begin{array}{l} 6\,g \\ 9\,m \end{array} \right. & & \begin{array}{r} 60\,g \\ 90\,m \end{array} \\
 & \overline{\quad\ } 15 & \overline{\quad\ } 150 \\
45 \times \left\{ \begin{array}{l} 6\,p \\ 4\,h \\ 8\,r \\ 8\,t \\ 8\,z \\ 2\,w \end{array} \right. & & \begin{array}{r} 270\,p \\ 180\,h \\ 360\,r \\ 360\,t \\ 360\,z \\ 90\,w \end{array} \\
 & \overline{\quad\ } 36 & \overline{\quad\ } 1620 \\
 & & \quad 1770.
\end{array}
$$

For the six-partite arrangement any pentad such as $ABHJN$ gives 45 intersections; $6 \times 45 = 270$; and any two pentads such as $ABHJN$ and $AEGMI$ give 100 intersections; $15 \times 100 = 1500$; and these are

$$
\begin{array}{rll}
6 \times \left\{ \begin{array}{l} 30\,h \\ 15\,m \end{array} \right. & & \begin{array}{r} 180\,h \\ 90\,m \end{array} \\
 & \overline{\quad\ } 45 & \overline{\quad\ } 270 \\
15 \times \left\{ \begin{array}{l} 4\,g \\ 18\,p \\ 24\,r \\ 24\,t \\ 24\,z \\ 6\,w \end{array} \right. & & \begin{array}{r} 60\,g \\ 270\,p \\ 360\,r \\ 360\,t \\ 360\,z \\ 90\,w \end{array} \\
 & \overline{\quad\ } 100 & \overline{\quad\ } 1500 \\
 & & \quad 1770.
\end{array}
$$

I analyse the intersections of a Pascalian line, say AE, by the remaining 59 Pascalian lines as follows:

Observe that AE belongs to the triad AEH, the complementary triad whereof is CKO; it also belongs to the pentad $AEIMG$. We thus obtain, corresponding to AE, the arrangement

$$
\begin{array}{cc|ccc}
 & & H & H & H \\
\hline
H & A & B & N & J \\
H & E & F & L & D \\
\hline
 & & I & M & G \\
 & & K & C & O
\end{array}
$$

viz. *HABNJ*, is the pentad which contains *HA*, the arrangement of the last three letters *B*, *N*, *J* thereof being arbitrary; *HEFLD* is the pentad that contains *HE*, but the last three letters are so arranged that the columns *HBF*, *HNL*, *HJD* are each of them a triad, *IMG* is then the residue of the pentad *AEIMG*, and *KCO* is the complementary triad to *AEH*, but the arrangement of the letters *IMG*, and of the letters *KCO*, are each of them determinate; viz. these are such that we have *BFICO*, *NLMKO*, *JDGCK*, each of them a pentad.

And this being so we derive from the arrangement

2 g	*AH*, *EH*;
3 m	*KC*, *KO*, *CO*;
6 h	*AI*, *AM*, *AG*; *EI*, *EM*, *EG*;
12 z	*IB*, *IF*, *MN*, *ML*, *GJ*, *GD*; *HB*, *HF*, *HN*, *HL*, *HJ*, *HD*;
9 p	*AB*, *AN*, *AJ*; *EF*, *EL*, *ED*; *BF*, *NL*, *JD*;
12 r	*CB*, *CF*, *CJ*, *CD*; *OB*, *OF*, *ON*, *OL*; *KN*, *KL*, *KJ*, *KD*;
12 t	*FL*, *FD*, *LD*; *BN*, *BJ*, *NJ*; *IC*, *IO*; *MK*, *MO*; *GK*, *GC*;
3 w	*IM*, *IG*, *MG*;
59	

viz. the line *AE* in question meets *AH*, *EH* each of them in a point g; *KC*, *KO*, *CO* each in a point m; and so on. By constructing in the same way an arrangement for each of the lines *AH*, &c., we find the nature of the point of intersection of any two of the lines *AB*, *AE*, *AH*, &c.; and we may then present the results in a table (see Plate), which shows at a glance what is the point of intersection (whether a point g, m, h, z, p, r, t, or w) of any two of the Pascalian lines.

I further remark that representing the 45 Pascalian points as follows:

12 . 34 = a	13 . 24 = g	14 . 23 = m	15 . 23 = s	16 . 23 = y
12 . 35 = b	13 . 25 = h	14 . 25 = n	15 . 24 = t	16 . 24 = z
12 . 36 = c	13 . 26 = i	14 . 26 = o	15 . 26 = u	16 . 25 = α
12 . 45 = d	13 . 45 = j	14 . 35 = p	15 . 34 = v	16 . 34 = β
12 . 46 = e	13 . 46 = k	14 . 36 = q	15 . 36 = w	16 . 35 = γ
12 . 56 = f	13 . 56 = l	14 . 56 = r	15 . 46 = x	16 . 45 = δ

23 . 45 = ϵ	25 . 34 = λ	34 . 56 = ρ
23 . 46 = ζ	25 . 36 = μ	35 . 46 = σ
23 . 56 = η	25 . 46 = ν	36 . 45 = τ
24 . 35 = θ	26 . 34 = ξ	
24 . 36 = ι	26 . 35 = ω	
24 . 56 = κ	26 . 45 = π	

the sixty hexagons and their Pascalian lines then are

AE	123456	12 . 45 23 . 56 34 . 61	$d\eta\beta$
AH	125634	12 . 63 25 . 34 56 . 41	$c\lambda r$
EH	145236	14 . 23 45 . 36 52 . 61	$m\tau\alpha$
CK	123654	12 . 65 23 . 54 36 . 41	$f\epsilon q$
CO	143256	14 . 25 43 . 56 32 . 61	$n\rho y$
KO	125436	12 . 43 25 . 36 54 . 61	$a\mu\delta$
AM	126534	12 . 53 26 . 34 65 . 41	$b\xi r$
AG	125643	12 . 64 25 . 43 56 . 31	$e\lambda l$
AI	124365	12 . 36 24 . 65 43 . 51	$c\kappa v$
EG	132546	13 . 54 32 . 46 25 . 61	$j\zeta\alpha$
DF	126435	12 . 43 26 . 35 64 . 51	$a\omega x$
FL	124653	12 . 65 24 . 53 46 . 31	$f\theta k$
DL	134265	13 . 26 34 . 65 42 . 51	$i\rho t$
BN	132645	13 . 64 32 . 45 26 . 51	$k\epsilon u$
BJ	135426	13 . 42 35 . 26 54 . 61	$g\omega\delta$
JN	153246	15 . 24 53 . 46 32 . 61	$t\sigma y$
GK	125463	12 . 46 25 . 63 54 . 31	$e\mu j$
KM	126354	12 . 35 26 . 54 63 . 41	$b\pi q$
IO	152436	15 . 43 52 . 36 24 . 61	$v\mu z$
MO	143526	14 . 52 43 . 26 35 . 61	$n\xi\gamma$
EM	145326	14 . 32 45 . 26 53 . 61	$m\pi\gamma$
EI	154236	15 . 23 54 . 36 42 . 61	$s\tau z$
AN	123465	12 . 46 23 . 65 34 . 51	$e\eta v$
AJ	124356	12 . 35 24 . 56 43 . 61	$b\kappa\beta$
AB	126543	12 . 54 26 . 43 65 . 31	$d\zeta l$
DE	154326	15 . 32 54 . 26 43 . 61	$s\pi\beta$
EL	132456	13 . 45 32 . 56 24 . 61	$j\eta z$
EF	123546	12 . 54 23 . 46 35 . 61	$d\zeta\gamma$
CD	143265	14 . 26 43 . 65 32 . 51	$o\rho s$
CF	123564	12 . 56 23 . 64 35 . 41	$f\zeta p$

CG	132564	13.56	32.64	25.41		$l\zeta n$
CI	142365	14.36	42.65	23.51		$q\kappa s$
MN	146235	14.23	46.35	62.51		$m\sigma u$
GJ	135246	13.24	35.46	52.61		$g\sigma\alpha$
BI	136245	13.24	36.45	62.51		$g\tau u$
DG	134625	13.62	34.25	46.51		$i\lambda x$
LM	135624	13.62	35.24	56.41		$i\theta r$
FI	124635	12.63	24.35	46.51		$c\theta x$
BH	136254	13.25	36.54	62.41		$h\tau o$
FH	125364	12.36	25.64	53.41		$c\nu p$
FO	125346	12.34	25.46	53.61		$a\nu\gamma$
LO	134256	13.25	34.56	42.61		$h\rho z$
DK	126345	12.34	26.45	63.51		$a\pi w$
KL	124563	12.56	24.63	45.31		$f\iota j$
BO	134526	13.52	34.26	45.61		$h\xi\delta$
NO	152346	15.34	52.46	23.61		$v\nu y$
BC	132654	13.65	32.54	26.41		$l\epsilon o$
CJ	142356	14.35	42.56	23.61		$p\kappa y$
JK	124536	12.53	24.36	45.61		$b\iota\delta$
KN	123645	12.64	23.45	36.51		$e\epsilon w$
DH	143625	14.62	43.25	36.51		$o\lambda w$
HJ	142536	14.53	42.36	25.61		$p\iota\alpha$
HL	136524	13.52	36.24	65.41		$h\iota r$
HN	146325	14.32	46.25	63.51		$m\nu w$
BF	126453	12.46	26.53	64.31		$d\omega k$
DJ	153426	15.42	53.26	34.61		$t\omega\beta$
LN	132465	13.46	32.65	24.51		$k\eta t$
GM	135264	13.26	35.64	52.41		$i\sigma n$
IM	142635	14.63	42.35	26.51		$q\theta u$
GI	136425	13.42	36.25	64.51		$g\mu x$

Each Pascalian point belongs to four different hexagons; viz. *a* to the hexagons *KD*, *KO*, *FD*, *FO*; and so for the other points, thus:

a	$(K, F)(D, O)$	x	$(D, I)(F, G)$
b	$(A, K)(M, J)$	y	$(C, N)(J, O)$
c	$(A, F)(H, I)$	z	$(E, O)(I, L)$
d	$(A, F)(B, E)$	α	$(E, J)(G, H)$
e	$(A, K)(G, N)$	β	$(A, D)(E, J)$
f	$(C, L)(K, F)$	γ	$(E, O)(F, M)$
g	$(B, G)(I, J)$	δ	$(B, K)(J, O)$
h	$(B, L)(H, O)$	ϵ	$(B, K)(C, N)$
i	$(D, M)(G, L)$	ζ	$(C, E)(F, G)$
j	$(E, K)(G, L)$	η	$(A, L)(E, N)$
k	$(B, L)(F, N)$	θ	$(F, M)(I, L)$
l	$(A, C)(B, G)$	ι	$(H, K)(J, L)$
m	$(E, N)(H, M)$	κ	$(A, C)(I, J)$
n	$(C, M)(G, O)$	λ	$(A, D)(G, H)$
o	$(B, D)(C, H)$	μ	$(G, O)(I, K)$
p	$(C, H)(F, J)$	ν	$(F, N)(H, O)$
q	$(C, M)(I, K)$	ξ	$(A, O)(B, M)$
r	$(A, L)(H, M)$	ω	$(B, D)(F, J)$
s	$(C, E)(D, I)$	π	$(D, M)(E, K)$
t	$(J, L)(D, N)$	ρ	$(C, L)(D, O)$
u	$(B, M)(I, N)$	σ	$(G, N)(J, M)$
v	$(A, O)(N, I)$	τ	$(B, E)(H, I)$
w	$(D, N)(H, K)$		

I have constructed on a very large scale a figure of the sixty Pascalian lines, and the forty-five Pascalian points, marking them according to the foregoing notation; but the figure is from its complexity, and the inconvenient way in which the points are either crowded together or fly off to a great distance, almost unintelligible.