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NOTICES OF COMMUNICATIONS TO THE LONDON MATHEMATICAL SOCIETY.

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December 13, 1866. pp. 6—7.

PROF. CAYLEY exhibited and explained some geometrical drawings. Thinking that the information might be convenient for persons wishing to make similar drawings, he noticed that the paper used was a tinted drawing paper, made in continuous lengths up to 24 yards, and of the breadth of about 56 inches [1]; the half-breadth being therefore sufficient for ordinary figures, and the paper being of a good quality and taking colour very readily. Among the drawings was one of the conics through four points forming a convex quadrangle. The plane is here divided into regions by the lines joining each of the six pairs of points, and by the two parabolas through the four points; and the regions being distinguished by different colours, the general form of the conics of the system is very clearly seen. (Prof. Cayley remarked that it would be interesting to make the figures of other systems of conics satisfying four conditions; and in particular for the remaining elementary systems of conics, where the conics pass through a number 3, 2, 1 or 0 of points and touch a number 1, 2, 3 or 4 of lines: the construction of some of these figures is, however, practically a great deal more difficult.) Other figures related to Cartesians and Bicircular Quartics. One of these was a figure of a system of triconfocal Cartesians; and derived from this by inversion in regard to a circle, there was a figure of a system of quadriconfocal bicircular quartics: in the assumed position of the inverting circle, each quartic consists (like the Cartesian which gives rise to it) of an exterior and an interior continuous curve, and the general aspect of the figure is that of a distortion of the original figure of the Cartesians. Another figure was that of the bicircular quartic, for which the

[1] Sold at Messrs Lechertier-Barbe's, Regent Street, at 6*d*. per yard, or 9*s*. the piece.

algebraical sum of the distances of a point thereof from three given foci is $= 0$ (this was selected for facility of construction, by the intersections of circles and confocal conics). The quartic consists of two equal and symmetrically situated pear-shaped curves, exterior to each other, and including the one of them two of the three given foci, the other of them the third given focus, and a fourth focus lying in a circle with the given foci: by inversion in regard to a circle having its centre at a focus the two pear-shaped curves became respectively the exterior and the interior ovals of a Cartesian. There was also a figure of the two circular cubics, having for foci four given points on a circle; and a figure (coloured in regions) in preparation for the construction of the analogous sextic curve derived from four given points not in a circle.

March 28, 1867. pp. 25—26.

Professor Cayley mentioned a theorem included in Prof. Sylvester's theory of derivation of the points of a cubic curve. Writing down the series of numbers 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, &c., viz., all the numbers not divisible by 3, then (repetitions of the same number being permissible) taking any two numbers of the series, we have in the series a third number, which is the sum or else the difference of the two numbers (for example, 2, 2 give their sum 4, but 2, 7 give their difference 5), and we have thus a series of triads, in each of which one number is the sum of the other two. The theorem is, that it is possible on a cubic curve to construct a series of points, such that denoting them by the above numbers respectively, then for any triad of numbers as aforesaid the points denoted by the three numbers respectively lie *in lineâ*. And the theorem gives its own construction: in fact the series of triads is 112, 224, 145, 257, 178, 248, &c. Take 1, an arbitrary point on the cubic, then (by the theorem) the triad 112 shows that 2 is the tangential of 1; 224 shows that 4 is the tangential of 2; 145 that 5 is the third point of 1 and 4; 257 that 7 is the third point of 2 and 5. So far we have no theorem; we have merely, starting from the point 1, constructed by an arbitrary process the points 2, 4, 5, and 7. But going a step further; 178 and 448 show, the first of them, that 8 is the third point of 1 and 7, the second of them, that 8 is the tangential of 4. We have here the theorem that the third point of 1 and 7 is also the tangential of 4. Similarly, 10, 11, 13 are each of them (like 8) determined by two constructions; 14, 16, 17, 19, each of them by three constructions, and so on; the number of constructions increasing by unity for each group of four numbers. And the theorem is, that these constructions, 2, 3, or more, as the case may be, give always one and the same point. Prof. Cayley mentioned that on a large figure of a cubic curve he had, in accordance with the theorem, constructed the series of points 1, 2, 4, 5, 7, 8, 10, 11, 13, 14.

April 15, 1867. p. 29.

Prof. Cayley communicated a theorem relating to the locus of the ninth of the points of intersection of two cubics, seven of these points being fixed, while the eighth moves on a straight line.

March 26, 1868. pp. 61—63.

Prof. Cayley made some remarks on a mode of generation of a sibi-reciprocal surface, that is, a surface the reciprocal of which is of the same order and has the same singularities as the original surface.

If a surface be considered as the envelope of a *plane* varying according to given conditions, this is a mode of generation which is essentially not sibi-reciprocal; the reciprocal surface is given as the locus of a *point* varying according to the reciprocal conditions. But if a surface be considered as the envelope of a *quadric surface* varying according to given conditions, then the reciprocal surface is given as the envelope of a *quadric surface* varying according to the reciprocal conditions; and if the conditions be sibi-reciprocal, it follows that the surface is a sibi-reciprocal surface. For instance, considering the surface which is the envelope of a quadric surface touching each of 8 given lines; the reciprocal surface is here the envelope of a quadric surface touching each of 8 given lines; that is, the surface is sibi-reciprocal. So again, when a quadric surface is subjected to the condition that 4 given points shall be in regard thereto a conjugate system, this is equivalent to the condition that 4 given planes shall be in regard thereto a conjugate system—or the condition is sibi-reciprocal; analytically the quadric surface $ax^2 + by^2 + cz^2 + dw^2 = 0$ is a quadric surface subjected to a sibi-reciprocal system of six conditions. Impose on the quadric surface two more sibi-reciprocal conditions,—for instance, that it shall pass through a given point and touch a given plane,—the envelope of the quadric will be a sibi-reciprocal surface. It was noticed that in this case the envelope was a surface of the order (=class) 12, and having (besides other singularities) the singularities of a conical point with a tangent cone of the class 3, and of a curve of plane contact of the order 3. In the foregoing instances the number of conditions imposed upon the quadric surface is 8; but it may be 7, or even a smaller number. An instance was given of the case of 7 conditions, viz.,—the quadric surface is taken to be $ax^2 + by^2 + cz^2 + dw^2 = 0$ (6 conditions) with a relation of the form

$$Abc + Bca + Cab + Fad + Gbd + Hcd = 0$$

between the coefficients (1 condition); this last condition is at once seen to be sibi-reciprocal; and the envelope is consequently a sibi-reciprocal surface—viz., it is a surface of the order (=class) 4, with 16 conical points and 16 conics of plane contact. It is the surface called by Prof. Cayley the "tetrahedroid," (see his paper "Sur la surface des ondes," *Liouv.* tom. XI. (1846), pp. 291—296 [47]), being in fact a homographic transformation of Fresnel's Wave Surface.

{Prof. Cayley adds an observation which has since occurred to him. If the quadric surface $ax^2 + by^2 + cz^2 + dw^2 = 0$, be subjected to touch a given line, this imposes on the coefficients a, b, c, d, a relation of the above form, viz., the relation is

$$A^2bc + B^2ca + C^2ab + F^2ad + G^2bd + H^2cd = 0\,;$$

where A, B, C, F, G, H are the "six coordinates" of the given line, and satisfy therefore the relation $AF + BG + CH = 0$. It is easy to see that there are 8 lines for which the squared coordinates have the same values A^2, B^2, C^2, F^2, G^2, H^2; these 8 lines are symmetrically situate in regard to the tetrahedron of coordinates, and moreover they lie in a hyperboloid. The quadric surface, instead of being defined as above, may, it is clear, be defined by the equivalent conditions of touching each of the 8 given lines: that is, we have the envelope of a quadric surface touching each of 8 given lines; these lines not being arbitrary lines, but being a system of a very special form. By what precedes, the envelope is a quartic surface. It appears, however, that in virtue of the relation $AF + BG + CH = 0$, this is no longer a proper quartic surface, but that it resolves itself into the above-mentioned hyperboloid taken twice. That is, restoring the original A, B, &c., in place of A^2, B^2, &c., the envelope of the quadric $ax^2 + by^2 + cz^2 + dw^2 = 0$, where a, b, c, d vary, subject to the condition $Abc + Bca + Cab + Fad + Gbd + Hcd = 0$, (which is in general a tetrahedroid), is when A, B, C, F, G, H are the squared coordinates of a line (or, what is the same thing, when $\sqrt{AF} + \sqrt{BG} + \sqrt{CH} = 0$) a hyperboloid taken twice, viz., this is the hyperboloid passing through the given line and through the symmetrically situate seven other lines.}

November 12, 1868. pp. 103, 104.

Professor Cayley gave an account to the Meeting of a Memoir by Herr Listing, "Census räumlicher Complexe oder Verallgemeinerung des Euler'schen Satzes von den Polyedern," published in the *Göttingen Transactions* for 1862. The fundamental theorem is a relation $a - (b - \kappa) + (c - \kappa' + \pi) - (d - \kappa'' + \pi' - \omega) = 0$ existing in any figure whatever between a the number of points, b the number of lines, c the number of areas, d the number of spaces, and certain supplementary quantities κ, κ', κ'', π, π', ω. In an extensive class of figures these last are each $= 0$, and the relation is $a - b + c - d = 0$; thus, in a closed box, $a = 8$, $b = 12$, $c = 6$, $d = 2$ (viz., there is the finite space inside, and the infinite space outside, the box): if the box be opened, $a = 10$, $b = 15$, $c = 6$, $d = 1$; if the lid be taken away, $a = 8$, $b = 12$, $c = 5$, $d = 1$; in each case, $a - b + c - d = 0$. If the bottom be also taken away, $a = 8$, $b = 12$, $c = 4$, $d = 1$; but here one of the supplementary quantities comes in, $\kappa'' = 1$, and the theorem is $a - b + c - (d - \kappa'') = 0$. The chief difficulty and interest of the Memoir consist in the determination of the supplementary quantities κ, κ', κ'', π, π', ω.

December 10, 1868. pp. 123—125. Appended to Paper by Mr T. Cotterill "On a Correspondence of Points etc."

Observations by Professor Cayley and Mr W. K. Clifford on the connexion of the transformation with Cremona's general theory, and the analytical formulæ.

According to Cremona's general theory,—taking (x_1, y_1, z_1) and (x_2, y_2, z_2) as current coordinates in the two planes respectively,—if we take in the first plane, any three points 1, 2, 3, and any other three points 4′, 5′, 6′, then if $X_1 = 0$, $Y_1 = 0$, $Z_1 = 0$ are quartic curves, each having the double points 1, 2, 3, and the simple points 4′, 5′, 6′, we have a transformation $x_2 : y_2 : z_2 = X_1 : Y_1 : Z_1$ leading to a converse system

$$x_1 : y_1 : z_1 = X_2 : Y_2 : Z_2$$

of the like form; viz., there will be in the second plane three points 4, 5, 6, and three other points 1′, 2′, 3′, such that $X_2 = 0$, $Y_2 = 0$, $Z_2 = 0$, are quartics having the double points 4, 5, 6, and the simple points 1′, 2′, 3′.

Analytically, Cremona's transformation is obtained by assuming the reciprocals of x_2, y_2, z_2 to be proportional to linear functions of the reciprocals of x_1, y_1, z_1—(of course, this being so, the reciprocals of x_1, y_1, z_1 will be proportional to linear functions of the reciprocals of x_2, y_2, z_2). Solving this under the theory as above explained, write

$$\left.\begin{array}{l}\dfrac{1}{x_2}\\[2ex] :\dfrac{1}{y_2}\\[2ex] :\dfrac{1}{z_2}\end{array}\right\} = \left\{\begin{array}{l}\dfrac{a}{x_1}+\dfrac{b}{y_1}+\dfrac{c}{z_1}\\[2ex] :\dfrac{d}{x_1}+\dfrac{e}{y_1}+\dfrac{f}{z_1}\\[2ex] :\dfrac{g}{x_1}+\dfrac{h}{y_1}+\dfrac{i}{z_1}\end{array}\right\} = \left\{\begin{array}{l}P_1\\[2ex] :Q_1\\[2ex] :R_1\end{array}\right.$$

if

$$P_1 = ay_1z_1 + bz_1x_1 + cx_1y_1,$$

$$Q_1 = dy_1z_1 + ez_1x_1 + fx_1y_1,$$

$$R_1 = gy_1z_1 + hz_1x_1 + ix_1y_1.$$

Hence

$$x_2 : y_2 : z_2 = Q_1R_1 : R_1P_1 : P_1Q_1.$$

$Q_1R_1 = 0$, &c., are quartics, or generally $\alpha Q_1R_1 + \beta R_1P_1 + \gamma P_1Q_1 = 0$ is a quartic, having three double points $(y_1 = 0,\ z_1 = 0)$, $(z_1 = 0,\ x_1 = 0)$, $(x_1 = 0,\ y_1 = 0)$, and having besides the three points which are the remaining points of intersection of the conics $(Q_1 = 0,\ R_1 = 0)$, $(R_1 = 0,\ P_1 = 0)$, $(P_1 = 0,\ Q_1 = 0)$ respectively; viz., these last are the points

$$\frac{1}{x_1} : \frac{1}{y_1} : \frac{1}{z_1} = ei - hf : fg - id : dh - ge, \text{ \&c. \&c.}$$

The double and simple points are fixed points (that is, independent of α, β, γ), and the formulæ come under Cremona's theory. It is, however, necessary to show that if the points $4'$, $5'$, $6'$ are in a line, the points $1'$, $2'$, $3'$ are also in a line. This may be done as follows:

Let there be three planes A, B, C, and let the points of the first two correspond by ordinary triangular inversion in respect of the triangle α_1 on the plane A, and β_1 on the plane B. Let also the planes B, C correspond by ordinary triangular inversion in respect of the triangle β_2 on the plane B, and γ_2 on the plane C. Then the correspondence between A and C is the one considered, the points $1\,2\,3$ forming the triangle α_1 and the points $4\,5\,6$ forming the triangle γ_2. The points $4'5'6'$ and $1'2'3'$ in the planes A, C respectively correspond to the triangles β_1, β_2; and the conditions that $4'$, $5'$, $6'$ shall be in a line and that $1'$, $2'$, $3'$ shall be in a line, are the same condition, namely, that the triangles β_1, β_2 shall be inscribed in the same conic. Analogous properties must apparently belong to Cremona's other transformations, and the investigation of them will form an interesting part of the theory.

It is important, also, to notice the relation of the transformation to Hesse's "Uebertragungsprincip," *Crelle*, tom. LXVI. p. 15, which establishes a correspondence between the points of a plane and the point-pairs on a line. If $Ax^2 + 2Bxy + Cy^2 = 0$ is the equation of a point-pair, the coordinates in the plane are taken by Hesse *directly*, but in the present Paper *inversely* proportional to A, B, C.