XV.

INTEGRATION OF PARTIAL DIFFERENTIAL EQUATIONS BY THE CALCULUS OF VARIATIONS*

[1836.]

[Note Book 42.]

[CONTENTS

[*The partial differential equation of the first order.*]

(April 22, 1836.)

[1.] (A) $$0=F(x, y, z, p, q),$$

(B) $$z'=px'+qy';$$

(C) $$\begin{cases} 0=\lambda\delta(px'+qy'-z')+\mu\delta F=\mu F'(x)\,\delta x+\lambda p\,\delta x'+\mu F'(p)\,\delta p+\lambda x'\delta p \\ \qquad +\mu F'(y)\,\delta y+\lambda q\,\delta y'+\mu F'(q)\,\delta q+\lambda y'\delta q \\ \qquad +\mu F'(z)\,\delta z-\lambda\delta z' \\ \qquad =(\lambda p\,\delta x+\lambda q\,\delta y-\lambda\delta z)', \end{cases}$$

if

(1) $$\mu F'(p)+\lambda x'=0,$$

(2) $$\mu F'(q)+\lambda y'=0,$$

(3) $$\mu F'(x)=(\lambda p)',$$

(4) $$\mu F'(y)=(\lambda q)',$$

(5) $$\mu F'(z)=-\lambda'.$$

The two first conditions give

(6) $$0=\mu\{pF'(p)+qF'(q)\}+\lambda z'.$$

Therefore

(7) $$\frac{x'}{z'}=\frac{F'(p)}{pF'(p)+qF'(q)},$$

(8) $$\frac{y'}{z'}=\frac{F'(q)}{pF'(p)+qF'(q)};$$

also (3) and (4) give

$$\mu\{F'(x)\,F'(p)+F'(y)\,F'(q)\}=\lambda'\{pF'(p)+qF'(q)\}+\lambda\{p'F'(p)+q'F'(q)\};$$

* [Hamilton's method for solving partial differential equations of the first order is very similar to Cauchy's method of characteristics. See Appendix, Note 10, p. 634.]

and

(9) $$0 = F' = x'F'(x) + y'F'(y) + z'F'(z) + p'F'(p) + q'F'(q).$$

Hence

(10) $$\lambda'\{pF'(p) + qF'(q)\} = \lambda z'F'(z),$$

agreeing with (5) and (6). Consequently from (3), (6) and (10)

(11) $$\frac{p'}{z'} = -\frac{F'(x) + pF'(z)}{pF'(p) + qF'(q)},$$

and similarly*

(12) $$\frac{q'}{z'} = -\frac{F'(y) + qF'(z)}{pF'(p) + qF'(q)}.$$

Let x, y, p be considered as three functions of z, to be determined by any three of the differential equations of the first order (7), (8), (11), (12), q being previously eliminated by (A). The expressions of these three functions will involve three arbitrary constants, which may be chosen so as to be the initial values a, b, j of the same three functions, corresponding to some assumed initial value c of z; and, if γ denote the corresponding initial value of λ, we can express λ by (10) as follows:

(13) $$\lambda = \gamma e^{\int_c^z \frac{F'(z)\,dz}{pF'(p) + qF'(q)}}.$$

Then (C) will give by integration

(14) $$\lambda(p\,\delta x + q\,\delta y - \delta z) = \gamma(j\,\delta a + k\,\delta b - \delta c),$$

k being the initial value of q, so that

(15) $$0 = F(a, b, c, j, k).$$

Therefore, by (13), we have

(16) $$p\,\delta x + q\,\delta y - \delta z = e^{-\int_c^z \frac{F'(z)\,dz}{pF'(p) + qF'(q)}}(j\,\delta a + k\,\delta b - \delta c);$$

and, if we suppose for simplicity $c = 0$, $\delta c = 0$, we shall then have the expressions

(D) $$p\,\delta x + q\,\delta y - \delta z = e^{-\int_0^z \frac{F'(z)\,dz}{pF'(p) + qF'(q)}}(j\,\delta a + k\,\delta b)$$

and

(17) $$0 = F(a, b, 0, j, k).$$

Here a, b, j, k, q may be regarded as known functions of x, y, z, p; or, reciprocally, x, y, p, q, k may be regarded as known functions of a, b, j, z; k indeed being a function of a, b, j alone. Considering then x and y as functions of a, b, j, z, we have by (D), between their partial differential coefficients, the relations

(E) $$\begin{cases} px'(a) + qy'(a) = je^{-\int_0^z \frac{F'(z)\,dz}{pF'(p)+qF'(q)}}, & px'(b) + qy'(b) = ke^{-\int_0^z \frac{F'(z)\,dz}{pF'(p)+qF'(q)}}, \\ px'(j) + qy'(j) = 0, & px'(z) + qy'(z) = 1; \end{cases}$$

so that, if we substitute for x and y and also for p and q their values as functions of a, b, j, z in the equation (B), we shall get

(F) $$0 = ja' + kb'.$$

* [Equations (7), (8), (11), (12) are of course the differential equations of the characteristics.]

More generally, by (D) we have

$$\text{(G)} \qquad px' + qy' - z' = (ja' + kb')\, e^{-\int_0^z \frac{F'(z)\,dz}{pF'(p)+qF'(q)}}.$$

If then we establish between the two known functions a and b of x, y, z, p any arbitrary relation

$$\text{(H)} \qquad b = \psi(a),$$

and put, besides,

$$\text{(I)} \qquad -\frac{j}{k} = \psi'(a),$$

we shall reproduce the equations (B) and (A); and consequently the system (H), (I) is the general integral for the partial differential equation (A). A particular integral with two arbitrary constants may be deduced from (F) by treating a and b as each separately constant and then eliminating p.*

As an example, let the proposed partial differential equation be

$$0 = p + \tfrac{1}{2}q^2 - gy = F.$$

Here the three auxiliary total differential equations (7), (8), (11) become

$$x' = \frac{z'}{p+q^2}, \quad y' = \frac{qz'}{p+q^2}, \quad p' = 0,$$

to which we may join, as a combination of the same, the equation (12),

$$q' = \frac{gz'}{p+q^2}.$$

Hence

$$q' = gx', \quad qq' = gy'.$$

Therefore, by integration,

$$p = j, \quad q - gx = k - ga, \quad \tfrac{1}{2}q^2 - gy = \tfrac{1}{2}k^2 - gb = -j;$$

also

$$y' = qx' = gxx' + (k - ga)\,x',$$

and so

$$y - b = \tfrac{1}{2}g\,(x^2 - a^2) + (k - ga)(x - a) = k\,(x - a) + \tfrac{1}{2}g\,(x - a)^2.$$

Also from

$$z' = px' + qy' = jx' + \frac{q^2q'}{g},$$

we have

$$z = j\,(x - a) + \frac{q^3 - k^3}{3g}.$$

Thus, to determine x, y, p, q as functions of a, b, j, k and z, we have the four equations

$$p = j, \quad q - k = g\,(x - a), \quad y - b = k\,(x - a) + \tfrac{1}{2}g\,(x - a)^2, \quad z = j\,(x - a) + \frac{q^3 - k^3}{3g},$$

to which we may join

$$0 = p + \tfrac{1}{2}q^2 - gy, \quad 0 = j + \tfrac{1}{2}k^2 - gb.$$

We may also conveniently consider the four equations just mentioned as serving to determine

* The logic is more fully explained in pages 395, 396.

p, q, y and z as functions of j, k, a, b, x; j being itself a known function of k and b. If now we treat j, k, a, b as variables, these equations give

$$y' = b' + k(x'-a') + (x-a)k' + g(x-a)(x'-a'), \quad z' = j(x'-a') + (x-a)j' + \frac{q^2q' - k^2k'}{g},$$

$$\begin{aligned} px' + qy' - z' &= jx' + q^2(x'-a') + \{k + g(x-a)\}\{b' + (x-a)k'\} - j(x'-a') \\ &\qquad + (x-a)(kk' - gb') - q^2(x'-a') + \frac{k^2-q^2}{g}k' \\ &= ja' + kb' + k'\left\{2k(x-a) + g(x-a)^2 + \frac{k^2-q^2}{g}\right\} = ja' + kb' = (gb - \tfrac{1}{2}k^2)a' + kb'; \end{aligned}$$

therefore

$$(gy - \tfrac{1}{2}q^2)x' + qy' - z' = (gb - \tfrac{1}{2}k^2)a' + kb'$$

identically, by the nature of the relations

$$q - k = g(x-a), \quad y - b = k(x-a) + \tfrac{1}{2}g(x-a)^2, \quad z = (gb - \tfrac{1}{2}k^2)(x-a) + \frac{q^3 - k^3}{3g},$$

from which three last equations we must conceive a, b, k deduced as functions of x, y, z and q. If, then, between the functions so deduced, we establish the two relations

$$b = \psi(a), \quad \tfrac{1}{2}k - \frac{gb}{k} = \psi'(a),$$

we shall have, as a consequence of them, this other:

$$z' = qy' + (gy - \tfrac{1}{2}q^2)x',$$

in which, by the two relations established between a and b, z and q may be considered as functions of the two independent variables x and y. Hence the partial differential coefficients of z, of the first order, are $gy - \tfrac{1}{2}q^2$ and q, and therefore the proposed partial differential equation is satisfied whatever the form of the arbitrary function ψ may be. A less general integral, but containing two arbitrary constants, may be obtained by supposing a and b to be both constants and by eliminating q and k between the three relations which connect a, b, k, x, y, z, q. This last corresponds to the *principal integral* of the total differential equation

$$z' = gyx' + \frac{y'^2}{2x'},$$

and is

$$z = \tfrac{1}{2}g(y+b)(x-a) + \frac{(y-b)^2}{2(x-a)} - \tfrac{1}{24}g^2(x-a)^3.$$

(April 23, 1836.)

Let

(1) $$q = f(x, y, z, p)$$

be any proposed partial differential equation of the first order. This equation gives

(2) $$z' = px' + f(x, y, z, p)y',$$

x', y', z' being the total derivatives of x, y, z, considered as functions of any one independent variable, of which p also may be considered as a function. Reciprocally, if we can discover any two such relations between the four variables x, y, z, p as to satisfy the total differential equation (2), independently of any third relation between the same four variables, we shall then, by eliminating p, deduce a relation between x, y, z which will be an integral of the partial differential equation (1). In this manner we are led to consider the modes of satisfying the total differential equation (2); and therefore (according to the general spirit of analysis) we are led to consider the consequences of that total differential equation.

One consequence is that which is obtained by taking the variation of the equation, namely,

(3) $$0 = p\,\delta x' + f(x, y, z, p)\,\delta y' - \delta z' + x'\delta p + y'\{f'(x)\,\delta x + f'(y)\,\delta y + f'(z)\,\delta z + f'(p)\,\delta p\}.$$

The meaning and necessity of this consequence may be explained as follows. Since the equation (2) is not by itself sufficient to determine the four variables x, y, z, p as functions of any one independent variable, or even to determine any three of those variables as a function of the fourth, we may, in general, conceive that besides some one set of four variables x, y, z, p, which satisfies the equation, another set also, such as $x+\epsilon\xi$, $y+\epsilon\eta$, $z+\epsilon\zeta$, $p+\epsilon\varpi$, satisfies the same equation, ϵ being a small and constant, but arbitrary, multiplier. Then, along with the equation (2) itself, we shall have also the varied equation:

(4) $$0 = (p+\epsilon\varpi)(x'+\epsilon\xi') + f(x+\epsilon\xi,\ y+\epsilon\eta,\ z+\epsilon\zeta,\ p+\epsilon\varpi)\,.\,(y'+\epsilon\eta') - (z'+\epsilon\zeta'),$$

which, when developed according to powers of ϵ, reduced by (2) itself, divided by ϵ, and made to tend to a limiting state by making ϵ tend to zero, becomes ultimately

(5) $$0 = p\xi' + f(x, y, z, p)\,\eta' - \zeta' + x'\varpi + y'\{f'(x)\,\xi + f'(y)\,\eta + f'(z)\,\zeta + f'(p)\,\varpi\};$$

and this is the consequence which with only a different notation is expressed by (3).

If we multiply (5) by a new function λ and integrate by parts, we find

(6) $$0 = \{\lambda p\xi + \lambda f(x, y, z, p)\,\eta - \lambda\zeta\}' + \{\lambda y'f'(x) - (\lambda p)'\}\,\xi + \{\lambda y'f'(y) - (\lambda f)'\}\,\eta$$
$$+ \{\lambda y'f'(z) + \lambda'\}\,\zeta + \lambda\{x' + y'f'(p)\}\,\varpi;$$

which will reduce itself to

(7) $$0 = (\lambda p\xi + \lambda f\eta - \lambda\zeta)',$$

if we can put, consistently with (2),

(8) $$x' = -y'f'(p),$$

(9) $$\lambda' = -\lambda y'f'(z),$$

(10) $$(\lambda f)' = \lambda y'f'(y),$$

(11) $$(\lambda p)' = \lambda y'f'(x).$$

Expanding these equations, we find by (9) and (10)

(12) $$0 = x'f'(x) + z'f'(z) + p'f'(p) - f\,.\,y'\,.\,f'(z);$$

also by (9) and (11)

(13) $$0 = p' - y'f'(x) - p\,.\,y'\,.\,f'(z),$$

which latter equation also results by elimination of x', z' between the three equations (2), (8), (12). For the equations (2) and (8) give

(14) $$z' = \{f - pf'(p)\}\,y',$$

and when we substitute these expressions (8) and (14) for x', z' in (12) and divide by $f'(p)$, we come to equation (13). Reciprocally (12) results from (2), (8) and (13). Thus we shall satisfy all the five equations (2), (8), (9), (10), (11) if we satisfy the four equations (8), (9), (13), (14). These four differential equations may in general be conceived to give, by integration, x, y, p and

λ/λ_0 as functions of z, a, b, j, if a, b, j and λ_0 denote the initial values of x, y, p and λ, corresponding to some assumed initial value, such as 0, of z; a, b, j, λ_0 being here treated as constants in forming the expressions of x', y', p', λ', which are to satisfy the differential equations. Now, just as (4) was formed from (2), so we must form three other analogously varied differential equations from (8), (9) and (13); and, integrating these, we must obtain $x+\epsilon\xi$, $y+\epsilon\eta$, $z+\epsilon\zeta$, $p+\epsilon\varpi$ and $\dfrac{\lambda+\epsilon\omega}{\lambda_0+\epsilon\omega_0}$ as functions of $z+\epsilon\zeta$, $a+\epsilon\alpha$, $b+\epsilon\beta$, $j+\epsilon\iota$, in which $\alpha, \beta, \iota, \omega_0$ are the initial values of $\xi, \eta, \varpi, \omega$, corresponding to the initial value 0 of $z+\epsilon\zeta$; which four functions will be precisely the same in form as those which expressed the dependence of x, y, p and $\dfrac{\lambda}{\lambda_0}$ on z, a, b, j. And the limiting equation (7) will give, by integration, (at the limit $\epsilon=0$)

$$\text{(15)} \qquad \lambda(p\xi+f\eta-\zeta)=\lambda_0(j\alpha+k\beta),$$

in which

$$\text{(16)} \qquad k=f_0=f(a, b, 0, j);$$

also, for the reason just mentioned,

$$\text{(17)} \qquad \xi=x'(z)\zeta+x'(a)\alpha+x'(b)\beta+x'(j)\iota, \quad \eta=y'(z)\zeta+y'(a)\alpha+y'(b)\beta+y'(j)\iota,$$

$x'(z), x'(a), x'(b), x'(j), y'(z), y'(a), y'(b), y'(j)$ being the partial differential coefficients of the first order of x and y considered as functions of z, a, b, j. Therefore, since $\alpha, \beta, \iota, \zeta$ are independent, the equation (15) resolves itself into the four following:

$$\text{(18)} \quad px'(z)+fy'(z)=1, \quad px'(j)+fy'(j)=0, \quad px'(a)+fy'(a)=\frac{\lambda_0}{\lambda}j, \quad px'(b)+fy'(b)=\frac{\lambda_0}{\lambda}k.$$

Having discovered these four relations (18) between the partial differential coefficients of x and y considered as functions of z, a, b, j, we may now deduce the following relation between the total differential coefficients of the same two functions, treating a, b, j as variables,

$$\text{(19)} \qquad px'+fy'=z'+\frac{\lambda_0}{\lambda}(ja'+kb'),$$

which shows that (2) gives

$$\text{(20)} \qquad 0=ja'+kb'.$$

Reciprocally, if in the two following equations,

$$\text{(21)} \qquad b=\psi(a), \quad -\frac{j}{k}=\psi'(a),$$

we consider a, b and j as known functions of x, y, z, p, deduced by elimination from the integrals of the equations (8), (13), (14), we shall have solved the problem proposed, namely, that of discovering two relations between x, y, z, p, which are sufficient to conduct to the total differential equation (2), and which therefore give, by elimination of p, an integral of the partial differential equation (1). This integral system (21) contains an arbitrary function ψ; another integral of the same partial differential equation, containing no arbitrary function but containing two arbitrary constants, may be had by eliminating p between the expressions of a and b.

[The partial differential equation of the second order.]

[2.] Passing now to partial differential equations of the second order, let

(1) $$t=f(x, y, z, p, q, r, s)$$

be such an equation. We have now the three following total differential equations of the first order:

(2) $$0=px'+qy'-z',$$

(3) $$0=rx'+sy'-p',$$

(4) $$0=sx'+f(x, y, z, p, q, r, s)\,y'-q',$$

x', y', z', p', q' being here the total derivatives of x, y, z, p, q, considered as functions of any one independent variable of which r and s may also be conceived to be functions. And if we can discover any five relations between x, y, z, p, q, r and s, which, without any other relation being required, conduct to the system of the three total differential equations (2), (3), (4), we shall then be able to conclude that the partial differential coefficients of the first and second orders of z considered as a function of x and y (deduced from the five supposed relations by elimination of p, q, r, s) may be expressed as follows:

(5) $$z'(x)=p,\quad z'(y)=q,\quad z''(x)=r,\quad z'{}'(x,y)=s,\quad z''(y)=f(x, y, z, p, q, r, s);$$

and therefore that this function z of x and y will satisfy and be an integral of the proposed partial differential equation (1).

Since the three equations (2), (3), (4), even if freed from differentials, would not be in general sufficient to determine the five sought relations between x, y, z, p, q, r, s, we may conceive that $x+\epsilon\xi$, $y+\epsilon\eta$, $z+\epsilon\zeta$, $p+\epsilon\varpi$, $q+\epsilon\kappa$, $r+\epsilon\rho$, $s+\epsilon\sigma$ are also functions of some one independent variable, and satisfy as such three equations similar to (2), (3), (4), namely,

(6) $$0=(p+\epsilon\varpi)(x'+\epsilon\xi')+(q+\epsilon\kappa)(y'+\epsilon\eta')-(z'+\epsilon\zeta'),$$

(7) $$0=(r+\epsilon\rho)(x'+\epsilon\xi')+(s+\epsilon\sigma)(y'+\epsilon\eta')-(p'+\epsilon\varpi'),$$

(8) $$0=(s+\epsilon\sigma)(x'+\epsilon\xi')+(t+\epsilon\tau)(y'+\epsilon\eta')-(q'+\epsilon\kappa'),$$

in which we have put for abridgment

(9) $$t+\epsilon\tau=f(x+\epsilon\xi,\ y+\epsilon\eta,\ z+\epsilon\zeta,\ p+\epsilon\varpi,\ q+\epsilon\kappa,\ r+\epsilon\rho,\ s+\epsilon\sigma).$$

Developing according to powers of ϵ, reducing by the equations (2), (3), (4), dividing by ϵ, and passing to the limit at which ϵ vanishes, we find

(10) $$0=p\xi'+q\eta'-\zeta'+\varpi x'+\kappa y',$$

(11) $$0=r\xi'+s\eta'-\varpi'+\rho x'+\sigma y',$$

(12) $$0=s\xi'+t\eta'-\kappa'+\sigma x'+\tau y';$$

in which t is to be considered as representing the expression $f(x, y, z, p, q, r, s)$, and therefore by (9)

(13) $$\tau=\xi f'(x)+\eta f'(y)+\zeta f'(z)+\varpi f'(p)+\kappa f'(q)+\rho f'(r)+\sigma f'(s).$$

We may multiply the three differential equations (10), (11), (12) by λ, μ, ν respectively and add them, thus obtaining

(14) $$0=(\lambda p+\mu r+\nu s)\,\xi'+(\lambda q+\mu s+\nu t)\,\eta'-\lambda\zeta'-\mu\varpi'-\nu\kappa' \\ +\lambda x'\varpi+\lambda y'\kappa+\mu x'\rho+(\mu y'+\nu x')\,\sigma+\nu y'\tau.$$

This result may be put under the form:

$$(15)\quad 0=\{(\lambda p+\mu r+\nu s)\,\xi+(\lambda q+\mu s+\nu t)\,\eta-\lambda\zeta-\mu\varpi-\nu\kappa\}'$$
$$+\xi\{\nu y'f'(x)-(\lambda p+\mu r+\nu s)'\}+\eta\{\nu y'f'(y)-(\lambda q+\mu s+\nu t)'\}$$
$$+\zeta\{\nu y'f'(z)+\lambda'\}+\varpi\{\nu y'f'(p)+\mu'+\lambda x'\}+\kappa\{\nu y'f'(q)+\nu'+\lambda y'\}$$
$$+\rho\{\mu x'+\nu y'f'(r)\}+\sigma\{\mu y'+\nu x'+\nu y'f'(s)\}.$$

It will therefore give

$$(16)\quad (\lambda p+\mu r+\nu s)\,\xi+(\lambda q+\mu s+\nu t)\,\eta-\lambda\zeta-\mu\varpi-\nu\kappa$$
$$=(\lambda_0p_0+\mu_0r_0+\nu_0s_0)\,\xi_0+(\lambda_0q_0+\mu_0s_0+\nu_0t_0)\,\eta_0-\mu_0\varpi_0-\nu_0\kappa_0,$$

in which $\lambda_0, \mu_0, \nu_0, \xi_0, \eta_0, \varpi_0, \kappa_0, p_0, q_0, r_0, s_0, t_0$ are the initial values of $\lambda, \mu, \nu, \xi, \eta, \varpi, \kappa, p, q, r, s, t$, corresponding to the assumed initial value 0 of z, if we suppose that the following differential equations are satisfied, as well as equations (2), (3), (4):

$$(17)\quad 0=\mu y'+\nu x'+\nu y'f'(s),$$

$$(18)\quad 0=\mu x'+\nu y'f'(r),$$

$$(19)\quad 0=\nu y'f'(q)+\nu'+\lambda y',$$

$$(20)\quad 0=\nu y'f'(p)+\mu'+\lambda x',$$

$$(21)\quad 0=\nu y'f'(z)+\lambda',$$

$$(22)\quad 0=\nu y'f'(y)-(\lambda q+\mu s+\nu t)',$$

$$(23)\quad 0=\nu y'f'(x)-(\lambda p+\mu r+\nu s)'.$$

Multiplying (23), (22), (21), (20), (19), (18) and (17) by $x', y', z', p', q', r', s'$ respectively and adding the products, we find

$$(24)\quad 0=\lambda'(z'-px'-qy')+\mu'(p'-rx'-sy')+\nu'(q'-sx'-ty').$$

We may therefore on this account subtract one from the number of equations (2)–(4), (17)–(23), and consider these 10 equations as not equivalent to more, at most, than 9 distinct equations, adopted to determine the 9 variables $x, y, p, q, r, s, \lambda, \mu, \nu$ as functions of z and of constants. We may also eliminate $\lambda, \lambda', \mu', \nu'$ between the five equations (19)–(23) and so obtain a single equation giving the ratio of ν to μ; which, when compared with (17) and (18), will give two equations entirely freed from λ, μ, ν, but differential of the first order between x, y, z, p, q, r, s.

As an example, to illustrate these formulae, let the given partial differential equation be $t=0$. Here the ten auxiliary total differential equations (equivalent only to nine distinct ones) are the following:

$$z'=px'+qy',\quad p'=rx'+sy',\quad q'=sx',\quad 0=(\lambda p+\mu r+\nu s)',\quad 0=(\lambda q+\mu s)',$$
$$0=\lambda',\quad 0=\mu'+\lambda x',\quad 0=\nu'+\lambda y',\quad 0=\mu x',\quad 0=\mu y'+\nu x'.$$

These equations give by integration $x, y, z, p, q, r, s, \lambda, \mu, \nu$ each constant, or else x, y, z, p, q, λ each constant and $\mu=0$, $\nu=0$. But each alternative seems to render equation (16) identical.

As another example, let $t=r$. Here the ten total differential equations are:

$$z'=px'+qy',\quad p'=rx'+sy',\quad q'=sx'+ry',\quad 0=(\lambda p+\mu r+\nu s)',\quad 0=(\lambda q+\mu s+\nu r)',$$
$$0=\lambda',\quad 0=\mu'+\lambda x',\quad 0=\nu'+\lambda y',\quad 0=\mu x'+\nu y',\quad 0=\mu y'+\nu x';$$

the ninth alone differing from the corresponding equation of the system of the last example. The two last of these ten equations give either

(I) $$\mu=\nu, \quad x'+y'=0,$$

or

(II) $$\mu=-\nu, \quad x'-y'=0.$$

If we adopt the first alternative, then the 7th and 8th equations give $\lambda(x'-y')=0$. Therefore either

(I, 1) $$x'=0,\ y'=0,\ \mu'=0,\ \nu'=0,$$

or else

(I, 2) $$\lambda=0,\ \mu'=0,\ \nu'=0,\ \mu=\nu,\ x'+y'=0.$$

In like manner we have

(II, 1) $$x'=0,\ y'=0,\ \mu'=0,\ \nu'=0,$$

or

(II, 2) $$\lambda=0,\ \mu'=0,\ \nu'=0,\ \mu=-\nu,\ x'-y'=0.$$

Thus we have, at all events, $\mu'=0$, $\nu'=0$, $\lambda'=0$, and the only question is whether $x'=0, y'=0$ (a), or $\lambda=0$, $x'^2-y'^2=0$ (b). In the alternative (a) we have

$$z'=0,\ p'=0,\ q'=0,\ 0=\mu r'+\nu s',\ 0=\mu s'+\nu r',$$

and in (b) we have

$$z'=px'+qy',\ p'=rx'+sy',\ q'=sx'+ry',\ 0=\mu r'+\nu s',\ 0=\mu s'+\nu r'.$$

If we adopt the supposition (I, 1), then $x, y, z, p, q, \lambda, \mu, \nu$ are each constant, $r+s$ is also constant, and $0=\mu r'+\nu s'$, $0=\mu s'+\nu r'$. Therefore one of the following alternatives is true:

(I, 1, 1) $$\mu=\nu, \quad r+s=\text{const.},$$

(I, 1, 2) $$\mu=-\nu, \quad r-s=\text{const.},$$

(I, 1, 3) $$r=\text{const.}, \quad s=\text{const.},$$

(I, 1, 4) $$\mu=0, \quad \nu=0.$$

In general, besides the two equations obtained by eliminating $\lambda, \mu, \nu, \lambda', \mu', \nu'$ from (17)–(23), which with the equations (2), (3), (4) make up a system of five differential equations equivalent only to four distinct equations of the first order between the variables x, y, z, p, q, r, s, we can deduce two other differential equations of the second order between the same 7 variables with the help of the derivatives of equations (17) and (18). And hence, in general, we can express x, y, p, q, r, s as functions of z and of eight arbitrary constants, such as $x_0, y_0, p_0, q_0, r_0, s_0, r_0', s_0'$, supposing $z_0=0$, $z'=1$, $z_0'=1$. We can also express $x_0', y_0', p_0', q_0', \frac{\mu_0}{\lambda_0}, \frac{\nu_0}{\lambda_0}, \frac{\lambda_0'}{\lambda_0}$ as functions of the same eight arbitrary constants, and $\frac{\lambda}{\lambda_0}, \frac{\mu}{\lambda_0}, \frac{\nu}{\lambda_0}$ as functions of them and of z. Hence, in (16),

$$(25)\quad \begin{cases} \xi=x'(z)\,\zeta+x'(x_0)\,\xi_0+x'(y_0)\,\eta_0+x'(p_0)\,\varpi_0+x'(q_0)\,\kappa_0+x'(r_0)\,\rho_0+x'(s_0)\,\sigma_0 \\ \qquad\qquad +x'(r_0')\,\rho_0'+x'(s_0')\,\sigma_0', \\ \eta=y'(z)\,\zeta+\dots, \\ \varpi=p'(z)\,\zeta+\dots, \\ \kappa=q'(z)\,\zeta+\dots, \end{cases}$$

and since $\zeta, \xi_0, \eta_0, \varpi_0, \kappa_0, \rho_0, \sigma_0, \rho_0', \sigma_0'$ may be treated as 9 independent and arbitrary data, we have the 9 equations:

$$(26)\quad \begin{cases} (\lambda p+\mu r+\nu s).x'(z) \;+(\lambda q+\mu s+\nu t).y'(z) \;-\mu p'(z) \;-\nu q'(z) \;=\lambda, \\ (\qquad\quad).x'(x_0)+(\qquad\quad).y'(x_0)-\mu p'(x_0)-\nu q'(x_0)=\lambda_0p_0+\mu_0r_0+\nu_0s_0, \\ (\qquad\quad).x'(y_0)+(\qquad\quad).y'(y_0)-\mu p'(y_0)-\nu q'(y_0)=\lambda_0q_0+\mu_0s_0+\nu_0t_0, \\ (\qquad\quad).x'(p_0)+(\qquad\quad).y'(p_0)-\mu p'(p_0)-\nu q'(p_0)=-\mu_0, \\ (\qquad\quad).x'(q_0)+(\qquad\quad).y'(q_0)-\mu p'(q_0)-\nu q'(q_0)=-\nu_0, \\ (\qquad\quad).x'(r_0)+(\qquad\quad).y'(r_0)-\mu p'(r_0)-\nu q'(r_0)=0, \\ (\qquad\quad).x'(s_0)+\quad\dots\quad-\quad\dots\quad-\quad\dots\quad=0, \\ (\qquad\quad).x'(r_0')+\quad\dots\quad-\quad\dots\quad-\quad\dots\quad=0, \\ (\qquad\quad).x'(s_0')+\quad\dots\quad-\quad\dots\quad-\quad\dots\quad=0. \end{cases}$$

Hence, treating all as variable,

$$(27)\quad (\lambda p+\mu r+\nu s)x'+(\lambda q+\mu s+\nu t)y'-\lambda z'-\mu p'-\nu q' \\ =(\lambda_0p_0+\mu_0r_0+\nu_0s_0).(x_0)'+(\lambda_0q_0+\mu_0s_0+\nu_0t_0).(y_0)'-\mu_0(p_0)'-\nu_0(q_0)'.$$

If then we establish the three equations (2), (3), (4), we shall have

$$(28)\qquad 0=(\lambda_0p_0+\mu_0r_0+\nu_0s_0).(x_0)'+(\lambda_0q_0+\mu_0s_0+\nu_0t_0).(y_0)'-\mu_0(p_0)'-\nu_0(q_0)'.$$

In the case of the second example of page 398 we have the twelve differential equations:

$$\begin{cases} z'=px'+qy',\quad p'=rx'+sy',\quad q'=sx'+ry',\quad \lambda'=0,\quad \mu'=-\lambda x',\quad \nu'=-\lambda y', \\ 0=\lambda p'+\mu r'+\nu s',\quad 0=\lambda q'+\mu s'+\nu r',\quad 0=\mu x'+\nu y',\quad 0=\nu x'+\mu y', \\ 0=\mu x''+\nu y''+\mu'x'+\nu'y',\quad 0=\nu x''+\mu y''+\nu'x'+\mu'y'. \end{cases}$$

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We can satisfy the 9th and 10th of these equations by supposing $x+y=a$, $\mu=\nu$, a being an arbitrary constant. Then the two equations $\mu'=-\lambda x'$, $\nu'=-\lambda y'$ give $\mu'+\nu'=-\lambda(x'+y')$, that is, $\mu'=0$, $\nu'=0$, and so $\mu=\nu=\text{const.}$ They give also $\lambda(x'-y')=0$, a condition which can be satisfied by supposing $\lambda=0$ and which must be satisfied unless we treat x and y as each separately constant. If we suppose $\lambda=0$, $\mu=\nu=\text{const.}$, $x+y=a$, the 4th, 11th and 12th of the above equations will be satisfied of themselves, as well as the 5th, 6th, 9th and 10th. Also the 7th and 8th equations will agree in giving $r+s=\text{const.}=c$; while the 2nd and 3rd will give $p+q=\text{const.}=b$, $p'-q'=(r-s)(x'-y')$; and the 1st will give $z'=\frac{1}{2}(p-q)(x'-y')$. Thus the whole twelve equations will be satisfied, if we suppose

$$\lambda=0,\quad \mu=\nu=\text{const.},\quad x+y=a,\quad p+q=b,\quad r+s=c,$$
$$p'-q'=(r-s)(x'-y'),\quad z'=\tfrac{1}{2}(p-q)(x'-y').$$

The last two conditions may be satisfied by supposing

$$\tfrac{1}{2}(r-s)=\psi''(x-y)+\gamma,\quad \tfrac{1}{2}(p-q)=\psi'(x-y)+\gamma(x-y)+\beta,$$
$$z=\psi(x-y)+\tfrac{1}{2}\gamma(x-y)^2+\beta(x-y)+\alpha,$$

α, β, γ being any arbitrary constants and ψ being any arbitrary function. Now the six equations

$$\begin{cases} x+y=a, \quad p+q=b, \quad r+s=c, \quad z=\psi(x-y)+\tfrac{1}{2}\gamma(x-y)^2+\beta(x-y)+\alpha, \\ \tfrac{1}{2}(p-q)=\psi'(x-y)+\gamma(x-y)+\beta, \quad \tfrac{1}{2}(r-s)=\psi''(x-y)+\gamma \end{cases}$$

are sufficient to determine the six variables x, y, p, q, r, s as functions of z and of the six constants $a, b, c, \alpha, \beta, \gamma$; and we may substitute these six functions in the three differential equations

$$z'=px'+qy', \quad p'-q'=(r-s)(x'-y'), \quad p'+q'=(r+s)(x'+y'),$$

and so deduce three other differential equations between z and the six constants rendered variable. These three new equations will evidently not contain z' on account of the nature of the process of integration by which they were deduced, but will be of the form of linear relations between $a', b', c', \alpha', \beta', \gamma'$. They are the following:

$$\alpha'+(x-y)\beta'+\tfrac{1}{2}(x-y)^2\gamma'=\tfrac{1}{2}ba', \quad 0=\beta'+(x-y)\gamma', \quad b'=ca';$$

of which the third gives

$$b=\phi(a), \quad c=\phi'(a);$$

the second gives

$$\beta=\chi(\gamma), \quad (x-y)=-\chi'(\gamma);$$

and the first gives

$$\alpha'-\tfrac{1}{2}\{\chi'(\gamma)\}^2\gamma'=\tfrac{1}{2}\phi(a)a'.$$

Therefore

$$\alpha=\tfrac{1}{2}\int\{\chi'(\gamma)\}^2 d\gamma+\tfrac{1}{2}\int\phi(a)\,da=\Xi(\gamma)+\Phi(a),$$

where

$$\Xi'(\gamma)=\tfrac{1}{2}\{\chi'(\gamma)\}^2, \quad \Phi'(a)=\tfrac{1}{2}\phi(a);$$

and hence

$$\alpha+\beta(x-y)+\tfrac{1}{2}\gamma(x-y)^2=\Xi(\gamma)+\Phi(a)-\chi(\gamma)\chi'(\gamma)+\gamma\Xi'(\gamma).$$

Consequently

$$\begin{cases} z=\psi\{-\chi'(\gamma)\}+\Xi(\gamma)+\tfrac{1}{2}\gamma\{\chi'(\gamma)\}^2-\chi(\gamma)\chi'(\gamma)+\Phi(a), \\ x=\tfrac{1}{2}a-\tfrac{1}{2}\chi'(\gamma), \quad y=\tfrac{1}{2}a+\tfrac{1}{2}\chi'(\gamma), \\ p=\Phi'(a)+\psi'\{-\chi'(\gamma)\}+\chi(\gamma)-\gamma\chi'(\gamma), \quad q=\Phi'(a)-\psi'\{-\chi'(\gamma)\}-\chi(\gamma)+\gamma\chi'(\gamma), \\ r=\Phi''(a)+\psi''\{-\chi'(\gamma)\}+\gamma, \quad s=\Phi''(a)-\psi''\{-\chi'(\gamma)\}-\gamma; \end{cases}$$

and it results from the nature of the foregoing investigation that this system of 7 relations between the 9 variables $x, y, z, p, q, r, s, a, \gamma$ may be considered as an integral of the proposed partial differential equation $t=r$, the functions Φ and ψ being altogether arbitrary and the functions Ξ and χ being subject only to the restriction $\Xi'(\gamma)=\tfrac{1}{2}\{\chi'(\gamma)\}^2$. Accordingly these 7 relations, considered with the derivatives of the first 5 of them, give

$$z'=px'+qy', \quad p'=rx'+sy', \quad q'=sx'+ry',$$

independent of any relation between x and y. But this integral system may be simplified; for, if we put for abridgement

$$\psi\{-\chi'(\gamma)\}+\Xi(\gamma)+\tfrac{1}{2}\gamma\{\chi'(\gamma)\}^2-\chi(\gamma)\chi'(\gamma)=\Psi\{-\chi'(\gamma)\},$$

and hence

$$\Psi'\{-\chi'(\gamma)\}=\psi'\{-\chi'(\gamma)\}-\gamma\chi'(\gamma)+\chi(\gamma), \quad \Psi''\{-\chi'(\gamma)\}=\psi''\{-\chi'(\gamma)\}+\gamma,$$

we can easily eliminate a and γ and deduce the 5 following relations between x, y, z, p, q, r, s:

$$z=\Psi(x-y)+\Phi(x+y),$$

$$p=\Psi'(x-y)+\Phi'(x+y),\quad q=-\Psi'(x-y)+\Phi'(x+y),$$

$$r=\Psi''(x-y)+\Phi''(x+y),\quad s=-\Psi''(x-y)+\Phi''(x+y);$$

of which the first is the sufficient and well-known integral of the equation $t=r$.

In general it seems likely enough that if we can integrate the system of the 3 total differential equations (2), (3), (4) and of their principal supplementaries (17)–(23), and thus, by elimination of λ, μ, ν, deduce x, y, p, q, r, s as functions of z and of constants; and if we then substitute these six functions instead of x, y, p, q, r, s in (2), (3), (4), treating now the constants as variable; the three resulting differential equations, which must evidently be linear relations between the differentials of the constants thus rendered variable, not involving the differential of z, will conduct by some new integration to the integral of the partial differential equation of the second order (1).

Now, if we substitute in (23) the values of λ', μ', ν' deduced from (21), (20), (19), we find

$$(29)\qquad 0=\nu y'\{f'(x)+pf'(z)+rf'(p)+sf'(q)\}-(\mu r'+\nu s');$$

and, similarly, from (22) we find

$$(30)\qquad 0=\nu y'\{f'(y)+qf'(z)+sf'(p)+tf'(q)\}-(\mu s'+\nu t'),$$

attending to (3) and (4). If we put

$$(31)\qquad r'=Rx'+Sy',$$

$$(32)\qquad s'=Sx'+Ty',$$

$$(33)\qquad t'=Tx'+Uy',$$

R, S, T, U being the partial differential coefficients of z of the third order, taken with respect to x and y, we shall have, by the proposed partial differential equation (1),

$$(34)\qquad T=f'(x)+pf'(z)+rf'(p)+sf'(q)+Rf'(r)+Sf'(s),$$

$$(35)\qquad U=f'(y)+qf'(z)+sf'(p)+tf'(q)+Sf'(r)+Tf'(s).$$

Hence (29) and (30) become

$$(36)\qquad 0=\nu y'\{T-Rf'(r)-Sf'(s)\}-\mu(Rx'+Sy')-\nu(Sx'+Ty'),$$

$$(37)\qquad 0=\nu y'\{U-Sf'(r)-Tf'(s)\}-\mu(Sx'+Ty')-\nu(Tx'+Uy');$$

and these are identically satisfied, by (17) and (18).

It seems then that we can in general dispense with the equations (22) and (23); or rather that we may substitute for them the equations (31)–(35), which by elimination of R, S, T, U will give in general a single differential equation of the first order between x, y, z, p, q, r, s, t.

The equations (17), (18) give

$$(38)\qquad \mu p'+\nu q'=\nu y'\{f-rf'(r)-sf'(s)\};$$

also by differentiation and by (19), (20)

$$(39)\qquad \mu y''+\nu[x''+y''f'(s)+y'\{f'(s)\}']=y'\{\lambda x'+\nu y'f'(p)\}+\{x'+y'f'(s)\}\{\lambda y'+\nu y'f'(q)\},$$

$$(40)\qquad \mu x''+\nu[y''f'(r)+y'\{f'(r)\}']=x'\{\lambda x'+\nu y'f'(p)\}+y'f'(r)\{\lambda y'+\nu y'f'(q)\}.$$

We have also

$$(41) \quad t' = x'\{f'(x) + pf'(z) + rf'(p) + sf'(q)\} + r'f'(r) + y'\{f'(y) + qf'(z) + sf'(p) + tf'(q)\} + s'f'(s),$$

and therefore may put equation (30) under the form

$$(42) \qquad 0 = \{\mu + \nu f'(s)\} s' + \nu r' f'(r) + \nu x' \{f'(x) + pf'(z) + rf'(p) + sf'(q)\},$$

which gives, when combined with (29),

$$(43) \qquad 0 = r'\{\nu y' f'(r) + \mu x'\} + s'\{\mu y' + \nu y' f'(s) + \nu x'\}.$$

But this is identically satisfied by (17) and (18). We may therefore consider the equation (30) as included in the three equations (17), (18), (29); which three last equations will give in general, by elimination of ν/μ, two differential equations of the first order between x, y, z, p, q, r, s, to be combined with the three equations (2), (3), (4). Another equation between the same variables, but of the second order, may be had by eliminating λ between (39) and (40) and then changing ν/μ to its value.

[*Examples.*]

[**3.**] In the example $t = r$, the equations (17), (18), (29) become

$$0 = \mu y' + \nu x', \quad 0 = \mu x' + \nu y', \quad 0 = \mu r' + \nu s',$$

and give, by elimination of ν/μ, the two following:

$$x'^2 = y'^2, \quad x's' = y'r'.$$

Also

$$x'r' = y's',$$

this latter being a consequence of the two former. In the same example, the equations (39) and (40) become

$$\mu y'' + \nu x'' = 2\lambda x'y', \quad \mu x'' + \nu y'' = \lambda(x'^2 + y'^2);$$

therefore

$$(\mu y'' + \nu x'')(x'^2 + y'^2) = 2(\mu x'' + \nu y'')x'y'.$$

Since

$$\frac{\nu}{\mu} = -\frac{y'}{x'} = -\frac{x'}{y'},$$

we have, by substitution of the first of these two equal values of ν/μ, the following differential equation of the second order:

$$\left(y'' - \frac{y'x''}{x'}\right)(x'^2 + y'^2) = 2\left(x'' - \frac{y'y''}{x'}\right)x'y';$$

which is identically satisfied, because the equation $x'^2 = y'^2$ gives both $x'x'' = y'y''$ and $x'y'' = y'x''$.

In the less particular example

$$t = Ar + Bs,$$

the equations (17) and (18) become

$$0 = \nu x' + (\mu + \nu B)y', \quad 0 = \mu x' + \nu A y';$$

equation (29) becomes $0 = \mu r' + \nu s'$, and (30) becomes $0 = \mu s' + \nu t'$. Thus, changing ν/μ to $-x'/Ay'$, we find

$$x'^2 + Bx'y' = Ay'^2, \quad Ay'r' = x's', \quad Ay's' = x'(Ar' + Bs'),$$

of which the last is a consequence of the two first since those give

$$Ay's' = \frac{s'(x'^2 + Bx'y')}{y'} = x'(Ar' + Bs').$$

Also equations (39) and (40) become

$$\mu y'' + \nu(x'' + y''B) = \lambda(2x'y' + By'^2), \quad \mu x'' + \nu Ay'' = \lambda(x'^2 + Ay'^2),$$

which give, by elimination of λ,

$$\{\mu y'' + \nu(x'' + y''B)\}(x'^2 + Ay'^2) = (\mu x'' + \nu Ay'')(2x'y' + By'^2).$$

Therefore, eliminating ν/μ,

$$\{Ay'y'' - x'(x'' + y''B)\}(x'^2 + Ay'^2) = (Ay'x'' - Ax'y'')(2x'y' + By'^2);$$

but both sides of this equation vanish in consequence of the relation

$$x'^2 + Bx'y' = Ay'^2,$$

which gives

$$y'/x' = \text{const.}, \quad (y'/x')' = 0, \quad \text{and} \quad Ay'y'' - x'x'' = \tfrac{1}{2}B(x'y')' = Bx'y''.$$

Still less particularly, let

$$t = Lr + Ms + N,$$

L, M, N being functions of x and y. Then

$$f'(x) = rL'(x) + sM'(x) + N'(x), \quad f'(y) = rL'(y) + sM'(y) + N'(y), \quad f'(z) = 0,$$

$$f'(p) = 0, \quad f'(q) = 0, \quad f'(r) = L, \quad f'(s) = M;$$

the equations (17) and (18) become

$$0 = \mu y' + \nu(x' + My'), \quad 0 = \mu x' + \nu Ly';$$

equation (29) becomes

$$0 = \nu y'\{rL'(x) + sM'(x) + N'(x)\} - (\mu r' + \nu s');$$

and equations (39), (40) become

$$\mu y'' + \nu(x'' + My'' + M'y') = \lambda(2x'y' + My'^2),$$

$$\mu x'' + \nu(Ly'' + L'y') = \lambda(x'^2 + Ly'^2).$$

Hence, by eliminating λ,

$$(x'^2 + Ly'^2)\{\mu y'' + \nu(x'' + My'' + M'y')\} = (2x'y' + My'^2)\{\mu x'' + \nu(Ly'' + L'y')\}.$$

Thus, by the elimination of ν/μ, we find the three following equations:

$$0 = Ly'^2 - x'^2 - Mx'y',$$

$$0 = Ly'r' - x's' + x'y'\{rL'(x) + sM'(x) + N'(x)\},$$

$$(x'^2 + Ly'^2)\{-Ly'y'' + x'x'' + x'My'' + x'M'y'\} = (2x'y' + My'^2)\{-Ly'x'' + Lx'y'' + L'x'y'\}.$$

The first of these equations gives

$$2Ly'y'' = 2x'x'' + M(x'y'' + y'x'') + M'x'y' - L'y'^2;$$

therefore

$$y'' = \frac{(2x' + My')x'' + (M'x' - L'y')y'}{2Ly' - Mx'} = y'\left\{\frac{x''}{x'} + \frac{M'x' - L'y'}{2Ly' - Mx'}\right\},$$

and

$$(x'^2 + Ly'^2)\{-x'^2(M'x' - L'y') + x'y'M'(2Ly' - Mx')\}$$
$$= (2x' + My')x'y'^2\{L(M'x' - L'y') + L'(2Ly' - Mx')\}.$$

Thus

$$0 = x'L'\{x'y'(x'^2 + Ly'^2) + y'^2(2x' + My')(Mx' - Ly')\}$$
$$+ x'M'\{(-x'^2 + 2Ly'^2 - Mx'y')(x'^2 + Ly'^2) - x'y'^2L(2x' + My')\}.$$

Hence, setting aside the factor $x'y'^2$,

$$0 = L'\{(2Ly' - Mx')x' + (2x' + My')(Mx' - Ly')\} + M'\{L(x'^2 + Ly'^2) - L(2x'^2 + Mx'y')\},$$

in which the coefficients of L' and M' both vanish in consequence of the equation

$$0 = Ly'^2 - x'^2 - Mx'y'.$$

In the present example, $t = Lr + Ms + N$, the equation (38) gives

$$\mu p' + \nu q' = \nu N y';$$

and, since (18) gives $0 = \mu x' + \nu L y'$, we have, by elimination of ν/μ,

$$Ly'p' - x'q' + Nx'y' = 0.$$

This equation may take the place of equation (4), so that, upon the whole, we may establish the system of the five following equations:

$$\begin{cases} z' = px' + qy', \quad p' = rx' + sy', \quad 0 = Ly'^2 - x'^2 - Mx'y', \\ 0 = Ly'p' - x'q' + Nx'y', \quad 0 = Ly'r' - x's' + x'y'\{rL'(x) + sM'(x) + N'(x)\}, \end{cases}$$

which are, if possible, to be integrated all together by discovering functions of z and of constants which, when substituted for x, y, p, q, r, s, shall satisfy them. If we can discover such functions, we may then put $\nu/\mu = -x'/Ly'$, and we shall therefore have

$$\frac{\nu}{\mu} = -\frac{y'}{x' + My'}.$$

Also (19) and (20) give

$$\left(\frac{\nu}{\mu}\right)' = \frac{\nu'}{\mu} - \frac{\nu\mu'}{\mu^2} = -\frac{\lambda}{\mu}y' + \frac{\nu\lambda}{\mu^2}x'.$$

Therefore

$$\frac{\lambda}{\mu} = \frac{-(\nu/\mu)'}{y' - (\nu/\mu)x'} = \frac{(x'/Ly')'}{y' + x'^2/Ly'} = \frac{x'' - x'(Ly'' + L'y')/Ly'}{Ly'^2 + x'^2}.$$

The same equations (19), (20) give

$$\lambda = -\frac{\nu'}{y'} = -\frac{\mu'}{x'}, \quad \nu = -\frac{\mu x'}{Ly'};$$

therefore

$$\lambda = \frac{\mu'x' + \mu x''}{Ly'^2} - \frac{\mu x'(Ly'' + L'y')}{L^2y'^3} = -\frac{\mu'}{x'}.$$

Thus

$$\lambda = -\frac{\mu'}{x'} = \frac{\mu}{Ly'^3}(y'x'' - x'y'') + \frac{x'}{L^2y'^2}(L\mu' - L'\mu) = \frac{x'}{Ly'^2}\left\{\mu' - \frac{L'\mu}{L} - \mu\frac{M'x' - L'y'}{2Ly' - Mx'}\right\},$$

by the value of y'' deduced by differentiation from the equation

$$Ly'^2 - x'^2 - Mx'y' = 0.$$

Hence

$$0 = (\lambda x' + \mu')L^2y'^2(2Ly' - Mx') = \mu'Ly'(2Ly' - Mx')^2 - \mu x'^2\{L'(Ly' - Mx') + M'Lx'\}$$
$$= x'^2\{\mu'Ly'(M^2 + 4L) - \mu LL'y' + \mu x'(ML' - LM')\},$$

and so

$$\frac{\mu'}{\mu}=\frac{LL'y'+(LM'-ML')x'}{L(M^2+4L)y'},\quad \frac{\lambda}{\mu}=-\frac{LL'y'+(LM'-ML')x'}{L(M^2+4L)x'y'}.$$

We had found before that

$$\frac{\nu}{\mu}=-\frac{x'}{Ly'}.$$

The equation (21) gives, in the present example, $\lambda'=0$; we ought therefore to have identically, or at least in virtue of the relation

$$0=Ly'^2-x'^2-Mx'y',$$

$$0=\frac{\{LL'y'+(LM'-ML')x'\}'}{LL'y'+(LM'-ML')x'}+\frac{LL'y'+(LM'-ML')x'}{L(M^2+4L)y'}-\frac{\{L(M^2+4L)x'y'\}'}{L(M^2+4L)x'y'}.$$

(If, to particularise a little further, we make

$$L=-\frac{x^2}{y^2},\quad M=-\frac{2xy}{y^2}=-\frac{2x}{y},$$

we fall on the case of equal roots and have $M^2+4L=0$; to which case we shall afterwards return.)

If we suppose, at random, $L=x^2$, $M=0$, we have

$$\frac{\mu'}{\mu}=\frac{L'}{4L}=\frac{x'}{2x},\quad \frac{\lambda}{\mu}=-\frac{1}{2x},\quad \frac{\lambda'}{\lambda}=\frac{\mu'}{\mu}-\frac{x'}{x}=-\frac{x'}{2x}.$$

It certainly seems that we have not in general $\lambda'=0$, without a new relation giving the ratio y'/x', which may not coincide with the relation $0=Ly'^2-x'^2-Mx'y'$; and perhaps there may thus arise an *equation of condition* restricting the application of the method.

In the case of the equation

$$t=-\frac{x^2}{y^2}r-\frac{2x}{y}s,$$

the equation $Ly'^2=x'^2+Mx'y'$ becomes $0=(yx'-xy')^2$, and gives

$$\frac{y'}{x'}=\frac{y}{x}=\text{const.}=a.$$

Also the equation $0=Ly'p'-x'q'+Nx'y'$ becomes $y'p'/x'q'=-y^2/x^2$. Therefore

$$\frac{p'}{q'}=-\frac{y}{x}=-a,$$

and hence

$$p+aq=\text{const.}=b.$$

The 5th of the five equations in the middle of page 405 becomes

$$0=-\frac{x^2}{y^2}y'r'-x's'-\frac{2x'y'}{y^2}(xr+ys),$$

that is,

$$0=xr'+ys'+2\frac{x'}{x}(xr+ys),$$

and so

$$0=\frac{r'+as'}{r+as}+2\frac{x'}{x},$$

or

$$(r+as)\,x^2=\text{const.}=c.$$

The first of the same equations gives

$$z'=(p+aq)\,x'=(b+2aq)\,x',$$

and hence

$$z-bx-\psi(x,\,a,\,b,\,c)=\text{const.}=\alpha,\quad \psi'(x)=2aq.$$

The second of the same equations gives

$$p'=(r+as)\,x'=\frac{cx'}{x^2},$$

and hence

$$p+\frac{c}{x}=\text{const.}=\beta.$$

Reciprocally if we differentiate these five integrals, treating a, b, c, α, β as variables, we find

$$y'=ax'+xa',\quad p'=-aq'-qa'+b',\quad (r'+as')\,x^2+2\,(r+as)\,xx'=c',$$

$$z'=bx'+xb'+\psi'(x)\,x'+\alpha',\quad p'=\frac{cx'}{x^2}-\frac{c'}{x}+\beta',$$

to which we are to join these five integral equations themselves and the equation $\psi'(x)=2aq$. Thus

$$0=px'+qy'-z'=bx'+q\,(ax'+y')-z'=(b+2aq)\,x'+qxa'-z'$$
$$=qxa'-xb'-\alpha';$$

$$0=rx'+sy'-p'=(r+as)\,x'+sxa'-\frac{cx'}{x^2}+\frac{c'}{x}-\beta'=sxa'+\frac{c'}{x}-\beta';$$

$$0=sx'+ty'-q'=(s+at)\,x'+txa'-q'$$

$$=\frac{1}{a}\left\{-(r+as)\,x'+txaa'+qa'-b'+\frac{cx'}{x^2}-\frac{c'}{x}+\beta'\right\}$$

$$=\frac{1}{a}\left\{a'\,(q+txa)-b'-\frac{c'}{x}+\beta'\right\}=\frac{1}{a}\{(q+txa+sx)\,a'-b'\}.$$

Therefore

$$\frac{b'}{a'}=q+x\,(s+ta)=q-\frac{x}{a}\,(r+as)=q-\frac{c}{ax}=q-\frac{c}{y},\quad \alpha'=x\,(qa'-b')=\frac{ca'}{a},$$

and so

$$\alpha=\chi(a),\quad c=a\chi'(a).$$