385.

ON THE CORRESPONDENCE OF TWO POINTS ON A CURVE.

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1. In a unicursal curve the coordinates (x, y, z) of any point of the curve are proportional to rational and integral functions of a variable parameter θ. Hence, if two points of the curve correspond in suchwise that to a given position of the first point there correspond α' positions of the second point, and to a given position of the second point α positions of the first point, the number of points which correspond each to itself is $=\alpha+\alpha'$. For let the two points be determined by their parameters θ, θ' respectively—then to a given value of θ there correspond α' values of θ', and to a given value of θ' there correspond α values of θ; hence the relation between (θ, θ') is of the form $(\theta, 1)^{\alpha}(\theta', 1)^{\alpha'}=0$; and writing therein $\theta'=\theta$, then for the points which correspond each to itself, we have an equation $(\theta, 1)^{\alpha+\alpha'}=0$ of the order $\alpha+\alpha'$; that is, the number of these points is $=\alpha+\alpha'$.

2. Hence for a unicursal curve we have a theorem similar to that of M. Chasles' for a line, viz., the theorem may be thus stated:

If two points of a unicursal curve have an (α, α') correspondence, the number of united points is $=\alpha+\alpha'$.

3. But a unicursal curve is nothing else than a curve with a deficiency $D=0$, and we thence infer

Theorem. If two points of a curve with deficiency D have an (α, α') correspondence, the number of united points is $=\alpha+\alpha'+2kD$; in which theorem $2k$ is a coefficient to be determined.

4. Suppose that the corresponding points are P, P' and imagine that when P is given, the corresponding points P' are the intersections of the given curve by a

curve Θ (the equation of the curve Θ will of course contain the coordinates of P as parameters, for otherwise the position of P' would not depend upon that of P). I find that if the curve Θ has with the given curve k intersections at the point P, then in the system of (P, P'), the number of united points is

$$\mathrm{a} = \alpha + \alpha' + 2kD,$$

whence in particular, if the curve Θ does not pass through the point P, then the number of united points is $= \alpha + \alpha'$, as in a unicursal curve.

4*. The foregoing theorem is easily proved in the particular case where the k intersections at the point P take place in consequence of the curve θ having a k-tuple point at P. Taking $U = (x, y, z)^m = 0$ as the equation of the given curve (which for greater simplicity is assumed to be a curve without singularities), then if we suppose (x, y, z) to be the coordinates of the point P, and (x', y', z') to be the coordinates of the point P', write $U = (x, y, z)^m$, $U' = (x', y', z')^m$, U' being what U becomes on writing therein (x', y', z') in place of (x, y, z); and

$$\Theta = (x, y, z)^a (x', y', z')^{a'} (yz' - y'z, zx' - z'x, xy' - x'y)^k,$$

viz., Θ is a function of the order k in $yz' - yz'$, $zx' - z'x$, $x'y - x'y$, the coefficients of the several powers and products of these quantities being functions of the order a in (x, y, z) and of the order a' in (x', y', z'), which functions are such that they do not all of them vanish, identically, or in virtue of the equation $U = 0$, on writing therein $(x', y', z') = (x, y, z)$. Taking for a moment (x, y, z) as current coordinates, suppose that the equation of the given curve is $U = 0$; then if (x, y, z) are the coordinates of the point P, we have $U = 0$, and similarly if (x', y', z') are the coordinates of the point P' we have $U' = 0$. The equation $\Theta = 0$, considering therein (x, y, z) as the coordinates of the given point P (and so as parameters satisfying the equation $U = 0$) and (x', y', z') as current coordinates, will be a curve having a k-tuple point at P, we have thus the case above supposed; and P being given, the corresponding points P' are given as the intersections of the curves $U' = 0$, $\Theta = 0$, which are respectively of the orders m and $a' + k$; the total number of intersections is thus $= m(a' + k)$, but inasmuch as the curve $\Theta = 0$ has a k-tuple point at P, k of these intersections coincide with the point P, and the number of the remaining intersections, that is the number of positions of the point P', is $= ma' + (m-1)k$. Similarly when P' is given, the number of positions of the point P is $= ma + (m-1)k$: and we have therefore

$$\alpha + \alpha' = m(a + a') + 2(m-1)k.$$

To find the united points, it is to be observed, that upon writing $(x', y', z') = (x, y, z)$, the function Θ becomes identically $= 0$; but if we suppose, in the first instance, that P', P, are consecutive points on the curve $U = 0$, then we have

$$yz' - y'z : zx' - z'x : xy' - x'y = \delta_x U ; \ \delta_y U ; \ \delta_z U ;$$

and the equation $\Theta = 0$ assumes the form

$$\Theta = (x, y, z)^a (x, y, z)^{a'} (\delta_x U, \delta_y U, \delta_z U)^k = 0,$$

which, (x, y, z) being current coordinates, is the equation of a curve of the order $a+a'+(m-1)k$; the united points are the intersection of this curve with the given curve $U=0$, and the number of the united points is thus

$$\mathrm{a}=m(\alpha+\alpha')+m(m-1)k.$$

Hence attending to the above-mentioned value of $\alpha+\alpha'$, we have

$$\mathrm{a}=\alpha+\alpha'+(m-1)(m-2)k.$$

But in the case of a curve $U=0$, without singularities, we have $2D=(m-1)(m-2)$, and we have thus the required formula

$$\mathrm{a}=\alpha+\alpha'+2kD.$$

The investigation in the case where the k intersections at P arise wholly or in part from a contact of the curve Θ, or any branch or branches thereof, with the given curve U, is more difficult, and I abstain from entering upon it.

I apply the theorem to some examples:

5. Investigation of the class of a curve of the order m with δ dps. Take as corresponding points on the curve two points, such that the line joining them passes through a fixed point O: the united points will be the points of contact of the tangents through O; that is, the number of the united points is equal to the class of the curve. The curve Θ is here the line OP, which has with the given curve a single intersection at P; that is, we have $k=1$. The points P' corresponding to a given position of P are the remaining $m-1$ intersections of OP with the curve; that is, we have $\alpha'=m-1$; and in like manner $\alpha=m-1$. Hence the class is $=2(m-1)+2D$, viz., this is $=(2m-2)+(m^2-3m+2-2\delta)$, which is $=m^2-m-2\delta$, as it should be.

6. It is in the foregoing example assumed that the dps are none of them cusps; if the curve has $\delta+\kappa$ dps, κ of which are cusps (or what is the same thing, δ nodes and κ cusps); then the number of united points is equal $2(m-1)+2D, =m^2-m-2\delta-2\kappa$; but in this case each of the cusps is reckoned as a united point, and we have, therefore, class $+\kappa=m^2-m-2\delta-2\kappa$, that is, class $=m^2-m-2\delta-3\kappa$. This will serve as an instance of the special considerations which are required in the case of a curve with cusps, but in what follows, I shall assume that the dps are none of them cusps and thus attend to the case of a curve of the order m, with δ dps, and therefore of the class $n=m^2-m-2\delta$, and of the deficiency $D=\frac{1}{2}(m-1)(m-2)-\delta, =\frac{1}{2}(n-2m+2)$.

7. Investigation of the number of inflexions. Taking the point P' to be a tangential of P (that is, an intersection of the curve by the tangent at P), the united points are the inflexions, and the number of the united points is equal to the number of inflexions. The curve Θ is here the tangent at P, having with the given curve two intersections at P; that is $k=2$. P' is any one of the $m-2$ tangentials of P, hence $\alpha'=m-2$; and P is the point of contact of any one of the $n-2$

tangents from P' to the curve, that is, $\alpha = n - 2$. Hence the number of inflexions is $= (m-2) + (n-2) + 4D, = m + n - 4 + 2(n - 2m + 2), = 3(n - m)$, which is right.

8. For the purpose of the next example it is necessary to present the fundamental equation $\mathrm{a} = \alpha + \alpha' + 2kD$ under a more general form. The curve Θ may intersect the given curve in a system of points P' each p times, a system of points Q' each q times, &c., in such manner that the points (P, P'), the points (P, Q'), &c., are pairs of points corresponding to each other according to distinct laws; and we shall then have the numbers $(\mathrm{a}, \alpha, \alpha')$, $(\mathrm{b}, \beta, \beta')$, &c., belonging to these pairs respectively; viz. (P, P') are points having an (α, α') correspondence, and the number of united points is $= \mathrm{a}$; similarly (P, Q') are points having a (β, β') correspondence, and the number of united points is $= \mathrm{b}$; and so on. The theorem then is

$$p(\mathrm{a} - \alpha - \alpha') + q(\mathrm{b} - \beta - \beta') + \ldots = 2kD.$$

9. Investigation of the number of double tangents:—Take P', an intersection with the curve of a tangent drawn from P to the curve (or what is the same thing, P, P' cotangentials of any point of the curve); the united points are here the points of contact of the several double tangents of the curve; or if τ be the number of double tangents, then the number of united points is $= 2\tau$. The curve Θ is the system of the $n-2$ tangents from P to the curve; each tangent has with the curve 1 intersection at P, that is, $k = n - 2$; each tangent, besides, meets the curve in the point of contact Q' twice, and in $(m-3)$ points P'. Hence, if $(\mathrm{a}, \alpha, \alpha')$ refer to the points (P, Q'), and $(2\tau, \beta, \beta')$ to the points (P, P'), we have

$$2(\mathrm{a} - \alpha - \alpha') + 2\tau - \beta - \beta' = 2(n-2)D.$$

Moreover, from the last example the value of $\mathrm{a} - \alpha - \alpha'$ is $= 4D$, and the formula thus becomes

$$2\tau - \beta - \beta' = 2(n-6)D;$$

but from above it appears that we have $\beta = \beta' = (n-2)(m-3)$, whence

$$\begin{aligned} 2\tau &= 2(n-2)(m-3) + 2(n-6)D, \\ &= 2(n-2)(m-3) + (n-6)(n-2m+2), \\ &= n^2 - 10n + 8m, \end{aligned}$$

which is right; in fact, observing that ι (the number of inflexions) is $= 3n - 3m$, the formula is equivalent to $2\tau + 3\iota = n^2 - n - m$, that is, $m = n^2 - n - 2\tau - 3\iota$.

In the foregoing examples the curve Θ is a line or system of lines; but I give an example in which Θ is a system of conics, and in which, as will appear, we have to consider the two characteristics (μ, ν) of the system.

10. Investigation of the number of conics which can be drawn, satisfying any four conditions, and touching the given curve; or say of the number of the conics $(4Z)(1)$. Take P', an intersection of the given curve by a conic $(4Z)$ passing through the point P, then the number of the united points is equal to that of the conics $(4Z)(1)$. The curve Θ is here the system of the conics $(4Z)$ which pass through P;

hence, if (μ, ν) be the characteristics of the system of conics $(4Z)$, the number of the conics through P is $=\mu$; each of these has with the given curve 1 intersection at P, and consequently $k=\mu$. Moreover, each of the conics besides meets the curve in $(2m-1)$ points, and consequently $\alpha=\alpha'=\mu(2m-1)$. Hence the formula gives the number of united points

$$=2\mu(2m-1)+\mu(n-2m+2),$$

$$=\mu(n+2m);$$

or, as this may be written,

$$=\mu n+\nu m+m(2\mu-\nu).$$

But the system of conics $(4Z)$ contains $(2\mu-\nu)$ point-pairs (*coniques infiniment aplaties*), each of which, regarded as a pair of coincident lines, meets the given curve in m pairs of coincident points; that is, the point-pair is to be considered as a conic *touching* the given curve in m points; and there is on this account a reduction $=m(2\mu-\nu)$ in the number of the united points; whence, finally, the number of the conics $(4Z)$ (1) is $=\mu n+\nu m$. It is hardly necessary to remark that it is assumed that the conditions $(4Z)$ are conditions having no special relation to the given curve.

11. As a final example, suppose that the point P on a given curve of the order m, and the point Q on a given curve of the order m', have an (α, α') correspondence, and let it be required to find the class of the curve enveloped by the line PQ. Take an arbitrary point O, join OQ, and let this meet the curve m in P', then (P, P') are points on the curve m, having a $(m'\alpha, m\alpha')$ correspondence; in fact, to a given position of P there correspond α' positions of Q, and to each of these m positions of P', that is, to each position of P there correspond $m\alpha'$ positions of P'; and similarly to each position of P' there correspond $m'\alpha$ positions of P. The curve Θ is the system of the lines drawn from each of the α' positions of Q to the point O, hence the curve Θ does not pass through P, and we have $k=0$. Hence the number of the united points (P, P'), that is, the number of the lines PQ which pass through the point O, is $=m\alpha'+m'\alpha$, or this is the class of the curve enveloped by PQ.

12. It may be remarked, that if the two curves are curves in space (plane or of double curvature), then the like reasoning shows that the number of the lines PQ which meet a given line O is $=m\alpha'+m'\alpha$, that is, the order of the scroll generated by the line PQ is $=m\alpha'+m'\alpha$.