394.

ON A LOCUS IN RELATION TO THE TRIANGLE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 264—277.]

If from any point of a circle circumscribed about a triangle perpendiculars are let fall upon the sides, the feet of the perpendiculars lie in a line; or, what is the same thing, the locus of a point, such that the perpendiculars let fall therefrom upon the sides of a given triangle have their feet in a line, is the circle circumscribed about the triangle.

In this well known theorem we may of course replace the circular points at infinity by any two points whatever; or the Absolute being a point-pair, and the terms perpendicular and circle being understood accordingly, we have the more general theorem expressed in the same words.

But it is less easy to see what the corresponding theorem is, when instead of being a point-pair, the Absolute is a proper conic; and the discussion of the question affords some interesting results.

Take $(x=0,\ y=0,\ z=0)$ for the equations of the sides of the triangle, and let the equation of the Absolute be

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=0,$$

then any two lines which are harmonics in regard to this conic (or, what is the same thing, which are such that the one of them passes through the pole of the other) are said to be perpendicular to each other, and the question is:

Find the locus of a point, such that the perpendiculars let fall therefrom on the sides of the triangle have their feet in a line.

Supposing, as usual, that the inverse coefficients are $(A,\ B,\ C,\ F,\ G,\ H)$, and that K is the discriminant, the coordinates of the poles of the three sides respectively are

(A, H, G), (H, B, F), (G, F, C). Hence considering a point P, the coordinates of which are (x, y, z), and taking (X, Y, Z) for current coordinates, the equation of the perpendicular from P on the side $X=0$ is

$$\left|\begin{matrix} X, & Y, & Z \\ x\,, & y\,, & z \\ A, & H, & G \end{matrix}\right| = 0,$$

and writing in this equation $X=0$, we find

$$(\quad 0 \quad, Ay - Hx, Az - Gx)$$

for the coordinates of the foot of the perpendicular. For the other perpendiculars respectively, the coordinates are

$$(Bx - Hy, \quad 0 \quad, Bz - Fy),$$

and

$$(Cx - Gz\,, Cy - Fz\,, \quad 0 \quad),$$

and hence the condition in order that the three feet may lie in a line is

$$\left|\begin{matrix} 0 \quad, & Ay - Hx, & Az - Gx \\ Bx - Hy, & 0 \quad, & Bz - Fy \\ Cx - Gz\,, & Cy - Fz\,, & 0 \end{matrix}\right| = 0\,;$$

or, what is the same thing,

$$(Ay - Hx)(Bz - Fy)(Cx - Gz) + (Az - Gx)(Bx - Hy)(Cy - Fz) = 0,$$

that is

$$\begin{aligned} 2\,(ABC - FGH)\,xyz & \\ + A\,(\;FH - \;BG)\,yz^2 + A\,(FG - CH)\,y^2z & \\ + B\,(\;FG - \;CH)\,zx^2 + B\,(GH - AF)\,z^2x & \\ + C\,(\;GH - \;AF)\,xy^2 + C\,(HF - BG)\,x^2y = 0, & \end{aligned}$$

which is the equation of the locus of P; the locus is therefore a cubic. Writing for a moment

$$(BC - F^2,\ CA - G^2,\ AB - H^2,\ GH - AF,\ HF - BG,\ FG - CH) = (A',\ B',\ C',\ F',\ G',\ H'),$$

and K' for the discriminant $ABC - AF^2 -$&c., the equation is

$$2\,(ABC - FGH)\,xyz + Ayz\,(H'y + G'z) + Bzx\,(H'x + F'z) + Cxy\,(G'x + F'y) = 0,$$

or as this may also be written

$$\frac{2}{F'G'H'}(ABC - FGH)\,xyz + \frac{A}{F'}\,yz\left(\frac{y}{G'} + \frac{z}{H'}\right) + \frac{B}{G'}\,zx\left(\frac{x}{F'} + \frac{z}{H'}\right) + \frac{C}{H'}\,xy\left(\frac{x}{F'} + \frac{y}{G'}\right) = 0,$$

that is

$$\left[\frac{2}{F'G'H'}(ABC-FGH)-\frac{A}{F'^2}-\frac{B}{G'^2}-\frac{C}{H'^2}\right]xyz+\left(\frac{A}{F'}yz+\frac{B}{G'}zx+\frac{C}{H'}xy\right)\left(\frac{x}{F'}+\frac{y}{G'}+\frac{z}{H'}\right)=0,$$

and the cubic will therefore break up into a line and conic if only

$$\frac{2}{F'G'H'}(ABC-FGH)-\frac{A}{F'^2}-\frac{B}{G'^2}-\frac{C}{H'^2}=0,$$

and it is easy to see that conversely this is the necessary and sufficient condition in order that the cubic may so break up.

The condition is

$$\Omega=2F'G'H'(ABC-FGH)-AG'^2H'^2-BH'^2F'^2-CF'^2G'^2=0,$$

we have

$$AA'+BB'+CC'=3ABC-AF^2-BG^2-CH^2,\ =K'+2(ABC-FGH),$$

and thence

$$\Omega=F'G'H'(AA'+BB'+CC'-K')-AG'^2H'^2-BH'^2F'^2-CF'^2G'^2,$$

that is

$$\begin{aligned}\Omega&=-AG'H'(G'H'-A'F')-BH'F'(H'F'-B'G')-CF'G'(F'G'-C'H')-K'F'G'H',\\ &=-AG'H'K'F-BH'F'K'G-CF'G'K'H-K'F'G'H',\\ &=-K'(AFG'H'+BGH'F'+CHF'G'+F'G'H'),\end{aligned}$$

so that the condition $\Omega=0$ is satisfied if $K'=0$, that is if the equation

$$(A,\ B,\ C,\ F,\ G,\ H\between\xi,\ \eta,\ \zeta)^2=0,$$

which is the line-equation of the Absolute breaks up into factors; that is, if the Absolute be a point-pair.

In the case in question we may write

$$(A,\ B,\ C,\ F,\ G,\ H\between\xi,\ \eta,\ \zeta)^2=2(\alpha\xi+\beta\eta+\gamma\zeta)(\alpha'\xi+\beta'\eta+\gamma'\zeta),$$

that is

$$(A,\ B,\ C,\ F,\ G,\ H)=(2\alpha\alpha',\ 2\beta\beta',\ 2\gamma\gamma',\ \beta\gamma'+\beta'\gamma,\ \gamma\alpha'+\gamma'\alpha,\ \alpha\beta'+\alpha'\beta),$$

whence also, putting for shortness,

$$(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta)=\ (\lambda,\ \mu,\ \nu),$$

we have

$$(A',\ B',\ C',\ F',\ G',\ H')=-(\lambda^2,\ \mu^2,\ \nu^2,\ \mu\nu,\ \nu\lambda,\ \lambda\mu),$$

and also

$$K'=0,\ 2(ABC-FGH)=AA'+BB'+CC',\ =-2(\alpha\alpha'\lambda^2+\beta\beta'\mu^2+\gamma\gamma'\nu^2).$$

The original cubic equation is

$$(\alpha\alpha'\lambda^2+\beta\beta'\mu^2+\gamma\gamma'\nu^2)\,xyz+\alpha\alpha'\lambda yz\,(\mu y+\nu z)+\beta\beta'\mu zx\,(\lambda x+\nu z)+\gamma\gamma'\nu xy\,(\lambda x+\mu y)=0,$$

and this in fact is

$$(\alpha\alpha'\lambda yz+\beta\beta'\mu zx+\gamma\gamma'\nu xy)\,(\lambda x+\mu y+\nu z)=0.$$

The equation $\lambda x+\mu y+\nu z=0$ is that of the line through the two points which constitute the Absolute; the other factor gives

$$\alpha\alpha'\lambda yz+\beta\beta'\mu zx+\gamma\gamma'\nu xy=0,$$

which is the equation of a conic through the angles of the triangle $(x=0,\ y=0,\ z=0)$, and which also passes through the two points of the Absolute; in fact, writing $(\alpha,\ \beta,\ \gamma)$ for $(x,\ y,\ z)$ the equation becomes $\alpha\beta\gamma\,(\alpha'\lambda+\beta'\mu+\gamma'\nu)=0$, and so also writing $(\alpha',\ \beta',\ \gamma')$ for $(x,\ y,\ z)$ it becomes $\alpha'\beta'\gamma'\,(\alpha\lambda+\beta\mu+\gamma\nu)=0$, which relations are identically satisfied by the values of $(\lambda,\ \mu,\ \nu)$. Hence we see that the Absolute being a point-pair, the locus is the conic passing through the angles of the triangle, and the two points of the Absolute; that is, it is the *circle* passing through the angles of the triangle.

But assuming that K' is not $=0$, or that the Absolute is a proper conic, the equation $\Omega=0$ will be satisfied if

$$AFG'H'+BGH'F'+CHF'G'+F'G'H'=0,$$

we have $F',\ G',\ H'=Kf,\ Kg,\ Kh$ respectively, or omitting the factor K^2, the equation becomes

$$AFgh+BGhf+CHfg+Kfgh=0,$$

which is

$$f^2g^2h^2-bcg^2h^2-cah^2f^2-abf^2g^2+2abcfgh=0,$$

or, as it may also be written,

$$abcf^2g^2h^2\left(\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}\right)=0.$$

I remark that we have $ABC-FGH=K\,(abc-fgh)$; substituting also for $F',\ G',\ H'$ the values $Kf,\ Kg,\ Kh$, the equation of the cubic curve is

$$2\,(abc-fgh)\,xyz+Ayz\,(hy+gz)+Bzx\,(hx+fy)+Cxy\,(gx+fy)=0,$$

and the transformed form is

$$\left[\frac{2}{fgh}\,(abc-fgh)-\frac{A}{f^2}-\frac{B}{g^2}-\frac{C}{h^2}\right]xyz+\left(\frac{A}{f}\,yz+\frac{B}{g}\,zx+\frac{C}{h}\,xy\right)\left(\frac{x}{f}+\frac{y}{g}+\frac{z}{h}\right)=0\,;$$

we have

$$\frac{2}{fgh}\,(abc-fgh)-\frac{A}{f^2}-\frac{B}{g^2}-\frac{C}{h^2}=abc\left(\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}\right),$$

so that the foregoing condition

$$\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}=0,$$

being satisfied, the cubic breaks up into the line $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$, and the conic

$$\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy=0.$$

It is to be remarked that in general a triangle and the reciprocal triangle are in perspective; that is, the lines joining corresponding angles meet in a point, and the points of intersections of opposite sides lie in a line; this is the case therefore with the triangle $(x=0,\ y=0,\ z=0)$, and the reciprocal triangle

$$(ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0);$$

and it is easy to see that the line through the points of intersection of corresponding sides is in fact the above mentioned line $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$. It is to be noticed also that the coordinates of the point of intersection of the lines joining the corresponding angles are $(F,\ G,\ H)$. The conic

$$\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy=0$$

is of course a conic passing through the angles of the triangle $(x=0,\ y=0,\ z=0)$; it is *not*, what it might have been expected to be, a conic having double contact with the Absolute $(a,\ b,\ c,\ f,\ g,\ h)(x,\ y,\ z)^2$.

I return to the condition

$$\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}=0,$$

this can be shown to be the condition in order that the sides of the triangle $(x=0,\ y=0,\ z=0)$, and the sides of the reciprocal triangle $(ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0)$ touch one and the same conic; in fact, using line coordinates, the coordinates of the first three sides are $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, $(0,\ 0,\ 1)$ respectively, and those of the second three sides are $(a,\ h,\ g)$, $(h,\ b,\ f)$, $(g,\ f,\ c)$ respectively; the equation of a conic touching the first three lines is

$$\frac{L}{\xi}+\frac{M}{\eta}+\frac{N}{\zeta}=0,$$

and hence making the conic touch the second three sides, we have three linear equations from which eliminating L, M, N, we find

$$\begin{vmatrix} \frac{1}{a}, & \frac{1}{h}, & \frac{1}{g} \\ \frac{1}{h}, & \frac{1}{b}, & \frac{1}{f} \\ \frac{1}{g}, & \frac{1}{f}, & \frac{1}{c} \end{vmatrix}=0,$$

which is the equation in question.

We know that if the sides of two triangles touch one and the same conic, their angles must lie in and on the same conic. The coordinates of the angles are (1, 0, 0), (0, 1, 0), (0, 0, 1) and (A, H, G), (H, B, F), (G, F, C) respectively, and the angles will be situate in a conic if only

$$\begin{vmatrix} \frac{1}{A}, & \frac{1}{H}, & \frac{1}{G} \\ \frac{1}{H}, & \frac{1}{B}, & \frac{1}{F} \\ \frac{1}{G}, & \frac{1}{F}, & \frac{1}{C} \end{vmatrix} = 0,$$

an equation which must be equivalent to the last preceding one; this is easily verified. In fact, writing for shortness

$$\nabla = \begin{vmatrix} \frac{1}{a}, & \frac{1}{h}, & \frac{1}{g} \\ \frac{1}{h}, & \frac{1}{b}, & \frac{1}{f} \\ \frac{1}{g}, & \frac{1}{f}, & \frac{1}{c} \end{vmatrix}, \quad \square = \begin{vmatrix} \frac{1}{A}, & \frac{1}{H}, & \frac{1}{G} \\ \frac{1}{H}, & \frac{1}{B}, & \frac{1}{F} \\ \frac{1}{G}, & \frac{1}{F}, & \frac{1}{C} \end{vmatrix},$$

we have

$$-\square = \frac{1}{ABCF^2}(BC-F^2) + \frac{1}{CFGH^2}(FG-CH) + \frac{1}{BFG^2H}(HF-BG),$$

$$= \frac{K}{ABCF^2G^2H^2}(aG^2H^2 + hABFG + gCAHF),$$

and the second factor is

$$= aGH(AF+Kf) + AFhBG + AFgCH,$$

$$= AF(aGH + hBG + gCH) + KafGH.$$

But

$$\begin{aligned} aGH + hBG + gCH &= G(aH+hB) + gCH = G - gF + gCH, \\ &= G - gF + gCH, \\ &= -g(FG-CH), \\ &= -ghK, \end{aligned}$$

so that the second factor is

$$= K(afGH - ghAF),$$

which is

$$= K(f^2g^2h^2 - bcg^2h^2 - cah^2f^2 - abf^2g^2 + 2abcfgh),$$

$$= Kabcf^2g^2h^2\left(\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh}\right),$$

$$= Kabcf^2g^2h^2\nabla,$$

so that we have identically

$$-ABCF^2G^2H^2\,\square = K^2abcf^2g^2h^2\nabla,$$

and the conditions $\nabla=0$, $\square=0$ are consequently equivalent.

The condition

$$\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}=0,$$

is the condition in order that the function

$$\left(\frac{1}{a},\ \frac{1}{b},\ \frac{1}{c},\ \frac{1}{f},\ \frac{1}{g},\ \frac{1}{h}\between ax,\ by,\ cz\right)^2,$$

may break up into linear factors; the function in question is

$$\left(a,\ b,\ c,\ \frac{bc}{f},\ \frac{ca}{g},\ \frac{ab}{h}\between x,\ y,\ z\right)^2,$$

which is

$$=(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2+2\left(\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy\right),$$

so that the condition is, that the conic

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2+2\left(\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy\right)=0,$$

(which is a certain conic passing through the intersections of the Absolute $(a, b, c, f, g, h\between x, y, z)^2=0$, and of the locus conic $\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy=0$) shall be a pair of lines. Writing the equation of the conic in question under the form

$$\left(a,\ b,\ c,\ \frac{bc}{f},\ \frac{ca}{g},\ \frac{ab}{h}\between x,\ y,\ z\right)^2=0,$$

the inverse coefficients A', B', C', F', G', H' of this conic, are

$$\left(-\frac{Abc}{f^2},-\frac{Bca}{g^2},-\frac{Cab}{h^2},-\frac{abc}{fgh}F,-\frac{abc}{fgh}G,-\frac{abc}{fgh}H\right),$$

so that we have $F' : G' : H' = F : G : H$. Hence, if in regard to this new conic we form the reciprocal of the triangle $(x=0,\ y=0,\ z=0)$, and join the corresponding angles of the two triangles, the joining lines meet in a point which is the same point as is obtained by the like process from the triangle and its reciprocal in regard to the Absolute. But I do not further pursue this part of the theory.

It is to be noticed that the conic

$$\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy=0,$$

contains the angles of the reciprocal triangle, and is thus in fact the conic in which are situate the angles of the two triangles. For the coordinates of one of the angles of the reciprocal triangle are (A, H, G); we should therefore have

$$\frac{A}{f}HG + \frac{B}{g}GA + \frac{C}{h}AH = 0,$$

which is

$$\frac{A}{fgh}(GHgh + BGhf + CHfg) = 0,$$

or attending only to the second factor and writing

$$GH = Kf + AF,$$

the condition is

$$Kfgh + AFgh + BGhf + CHfg = 0,$$

or substituting for K, A, B, C, F, G, H their values and reducing, this is

$$-abcf^2g^2h^2\left(\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh}\right) = 0,$$

which is satisfied: hence the three angles of the reciprocal triangle lie on the conic in question.

Partially recapitulating the foregoing results, we see in the case where the Absolute is not a point-pair, that the locus of a point such that the perpendiculars from it on the sides of the triangle have their feet in a line, is in general a *cubic curve* passing through the angles of the triangle: if, however, the condition

$$\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh} = 0$$

be satisfied, that is, if the triangle be such that the angles thereof and of the reciprocal triangle lie in a conic (or, what is the same thing, if the sides touch a conic) then the cubic locus breaks up into the line $\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0$, which is the line through the points of intersection of the corresponding sides of the two triangles, and into the conic

$$\frac{A}{f}yz + \frac{B}{g}zx + \frac{C}{h}xy = 0,$$

which is the conic through the angles of the two triangles.

The question arises, given a conic (the Absolute) to construct a triangle such that its angles, and the angles of the reciprocal triangle in regard to the given conic, lie in a conic.

I suppose that two of the angles of the triangle are given, and I enquire into the locus of the remaining angle. To fix the ideas, let A, B, C be the angles of the triangle, A', B', C' those of the reciprocal triangle; and let the angles A and B be given. We have to find the locus of the point C: I observe however, that the lines AA', BB', CC' meet in a point O, and I conduct the investigation in such manner as to obtain simultaneously the loci of the two points C and O. The lines $C'B'$, $C'A'$ are the polars of A, B respectively, let their equations be $x=0$, and $y=0$, and let the equation of the line AB be $z=0$; this being so, the equation of the given conic will be of the form

$$(a, b, c, 0, 0, h\between x, y, z)^2=0.$$

I take (α, β, γ) for the coordinates of O and (x, y, z) for those of C; the coordinates of either of these points being of course deducible from those of the other.

Observing that the inverse coefficients are

$$(bc, ca, ab-h^2, 0, 0, -ch),$$

we find

$$\begin{aligned} &\text{coordinates of } A \text{ are } (\;b, -h, 0),\\ &\quad ,, \qquad B \;\;,, \;\; (-h, \;\; a, 0). \end{aligned}$$

The points A' and B' are then given as the intersections of AO with $C'A'(y=0)$ and of BO with $C'B'(x=0)$; we find

$$\begin{aligned} &\text{coordinates of } A' \text{ are } (h\alpha+b\beta, \quad 0 \quad, h\gamma),\\ &\quad ,, \qquad B' \;\;,, \;\; (\quad 0 \quad, a\alpha+h\beta, h\gamma). \end{aligned}$$

Moreover,

$$\begin{aligned} &\text{coordinates of } C' \text{ are } (0, 0, 1),\\ &\quad ,, \qquad C \;\;,, \;\; (x, y, z). \end{aligned}$$

The six points A, B, C, A', B', C' are to lie in a conic; the equations of the lines $C'A$, $C'B$, AB are $hX+bY=0$, $aX+hY=0$, $Z=0$, and hence the equation of a conic passing through the points C', A, B is

$$\frac{L}{aX+hY}+\frac{M}{hX+bY}+\frac{N}{Z}=0.$$

Hence, making the conic pass through the remaining points A', B', C, we find

$$\frac{L}{a\,(h\alpha+b\beta)}+\frac{M}{h\,(h\alpha+b\beta)}+\frac{N}{h\gamma}=0,$$

$$\frac{L}{h\,(a\alpha+h\beta)}+\frac{M}{b\,(a\alpha+h\beta)}+\frac{N}{h\gamma}=0,$$

$$\frac{L}{ax+hy}+\frac{M}{hx+by}+\frac{N}{z}=0,$$

and eliminating the L, M, N, we find

$$\begin{vmatrix} \dfrac{1}{a}\ , & \dfrac{1}{h}\ , & h\alpha+b\beta \\ \dfrac{1}{h}\ , & \dfrac{1}{b}\ , & a\alpha+h\beta \\ \dfrac{1}{ax+hy}, & \dfrac{1}{hx+by}, & \dfrac{hy}{z} \end{vmatrix} = 0,$$

or developing and reducing, this is

$$-\frac{(ab-h^2)}{hab}\frac{y}{z}+\frac{1}{h}\frac{a\alpha+h\beta}{ax+hy}+\frac{1}{h}\frac{h\alpha+b\beta}{hx+by}$$
$$-\frac{1}{a}\frac{a\alpha+h\beta}{hx+by}-\frac{1}{b}\frac{h\alpha+b\beta}{ax+hy}=0.$$

We have still to find the relation between (α, β, γ) and (x, y, z); this is obtained by the consideration that the line $A'B'$, through the two points A', B' the coordinates of which are known in terms of (α, β, γ), is the polar of the point C, the coordinates of which are (x, y, z). The equation of $A'B'$ is thus obtained in the two forms

$$(a\alpha+h\beta)\,X+(h\alpha+b\beta)\,Y-\frac{(a\alpha+h\beta)\,(h\alpha+b\beta)}{hy}\,Z=0,$$

and

$$(ax+hy)\,X+(hx+by)\,Z+\qquad cz\qquad Z=0,$$

and comparing these, we have

$$x : y : z = \alpha : \beta : \frac{-(a\alpha+h\beta)\,(h\alpha+b\beta)}{chy},$$

or what is the same thing

$$\alpha : \beta : \gamma = x : y : \frac{-(ax+hy)\,(hx+by)}{chz},$$

(where it is to be observed that the equation $\alpha : \beta = x : y$ is the verification of the theorem that the lines AA', BB', CC' meet in a point O).

We may now from the above found relation eliminate either the (α, β, γ) or the (x, y, z); first eliminating the (α, β, γ), we find

$$-\frac{ab-h^2}{hab}\frac{Y}{Z}+\frac{2}{h}-\frac{1}{a}\frac{ax+hy}{hx+by}-\frac{1}{b}\frac{hx+by}{ax+by}=0,$$

where

$$\frac{Y}{Z}=-\frac{(ax+hy)\,(hx+by)}{chz^2},$$

or, completing the elimination,

$$\frac{ab-h^2}{ch}\,\frac{(ax+hy)^2\,(hx+by)^2}{z^2}=(hb,\ -ab,\ ha\between ax+hy,\ hx+by)^2=0,$$

which is a quartic curve having a node at each of the points

$$(z=0,\ ax+hy=0),\ (z=0,\ hx+by=0),\ (ax+hy=0,\ hx+by=0),$$

that is, at each of the points B, A, C'. The right-hand side of the foregoing equation is

$$= -(ab-h^2)(ha,\ ab,\ hb \between x,\ y)^2, = -(ab-h^2)\,h\left(ax^2+by^2+\frac{2ab}{h}xy\right),$$

so that the equation may also be written

$$(ax+hy)^2(hx+by)^2 + ch^2z^2\left(ax^2+by^2+\frac{2ab}{h}xy\right) = 0.$$

Secondly, to eliminate the (x, y, z), we have

$$-\frac{ab-h^2}{hab}\frac{Y}{Z}+\frac{2}{h}-\frac{1}{a}\frac{a\alpha+h\beta}{h\alpha+b\beta}-\frac{1}{b}\frac{h\alpha+b\beta}{a\alpha+h\beta}=0,$$

where

$$\frac{Y}{Z}=\frac{chy^2}{(a\alpha+h\beta)(h\alpha+b\beta)},$$

or, completing the elimination,

$$(ab-h^2)\,chy^2 = \quad (hb,\ -ab,\ ha \between a\alpha+h\beta,\ h\alpha+b\beta)^2$$

$$= -(ab-h^2)\,h\left(a\alpha^2+b\beta^2+\frac{2ab}{h}\alpha\beta\right),$$

that is

$$\left(a,\ b,\ c,\ 0,\ 0,\ \frac{ab}{h} \between \alpha,\ \beta,\ \gamma\right)^2 = 0.$$

Writing (x, y, z) in place of (α, β, γ), the locus of the point O is the conic

$$\left(a,\ b,\ c,\ 0,\ 0,\ \frac{ab}{h} \between x,\ y,\ z\right)^2 = 0,$$

which is a conic intersecting the Absolute

$$(a,\ b,\ c,\ 0,\ 0,\ h \between x,\ y,\ z)^2 = 0,$$

at its intersections with the lines $x=0$, $y=0$, that is the lines $C'B'$ and $C'A'$.

In regard to this new conic, the coordinates of the pole of $C'B'$ $(x=0)$ are at once found to be $(-h,\ a,\ 0)$, that is, the pole of $C'B'$ is B; and similarly the coordinates of the pole of $C'A'$ $(y=0)$ are $(b,\ -h,\ 0)$, that is, the pole of $C'A'$ is A. We may consequently construct the conic the locus of O, viz. given the Absolute and the points A and B, we have $C'A'$ the polar of B, meeting the Absolute in two points (a_1, a_2), and $C'B'$ the polar of A meeting the Absolute in the points (b_1 and b_2); the lines $C'A'$ and $C'B'$ meet in C'. This being so, the required conic passes through the points a_1, a_2, b_1, b_2, the tangents at these points being Aa_1, Aa_2, Bb_1, Bb_2 respectively; eight conditions, five of which would be sufficient to determine the conic. It is to be remarked that the lines $C'B'$, $C'A'$ (which in regard to the Absolute are the polars of A, B respectively) are in regard to the required conic the polars of B, A respectively.

The conic the locus of O being known, the point O may be taken at any point of this conic, and then we have A' as the intersection of $C'A'$ with AO, B' as the intersection of $C'B'$ with BO, and finally, C as the pole of the line $A'B'$ in regard to the Absolute, the point so obtained being a point on the line $C'O$. To each position of O on the conic locus, there corresponds of course a position of C; the locus of C is, as has been shown, a quartic curve having a node at each of the points C', A, B.

The foregoing conclusions apply of course to spherical figures; we see therefore that on the sphere the locus of a point such that the perpendiculars let fall on the sides of a given spherical triangle have their feet in a line (great circle), is a spherical cubic. If, however, the spherical triangle is such that the angles thereof and the poles of the sides (or, what is the same thing, the angles of the polar triangle) lie on a spherical conic; then the cubic locus breaks up into a line (great circle), which is in fact the circle having for its pole the point of intersection of the perpendiculars from the angles of the triangle on the opposite sides respectively, and into the before-mentioned spherical conic. Assuming that the angles A and B are given, the above-mentioned construction, by means of the point O, is applicable to the determination of the locus of the remaining angle C, in order that the spherical triangle ABC may be such that the angles and the poles of the sides lie on the same spherical conic, but this requires some further developments. The lines $C'B'$, $C'A'$ which are the polars of the given angles A, B respectively, are the cyclic arcs of the conic the locus of O, or say for shortness the conic O; and moreover these same lines $C'B'$, $C'A'$ are in regard to the conic O, the polars of the angles B, A respectively. If instead of the conic O we consider the polar conic O', it follows that A, B are the foci, and $C'A'$, $C'B'$ the corresponding directrices of the conic O'. The distance of the directrix $C'A'$ from the centre of the conic, measuring such distance along the transverse axis is clearly $= 90° -$ distance of the focus A; it follows that the transverse semi-axis is $= 45°$, or what is the same thing, that the transverse axis is $= 90°$; that is, the conic O' is a conic described about the foci A, B with a transverse axis (or sum or difference of the focal distances) $= 90°$. Considering any tangent whatever of this conic, the pole of the tangent is a position of the point O, which is the point of intersection of the perpendiculars let fall from the angles of the spherical triangle on the opposite sides; hence, to complete the construction, we have only through A and B respectively to draw lines AC, BC perpendicular to the lines BO, CO respectively; the lines in question will meet in a point C, which is such that CO will be perpendicular to AB, and which point C is the required third angle of the spherical triangle ABC. In order to ascertain whether a given spherical triangle ABC has the property in question (viz. whether it is such that the angles thereof and of the polar triangle lie in a spherical conic), we have only to construct as before the conic O' with the foci A, B and transverse axis $= 90°$, and then ascertain whether the polar of the point O, the intersections of the perpendiculars from the angles of the triangle on the opposite sides respectively, is a tangent of the conic O'. It is moreover clear, that given a triangle ABC having the property in question, if with the foci A, B and transverse axis $= 90°$ we describe a conic, and if in like manner with the foci A, C and the same transverse axis, and with the foci B, C and the same transverse axis, we describe two other conics; then that the three conics will have a common tangent the pole whereof will be the point of intersection of the perpendiculars from the angles of the triangle ABC on the opposite sides respectively.