XIII.

CALCULUS OF PRINCIPAL RELATIONS

[1836.]

[Note Book 42.]

[CONTENTS

PAGE

[*The principal integral or principal function.*]

(Jan. 20th, 1836.)

[1.] In general let

$$dS = \Phi(x_1, x_2, \ldots, x_i, dx_1, dx_2, \ldots, dx_i), \tag{1}$$

this function Φ being homogeneous of the first dimension with respect to $dx_1, dx_2, \ldots, dx_i$, so that*

$$dS = dx_1 \frac{\delta dS}{\delta dx_1} + dx_2 \frac{\delta dS}{\delta dx_2} + \ldots + dx_i \frac{\delta dS}{\delta dx_i}. \tag{2}$$

Then, by the first expression for dS,

$$\delta dS = \frac{\delta dS}{\delta x_1}\delta x_1 + \ldots + \frac{\delta dS}{\delta x_i}\delta x_i + \frac{\delta dS}{\delta dx_1}\delta dx_1 + \ldots + \frac{\delta dS}{\delta dx_i}\delta dx_i; \tag{3}$$

and, by the second expression for dS,

$$\delta dS = dx_1 \delta\frac{\delta dS}{\delta dx_1} + \ldots + dx_i \delta\frac{\delta dS}{\delta dx_i} + \frac{\delta dS}{\delta dx_1}\delta dx_1 + \ldots + \frac{\delta dS}{\delta dx_i}\delta dx_i; \tag{4}$$

and therefore, by comparing these equations, we find

$$0 = \frac{\delta dS}{\delta x_1}\delta x_1 - \delta\frac{\delta dS}{\delta dx_1} dx_1 + \ldots + \frac{\delta dS}{\delta x_i}\delta x_i - \delta\frac{\delta dS}{\delta dx_i} dx_i. \tag{5}$$

Also, by (3),

$$\delta S = \int \delta dS = \Delta\left(\frac{\delta dS}{\delta dx_1}\delta x_1 + \ldots + \frac{\delta dS}{\delta dx_i}\delta x_i\right) + \int\left\{\left(\frac{\delta dS}{\delta x_1} - d\frac{\delta dS}{\delta dx_1}\right)\delta x_1 + \ldots + \left(\frac{\delta dS}{\delta x_i} - d\frac{\delta dS}{\delta dx_i}\right)\delta x_i\right\}; \tag{6}$$

* $\left[\frac{\delta dS}{\delta dx_i}\right.$ stands for the partial derivative of dS with respect to $\left.dx_i.\right]$

and if we establish the i equations

$$\frac{\delta dS}{\delta x_1} = d\frac{\delta dS}{\delta dx_1}, \ldots, \frac{\delta dS}{\delta x_i} = d\frac{\delta dS}{\delta dx_i}, \tag{7}$$

(which are, by (5), equivalent only to $i-1$ distinct equations, because the general relation (5) gives, in particular,

$$0 = \left(\frac{\delta dS}{\delta x_1} - d\frac{\delta dS}{\delta dx_1}\right) dx_1 + \ldots + \left(\frac{\delta dS}{\delta x_i} - d\frac{\delta dS}{\delta dx_i}\right) dx_i),$$

the variation δS of the integral $\int dS$ will take the *simplest possible form*, (as being that form which is most independent of the variations $\delta x_1, \delta x_2, \ldots, \delta x_i$, since it depends only on their extreme and not on their intermediate values,) namely the form

$$\delta S = \Delta\left(\frac{\delta dS}{\delta dx_1}\delta x_1 + \ldots + \frac{\delta dS}{\delta dx_i}\delta x_i\right). \tag{8}$$

We shall call the integral $S = \int dS$, determined in this way, the *principal integral** of the given element dS, or of the function Φ, in equation (1) and shall denote it, for distinction, by the symbolic expression

$$S = \underline{\int} dS = \underline{\int} \Phi(x_1, \ldots, x_i, dx_1, \ldots, dx_i), \tag{9}$$

drawing a stroke under the sign $\int$ of integration.

If we denote by $a_1, a_2, \ldots, a_i$ the initial values (or values at the first limit of the integral) of the i variables $x_1, x_2, \ldots, x_i$, if also we put for abridgement

$$\frac{\delta dS}{\delta dx_1} = y_1, \frac{\delta dS}{\delta dx_2} = y_2, \ldots, \frac{\delta dS}{\delta dx_i} = y_i \tag{10}$$

and denote the initial values of $y_1, \ldots, y_i$ by $b_1, \ldots, b_i$, we may consider the *principal integral*, $S = \underline{\int} dS$, as a function of $x_1, x_2, \ldots, x_i, a_1, a_2, \ldots, a_i$, of which the variation is

$$\begin{aligned}\delta S &= \frac{\delta S}{\delta x_1}\delta x_1 + \ldots + \frac{\delta S}{\delta x_i}\delta x_i + \frac{\delta S}{\delta a_1}\delta a_1 + \ldots + \frac{\delta S}{\delta a_i}\delta a_i \\ &= y_1\delta x_1 + \ldots + y_i\delta x_i - b_1\delta a_1 - \ldots - b_i\delta a_i;\end{aligned} \tag{11}$$

so that we have the $2i$ following equations:

$$y_1 = \frac{\delta S}{\delta x_1}, \ldots, y_i = \frac{\delta S}{\delta x_i}, \tag{12}$$

$$b_1 = -\frac{\delta S}{\delta a_1}, \ldots, b_i = -\frac{\delta S}{\delta a_i}. \tag{13}$$

If the form of the function S, as depending on $x_1, \ldots, x_i, a_1, \ldots, a_i$, were known, we could substitute it in the i equations (13) and thus transform them into i relations between the i varying or final quantities $x_1, \ldots, x_i$, and the $2i$ initial data $a_1, \ldots, a_i, b_1, \ldots, b_i$, which i relations, with $2i$ arbitrary constants, would be forms for the i integrals of the i ordinary differential equations of the second order (7). And therefore the i relations between the $3i$ quantities $x_1, \ldots, x_i, a_1, \ldots, a_i, b_1, \ldots, b_i$, which might be found in one way by integrating the i ordinary differential equations of the second order (7), may also be deduced in another way from the one *principal relation* between the *principal function* S and the $2i$ quantities $x_1, \ldots, x_i, a_1, \ldots, a_i$ by

* [The definition of S is, of course, exactly analogous to that of the principal function in dynamics, to which, in fact, it would reduce if $\Phi = Ldt$, where L is the Lagrangian of the dynamical system.]

taking the partial differential coefficients (of the first order) of that one principal function with respect to the initial variables $a_1, \ldots, a_i$ and then equating these coefficients to $-b_1, \ldots, -b_i$ respectively; which is my chief result respecting the properties of this principal integral S, considered as depending on its limits, and my chief reason for calling that integral a *principal function*; and for giving to that new branch of Algebra, which proposes by new methods to find and to use the form of this principal function, the name of the CALCULUS OF PRINCIPAL RELATIONS.

[*The partial differential equation satisfied by the principal function.*]

(Jan. 21st, 1836.)

[2.] Among the chief methods for finding the form of the Principal Function S is the following, by a partial differential equation of the first order or by a pair of such equations. Since $y_1, y_2, \ldots, y_i$ are functions only of the ratios of $dx_1, \ldots, dx_i$, we can in general eliminate these $i-1$ ratios and obtain one relation between $y_1, \ldots, y_i$, involving also in general $x_1, \ldots, x_i$ and depending for its form upon the form of dS or of the function Φ in (1); and we may represent this relation as follows:

$$0 = \Psi(y_1, \ldots, y_i, x_1, \ldots, x_i). \tag{14}$$

In like manner we have by considering initial values

$$0 = \Psi(b_1, \ldots, b_i, a_1, \ldots, a_i), \tag{15}$$

the form of the function Ψ being the same as in (14). And if in these relations we substitute for $y_1, \ldots, y_i$ and $b_1, \ldots, b_i$ their values (12) and (13), we obtain the two partial differential equations

$$0 = \Psi\left(\frac{\delta S}{\delta x_1}, \ldots, \frac{\delta S}{\delta x_i}, x_1, \ldots, x_i\right) \tag{16}$$

and

$$0 = \Psi\left(-\frac{\delta S}{\delta a_1}, \ldots, -\frac{\delta S}{\delta a_i}, a_1, \ldots, a_i\right). \tag{17}$$

In integrating these equations we are to determine the arbitrary functions which may present themselves by the following conditions.

First, S must vanish when $x_1 - a_1, x_2 - a_2, \ldots, x_i - a_i$ all vanish—at least that form of S which corresponds to moderate values of those increments, and indeed every form of S excepting those cases of periodicity in which $x_1, x_2, \ldots, x_i$, being considered as functions of some one indefinitely and continuously increasing variable t, acquire all together the same values $a_1, \ldots, a_i$ for some new value $t = t_2$ which they had for the old or original value $t = t_1$. For, generally, if $x_1, x_2, \ldots, x_i$ be considered as so many functions of t while $a_1, a_2, \ldots a_i$ are considered as the values to which those functions reduce when t is made equal to 0, and if therefore the principal integral S be put under the form

$$S = \int_{t_1}^{t_2} \Phi(x_1, \ldots, x_i, x_1', \ldots, x_i')\, dt, \tag{18}$$

in which

$$x_1' = \frac{dx_1}{dt}, \ldots, x_i' = \frac{dx_i}{dt}, \tag{19}$$

then the function S by its integral nature must vanish when $t = t_1$. It is important to observe that the value of the integral S is not affected by the arbitrary form of x_i as a function of t, if the forms of $x_1, \ldots, x_{i-1}$ be deduced from this by the differential equations (7) and if the conditions at the limits be satisfied.

Secondly, at the origin of the progression, that is, when $t=t_1$, the general values of the partial differential coefficients $\frac{\delta S}{\delta x_1}, \ldots, \frac{\delta S}{\delta x_i}$ and at the same time those of $-\frac{\delta S}{\delta a_1}, \ldots, -\frac{\delta S}{\delta a_i}$ must reduce to those functions of $a_1, \ldots, a_i$ and of the ratios of $x_1-a_1, \ldots, x_i-a_i$ which may be otherwise deduced from the general values of $b_1, \ldots, b_i$ by changing the ratios of $da_1, \ldots, da_i$ to the ratios of $x_1-a_1, \ldots, x_i-a_i$, or from the general values of $y_1, \ldots, y_i$ by changing the ratios of $dx_1, \ldots, dx_i$ to those of $x_1-a_1, \ldots, x_i-a_i$ and at the same time changing $x_1, \ldots, x_i$ to $a_1, \ldots, a_i$.

Thirdly—and this condition includes the two former ones—at the origin of the progression or first limit of the integration ($t=t_1$) the principal function or integral S must bear the (nascent) ratio of unity or equality to the function formed from dS by changing the differentials $dx_1, \ldots, dx_i$ to the increments $x_1-a_1, \ldots, x_i-a_i$ and by changing $x_1, \ldots, x_i$ themselves to $a_1, \ldots, a_i$; that is,

$$\lim_{t=t_1} \frac{S}{t} = \lim_{t=t_1} \Phi\left(a_1, a_2, \ldots a_i, \frac{x_1-a_1}{t}, \ldots, \frac{x_i-a_i}{t}\right), \tag{20}$$

or, in other symbols,

$$1 = \lim_{t=t_1} \frac{S}{\Phi(a_1, \ldots, a_i, x_1-a_1, \ldots, x_i-a_i)}. \tag{21}$$

[*Solution of the partial differential equation by successive approximation.**]

[**3.**] We may in general consider S as a function of $a_1, a_2, \ldots, a_{i-1}, a_i, \frac{x_1-a_1}{x_i-a_i}, \ldots, \frac{x_{i-1}-a_{i-1}}{x_i-a_i}$, x_i-a_i, and for small or moderate values of $t-t_1$ and of $x_1-a_1, \ldots, x_i-a_i$ we may in general develope this function according to ascending integer powers of the small or moderate increment x_i-a_i (setting aside singular exceptions) in a series of the form

$$S = A(x_i-a_i) + B(x_i-a_i)^2 + C(x_i-a_i)^3 + \&c., \tag{22}$$

which may also be thus written, more simply and symmetrically,

$$S = S_1 + S_2 + S_3 + \&c., \tag{23}$$

S_n being a homogeneous function of the nth dimension of the i increments $x_1-a_1, \ldots, x_i-a_i$, involving also in general $a_1, \ldots, a_i$. We may now conceive this expression substituted in the partial differential equation (16) so as to give an equation of the following form:

$$0 = \Psi\left\{\frac{\delta S_1}{\delta x_1} + \frac{\delta S_2}{\delta x_1} + \&c., \ldots, \frac{\delta S_1}{\delta x_2} + \frac{\delta S_2}{\delta x_2} + \&c., \ldots, \frac{\delta S_1}{\delta x_i} + \frac{\delta S_2}{\delta x_i} + \&c., \right.$$
$$\left. a_1 + (x_1-a_1), \ldots, a_i + (x_i-a_i)\right\}, \tag{24}$$

in which $\frac{\delta S_n}{\delta x_k}$ is a homogeneous function of dimension $n-1$ of the increments $x_1-a_1, \ldots, x_i-a_i$. And we may in general develope this equation (24) by Taylor's theorem as follows:

$$0 = \Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \&c., \tag{25}$$

in which Ψ_n is homogeneous of dimension n with respect to the increments $x_1-a_1, \ldots, x_i-a_i$; and then may deduce from it the following indefinite series of separate equations in partial differential coefficients of the first order,

$$0 = \Psi_0, \quad 0 = \Psi_1, \quad 0 = \Psi_2, \quad 0 = \Psi_3, \quad \&c. \tag{26}$$

* [See Appendix, Note 9, p. 631.]

To develope these equations, let us write generally

$$\Psi(b_1+\beta_1, b_2+\beta_2, \ldots, b_i+\beta_i, a_1+\alpha_1, a_2+\alpha_2, \ldots, a_i+\alpha_i)$$
$$=\Psi(b_1, b_2, \ldots, b_i, a_1, a_2, \ldots, a_i)+\beta_1\Psi'(b_1)+\beta_2\Psi'(b_2)+\ldots+\beta_i\Psi'(b_i)$$
$$+\alpha_1\Psi'(a_1)+\alpha_2\Psi'(a_2)+\ldots+\alpha_i\Psi'(a_i)+\tfrac{1}{2}\beta_1^2\Psi''(b_1)+\beta_1\beta_2\Psi'{,}'(b_1,b_2)+\tfrac{1}{2}\beta_2^2\Psi''(b_2)+\ldots$$
$$+\tfrac{1}{6}\beta_1^3\Psi'''(b_1)+\tfrac{1}{2}\beta_1^2\beta_2\Psi''{,}'(b_1,b_2)+\tfrac{1}{2}\beta_1\beta_2^2\Psi'{,}''(b_1,b_2)+\&c. \qquad (27)$$

Adopting this notation which has been already used for similar purposes by Lagrange and other mathematicians, this second side of equation (24) may be thus developed:

$$\Psi\left(\frac{\delta S_1}{\delta x_1}+\frac{\delta S_2}{\delta x_1}+\&c., \ldots, \frac{\delta S_1}{\delta x_i}+\frac{\delta S_2}{\delta x_i}+\&c., a_1+x_1-a_1, \ldots, a_i+x_i-a_i\right)$$
$$=\Psi\left(\frac{\delta S_1}{\delta x_1}, \frac{\delta S_1}{\delta x_2}, \ldots, \frac{\delta S_1}{\delta x_i}, a_1, a_2, \ldots, a_i\right)$$
$$+\Psi'(a_1)(x_1-a_1)+\Psi'(a_2)(x_2-a_2)+\ldots+\Psi'(a_i)(x_i-a_i)$$
$$+\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\left(\frac{\delta S_2}{\delta x_1}+\frac{\delta S_3}{\delta x_1}+\&c.\right)+\Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\left(\frac{\delta S_2}{\delta x_2}+\frac{\delta S_3}{\delta x_2}+\&c.\right)+\ldots+\Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\left(\frac{\delta S_2}{\delta x_i}+\frac{\delta S_3}{\delta x_i}+\&c.\right)$$
$$+\tfrac{1}{2}\Psi''(a_1)(x_1-a_1)^2+\Psi'{,}'(a_1,a_2)(x_1-a_1)(x_2-a_2)+\ldots+\tfrac{1}{2}\Psi''(a_i)(x_i-a_i)^2$$
$$+\Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, a_1\right)\left(\frac{\delta S_2}{\delta x_1}+\&c.\right)(x_1-a_1)+\Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, a_2\right)\left(\frac{\delta S_2}{\delta x_1}+\&c.\right)(x_2-a_2)$$
$$+\Psi'{,}'\left(\frac{\delta S_1}{\delta x_2}, a_2\right)\left(\frac{\delta S_2}{\delta x_2}+\&c.\right)(x_2-a_2)+\&c.$$
$$+\tfrac{1}{2}\Psi''\left(\frac{\delta S_1}{\delta x_1}\right)\left(\frac{\delta S_2}{\delta x_1}+\&c.\right)^2+\Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, \frac{\delta S_1}{\delta x_2}\right)\left(\frac{\delta S_2}{\delta x_1}+\&c.\right)\left(\frac{\delta S_2}{\delta x_2}+\&c.\right)$$
$$+\ldots+\tfrac{1}{2}\Psi''\left(\frac{\delta S_1}{\delta x_i}\right)\left(\frac{\delta S_2}{\delta x_i}+\&c.\right)^2+\&c.; \qquad (28)$$

and thus the three first partial differential equations of the series (28) may be developed as follows:

$$0=(\Psi_0=)\,\Psi\left(\frac{\delta S_1}{\delta x_1}, \frac{\delta S_1}{\delta x_2}, \ldots, \frac{\delta S_1}{\delta x_i}, a_1, a_2, \ldots, a_i\right); \qquad (29)$$

$$0=(\Psi_1=)\,\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\frac{\delta S_2}{\delta x_1}+\Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\frac{\delta S_2}{\delta x_2}+\ldots+\Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\frac{\delta S_2}{\delta x_i}$$
$$+\Psi'(a_1)(x_1-a_1)+\Psi'(a_2)(x_2-a_2)+\ldots+\Psi'(a_i)(x_i-a_i); \qquad (30)$$

$$0=(\Psi_2=)\,\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\frac{\delta S_3}{\delta x_1}+\Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\frac{\delta S_3}{\delta x_2}+\ldots+\Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\frac{\delta S_3}{\delta x_i}$$
$$+\tfrac{1}{2}\Psi''\left(\frac{\delta S_1}{\delta x_1}\right)\left(\frac{\delta S_2}{\delta x_1}\right)^2+\Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, \frac{\delta S_1}{\delta x_2}\right)\frac{\delta S_2}{\delta x_1}\frac{\delta S_2}{\delta x_2}+\ldots+\tfrac{1}{2}\Psi''\left(\frac{\delta S_1}{\delta x_i}\right)\left(\frac{\delta S_2}{\delta x_i}\right)^2$$
$$+\Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, a_1\right)\frac{\delta S_2}{\delta x_1}(x_1-a_1)+\Psi'{,}'\left(\frac{\delta S_1}{\delta x_1}, a_2\right)\frac{\delta S_2}{\delta x_2}(x_2-a_2)+\ldots+\Psi'{,}'\left(\frac{\delta S_1}{\delta x_i}, a_i\right)\frac{\delta S_2}{\delta x_i}(x_i-a_i)$$
$$+\tfrac{1}{2}\Psi''(a_1)(x_1-a_1)^2+\Psi'{,}'(a_1,a_2)(x_1-a_1)(x_2-a_2)+\ldots+\tfrac{1}{2}\Psi''(a_i)(x_i-a_i)^2; \qquad (31)$$

and the others may be similarly developed.

We have next to integrate these equations; at least to discover functions S_1, S_2, S_3, &c. which shall satisfy them. It might seem that this integration would introduce in general an arbitrary function for every differential equation; and thus an infinite number of arbitrary functions into the general expression of the sum $S_1+S_2+S_3+\&c.=S$; but the conditions already mentioned enable us to foresee that the form of S_1 required for our present purpose must be

$$S_1=\Phi(a_1, a_2, \dots, a_i, x_1-a_1, x_2-a_2, \dots, x_i-a_i), \tag{32}$$

which form accordingly may be easily shown to satisfy the partial differential equation (29) (see below); and then the remaining functions S_2, S_3, &c. may be determined, as we are about to prove, by the remaining equations (30), (31) ... without any new integrations being required —a result of great importance in the Calculus of Principal Relations as enabling us to develope the Principal Function without ambiguity for the case of moderate increments of the variables $x_1, \dots, x_i$.

To show first of all that the form (32) for S_1 satisfies equation (29), we may observe that this form gives by partial differentiation for $\frac{\delta S_1}{\delta x_1}, \dots, \frac{\delta S_1}{\delta x_i}$ the same functions of $a_1, \dots, a_i$ and of the ratios of $x_1-a_1, \dots, x_i-a_i$, which might be otherwise deduced from the expressions for $b_1, \dots, b_i$ by changing the ratios of $da_1, \dots, da_i$ to the ratios of $x_1-a_1, \dots, x_i-a_i$; since then we had, independently of the ratios of $da_1, \dots, da_i$, the relation (15) between $b_1, \dots, b_i, a_1, \dots, a_i$ we must also have, independently of the ratios of $x_1-a_1, \dots, x_i-a_i$, the relation (29) between

$$\frac{\delta S_1}{\delta x_1}, \dots, \frac{\delta S_1}{\delta x_i}, a_1, \dots, a_i.$$

(Again, the equation (5) shows that the variations $\delta x_1, \dots, \delta x_i, \delta y_1, \dots, \delta y_i$ are connected by the relation

$$0=\frac{\delta dS}{\delta x_1}\delta x_1 - dx_1 \delta y_1 + \dots + \frac{\delta dS}{\delta x_i}\delta x_i - dx_i \delta y_i, \tag{33}$$

which may by (7) be put in the form

$$0=dy_1 \delta x_1 - dx_1 \delta y_1 + \dots + dy_i \delta x_i - dx_i \delta y_i; \tag{34}$$

since then, by (14), we have

$$0=\Psi'(x_1)\delta x_1 + \Psi'(y_1)\delta y_1 + \dots + \Psi'(x_i)\delta x_i + \Psi'(y_i)\delta y_i, \tag{35}$$

and since these two last expressions must both be satisfied independently of any other relation between $\delta x_1, \dots, \delta x_i$ and $\delta y_1, \dots, \delta y_i$, we see that we must have, separately,

$$\Psi'(y_1)=-Lx_1', \quad \Psi'(y_2)=-Lx_2', \quad \dots, \quad \Psi'(y_i)=-Lx_i', \tag{36}$$

x_1', etc. having the meanings (19) and L being some common multiplier; and in like manner, L being still the same common multiplier, we have

$$\Psi'(x_1)=+Ly_1', \quad \Psi'(x_2)=+Ly_2', \quad \dots, \quad \Psi'(x_i)=+Ly_i', \tag{37}$$

in which

$$y_1'=\frac{dy_1}{dt}, \dots, y_i'=\frac{dy_i}{dt}. \tag{38}$$

We might proceed in this way to determine the ratios of $\Psi'\left(\frac{\delta S_1}{\delta x_1}\right), \dots, \Psi'\left(\frac{\delta S_1}{\delta x_i}\right)$ and to show that they are the same as the ratios of $x_1 - a_1, \ldots, x_i - a_i$, but the following method is more simple.)

Since S_1 is a homogeneous function of the first dimension of $x_1 - a_1, \ldots, x_i - a_i$, it must satisfy the condition

$$S_1 = (x_1 - a_1)\frac{\delta S_1}{\delta x_1} + (x_2 - a_2)\frac{\delta S_1}{\delta x_2} + \ldots + (x_i - a_i)\frac{\delta S_1}{\delta x_i}; \tag{39}$$

which gives, by being varied with respect to $x_1, \ldots, x_i$,

$$0 = (x_1 - a_1)\,\delta\frac{\delta S_1}{\delta x_1} + (x_2 - a_2)\,\delta\frac{\delta S_1}{\delta x_2} + \ldots + (x_i - a_i)\,\delta\frac{\delta S_1}{\delta x_i}, \tag{40}$$

the quantities $a_1, \ldots, a_i$ being treated as constants. But on this last supposition, the equation (29) gives

$$0 = \Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\delta\frac{\delta S_1}{\delta x_1} + \Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\delta\frac{\delta S_1}{\delta x_2} + \ldots + \Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\delta\frac{\delta S_1}{\delta x_i}; \tag{41}$$

and since these two linear relations (40) and (41) between the variations $\delta\frac{\delta S_1}{\delta x_1}, \ldots, \delta\frac{\delta S_1}{\delta x_i}$ must in general hold together and be equivalent only to one relation, the coefficients in the one must be proportional to those in the other; so that, in general,

$$\Psi'\left(\frac{\delta S_1}{\delta x_1}\right) = \lambda(x_1 - a_1),\ \Psi'\left(\frac{\delta S_1}{\delta x_2}\right) = \lambda(x_2 - a_2),\ \ldots,\ \Psi'\left(\frac{\delta S_1}{\delta x_i}\right) = \lambda(x_i - a_i), \tag{42}$$

λ being some common multiplier of which the form can be found when those of S_1 and Ψ' are known.

Whatever this form of λ may be, we see now that

$$\Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\frac{\delta S_n}{\delta x_1} + \Psi'\left(\frac{\delta S_1}{\delta x_2}\right)\frac{\delta S_n}{\delta x_2} + \ldots + \Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\frac{\delta S_n}{\delta x_i}$$
$$= \lambda\left\{(x_1 - a_1)\frac{\delta S_n}{\delta x_1} + (x_2 - a_2)\frac{\delta S_n}{\delta x_2} + \ldots + (x_i - a_i)\frac{\delta S_n}{\delta x_i}\right\} = \lambda n S_n, \tag{43}$$

on account of the homogeneous form of S_n. Hence the equations (30), (31) and the other similar equations for S_4, S_5, &c. will determine (in general) the several functions S_2, S_3, S_4, S_5, &c. without any integration being required after the form of S_1 has been found by the equation (32): which is one of the most useful theorems in this Calculus.

In particular, equation (30) gives

$$S_2 = -\frac{1}{2\lambda}\{\Psi'(a_1)(x_1 - a_1) + \Psi'(a_2)(x_2 - a_2) + \ldots + \Psi'(a_i)(x_i - a_i)\}. \tag{44}$$

To transform this expression for the first correction S_2 of the first approximate value S_1 of S, we may observe that the equation (39) gives, when varied with respect to all the quantities x_1, &c. and a_1, &c.,

$$0 = (x_1 - a_1)\,\delta\frac{\delta S_1}{\delta x_1} + \ldots + (x_i - a_i)\,\delta\frac{\delta S_1}{\delta x_i} - \left(\frac{\delta S_1}{\delta x_1} + \frac{\delta S_1}{\delta a_1}\right)\delta a_1 - \ldots - \left(\frac{\delta S_1}{\delta x_i} + \frac{\delta S_1}{\delta a_i}\right)\delta a_i; \tag{45}$$

while the equation (32) gives in like manner

$$0 = \Psi'\left(\frac{\delta S_1}{\delta x_1}\right)\delta\frac{\delta S_1}{\delta x_1} + \ldots + \Psi'\left(\frac{\delta S_1}{\delta x_i}\right)\delta\frac{\delta S_1}{\delta x_i} + \Psi'(a_1)\,\delta a_1 + \ldots + \Psi'(a_i)\,\delta a_i; \tag{46}$$

and since these two last equations must coincide, we have in general along with the relations (42) the following other relations:

$$\Psi'(a_1) = -\lambda\left(\frac{\delta S_1}{\delta x_1} + \frac{\delta S_1}{\delta a_1}\right),\ \Psi'(a_2) = -\lambda\left(\frac{\delta S_1}{\delta x_2} + \frac{\delta S_1}{\delta a_2}\right),\ \ldots,\ \Psi'(a_i) = -\lambda\left(\frac{\delta S_1}{\delta x_i} + \frac{\delta S_1}{\delta a_i}\right). \quad (47)$$

And thus the expression (44) transforms itself into the following:

$$S_2 = \tfrac{1}{2}(x_1 - a_1)\left(\frac{\delta S_1}{\delta x_1} + \frac{\delta S_1}{\delta a_1}\right) + \tfrac{1}{2}(x_2 - a_2)\left(\frac{\delta S_1}{\delta x_2} + \frac{\delta S_1}{\delta a_2}\right) + \ldots + \tfrac{1}{2}(x_i - a_i)\left(\frac{\delta S_1}{\delta x_i} + \frac{\delta S_1}{\delta a_i}\right). \quad (48)$$

If then we neglect only terms which are of the third dimension with respect to the small increments $x_1 - a_1, \ldots, x_i - a_i$, the principal function S may be thus expressed:

$$\begin{aligned} S &= \int \Phi(x_1, x_2, \ldots, x_i, dx_1, dx_2, \ldots, dx_i) \\ &= S_1 + \tfrac{1}{2}(x_1 - a_1)\left(\frac{\delta S_1}{\delta x_1} + \frac{\delta S_1}{\delta a_1}\right) + \ldots + \tfrac{1}{2}(x_i - a_i)\left(\frac{\delta S_1}{\delta x_i} + \frac{\delta S_1}{\delta a_i}\right), \end{aligned} \quad (49)$$

in which, by (32),

$$S_1 = \Phi(a_1, a_2, \ldots, a_i, x_1 - a_1, x_2 - a_2, \ldots, x_i - a_i).$$

And it is remarkable that in the same order of approximation this expression (49) for the principal function S may be transformed as follows:

$$S = \Phi\left(\frac{x_1 + a_1}{2}, \frac{x_2 + a_2}{2}, \ldots, \frac{x_i + a_i}{2}, x_1 - a_1, x_2 - a_2, \ldots, x_i - a_i\right). \quad (50)$$

[*An alternative method of approximation.*]

[**4.**] Before proceeding further in this integration of the partial differential equation (16), let us observe that if we consider z as an independent and continuously flowing variable on which all the rest depend, and which is $= 0$ at the beginning and $= x$ at the end of the progression, we may in general denote the principal function or integral S as follows:

$$S = \int_a^x \Phi\left(z_1, z_2, \ldots, z_i, \frac{dz_1}{dz}, \frac{dz_2}{dz}, \ldots, \frac{dz_i}{dz}\right) dz, \quad (51)$$

$z_1, z_2, \ldots, z_i$ being functions of z which may be thus denoted

$$z_1 = f_1(z),\ z_2 = f_2(z),\ \ldots,\ z_i = f_i(z), \quad (52)$$

and which satisfy the i initial conditions

$$f_1(a) = a_1,\ f_2(a) = a_2,\ \ldots,\ f_i(a) = a_i, \quad (53)$$

and the i final conditions

$$f_1(x) = x_1,\ f_2(x) = x_2,\ \ldots,\ f_i(x) = x_i. \quad (54)$$

And if, as a first approximation, we make the supposition of uniformly flowing values, or linear forms, of the functions $z_1, z_2, \ldots, z_i$ so as to suppose

$$z_1 = a_1 + (z - a)\frac{x_1 - a_1}{x - a},\ z_2 = a_2 + (z - a)\frac{x_2 - a_2}{x - a},\ \ldots,\ z_i = a_i + (z - a)\frac{x_i - a_i}{x - a}, \quad (55)$$

and

$$\frac{dz_1}{dz} = f_1'(z) = \frac{x_1 - a_1}{x - a},\ \frac{dz_2}{dz} = f_2'(z) = \frac{x_2 - a_2}{x - a},\ \ldots,\ \frac{dz_i}{dz} = f_i'(z) = \frac{x_i - a_i}{x - a} \quad (56)$$

and therefore by (51)

$$S=\int_a^x \Phi\left(a_1+\overline{z-a}\,\frac{x_1-a_1}{x-a}, \ldots, a_i+\overline{z-a}\,\frac{x_i-a_i}{x-a}, \frac{x_1-a_1}{x-a}, \ldots, \frac{x_i-a_i}{x-a}\right) dz; \qquad (57)$$

we find, by developing the coefficient under the integral sign as far as the first power inclusive of $z-a$,

$$\frac{dS}{dz}=\Phi\left(a_1+\overline{z-a}\,\frac{x_1-a_1}{x-a}, \ldots, a_i+\overline{z-a}\,\frac{x_i-a_i}{x-a}, \frac{x_1-a_1}{x-a}, \ldots, \frac{x_i-a_i}{x-a}\right)$$

$$=\Phi\left(a_1, \ldots, a_i, \frac{x_1-a_1}{x-a}, \ldots, \frac{x_i-a_i}{x-a}\right)+\frac{z-a}{x-a}\{\Phi'(a_1)(x_1-a_1)+\ldots+\Phi'(a_i)(x_i-a_i)\}, \quad (58)$$

$\Phi'(a_1), \ldots, \Phi'(a_i)$ being here formed by varying $\Phi\left(a_1, a_2, \ldots, a_i, \frac{x_1-a_1}{x-a}, \frac{x_2-a_2}{x-a}, \ldots, \frac{x_i-a_i}{x-a}\right)$ as if $\frac{x_1-a_1}{x-a}$, etc. were constants; and therefore, by integration,

$$S=(x-a)\,\Phi\left(a_1, \ldots, a_i, \frac{x_1-a_1}{x-a}, \ldots, \frac{x_i-a_i}{x-a}\right)$$

$$+\tfrac{1}{2}(x-a)\{\Phi'(a_1)(x_1-a_1)+\ldots+\Phi'(a_i)(x_i-a_i)\}, \qquad (59)$$

that is, $S=\Phi(a_1, a_2, \ldots, a_i, x_1-a_1, x_2-a_2, \ldots, x_i-a_i)$

$$+\frac{x_1-a_1}{2}\left(\frac{\delta}{\delta a_1}+\frac{\delta}{\delta x_1}\right)\Phi(a_1, a_2, \ldots, a_i, x_1-a_1, x_2-a_2, \ldots, x_i-a_i)$$

+&c.

$$+\frac{x_i-a_i}{2}\left(\frac{\delta}{\delta a_i}+\frac{\delta}{\delta x_i}\right)\Phi(a_1, a_2, \ldots, a_i, x_1-a_1, x_2-a_2, \ldots, x_i-a_i); \qquad (60)$$

which agrees with the expression (49) and is therefore accurate as far as the second dimension inclusive, although $\frac{dz_1}{dz}, \ldots, \frac{dz_i}{dz}$ are not accurate as far as the first dimension inclusive with respect to the small quantities $x_1-a_1, x_2-a_2, \ldots, x_i-a_i$. The theory of this fact will soon be fully explained.*

[*The first method may be used without forming the partial differential equation.*]

(Jan. 22nd, 1836.)

[5.] Proceeding now to equation (31) and seeking to transform the expression which it gives for S_3 into one more commodious and especially into one more closely connected with the form of the original function Φ in the expression for the element dS in (1), we may suppose in general that the equation (29) has been so prepared, by resolving it with respect to $\frac{\delta S_1}{\delta x_i}$, as to be of the form

$$0=-\frac{\delta S_1}{\delta x_i}+\text{funct}^{\text{n}}\left(\frac{\delta S_1}{\delta x_1}, \frac{\delta S_1}{\delta x_2}, \ldots, \frac{\delta S_1}{\delta x_{i-1}}, a_1, a_2, \ldots, a_{i-1}, a_i\right); \qquad (61)$$

* [This alternative method of finding an approximate expression for S is very similar to that adopted by various writers on Rayleigh's Principle (Rayleigh, *Phil. Trans.* (1870), A, CLXI, p. 77; Ritz, *Crelle* (1908), CXXXV, p. 1). Approximate values which satisfy the end conditions (53) and (54) are substituted for $z_1, \ldots, z_n$ in the integral (57) and an approximate value thus found for the principal value, which can then be used to solve the original set of differential equations (7).]

and then we shall have

$$\Psi'\left(\frac{\delta S_1}{\delta x_i}\right) = -1; \tag{62}$$

$$\left.\begin{array}{l}\Psi''\left(\frac{\delta S_1}{\delta x_1}\right) = 0,\ \Psi','\left(\frac{\delta S_1}{\delta x_1}, \frac{\delta S_1}{\delta x_i}\right) = 0,\ \Psi','\left(\frac{\delta S_1}{\delta x_2}, \frac{\delta S_1}{\delta x_i}\right) = 0, \ldots, \\ \Psi','\left(\frac{\delta S_1}{\delta x_i}, a_1\right) = 0, \ldots, \Psi','\left(\frac{\delta S_1}{\delta x_i}, a_i\right) = 0;\end{array}\right\} \tag{63}$$

and $\Psi''\left(\frac{\delta S_1}{\delta x_1}\right), \ldots, \Psi','\left(\frac{\delta S_1}{\delta x_1}, a_1\right), \ldots, \Psi''(a_1), \ldots$ will be the partial differential coefficients of the second order of $\frac{\delta S_1}{\delta x_i}$ considered as a function of $\frac{\delta S_1}{\delta x_1}$, &c. If then we put for abbreviation

$$\frac{\delta S_1}{\delta x_1} = v_1,\ \frac{\delta S_1}{\delta x_2} = v_2,\ \ldots,\ \frac{\delta S_1}{\delta x_i} = v_i, \tag{64}$$

we shall have besides (62) and (63) the expressions

$$\left.\begin{array}{l}\Psi'\left(\frac{\delta S_1}{\delta x_1}\right) = \frac{\delta v_i}{\delta v_1},\ \Psi'\left(\frac{\delta S_1}{\delta x_2}\right) = \frac{\delta v_i}{\delta v_2},\ \ldots,\ \Psi'\left(\frac{\delta S_1}{\delta x_{i-1}}\right) = \frac{\delta v_i}{\delta v_{i-1}}; \\ \Psi'\left(\frac{\delta S_1}{\delta a_1}\right) = \frac{\delta v_i}{\delta a_1},\ \Psi'\left(\frac{\delta S_1}{\delta a_2}\right) = \frac{\delta v_i}{\delta a_2},\ \ldots,\ \Psi'\left(\frac{\delta S_1}{\delta a_i}\right) = \frac{\delta v_i}{\delta a_i};\end{array}\right\} \tag{65}$$

$\frac{\delta v_i}{\delta v_1}, \frac{\delta v_i}{\delta a_1}$, &c. denoting here the partial differential coefficients of the function v_i, taken with respect to v_1, a_1, &c.: we have, too,

$$\left.\begin{array}{l}\Psi''\left(\frac{\delta S_1}{\delta x_1}\right) = \frac{\delta^2 v_i}{\delta v_1^2},\ \Psi','\left(\frac{\delta S_1}{\delta x_1}, \frac{\delta S_1}{\delta x_2}\right) = \frac{\delta^2 v_i}{\delta v_1 \delta v_2},\ \ldots,\ \Psi''\left(\frac{\delta S_1}{\delta x_{i-1}}\right) = \frac{\delta^2 v_i}{\delta v_{i-1}^2}, \\ \Psi','\left(\frac{\delta S_1}{\delta x_1}, a_1\right) = \frac{\delta^2 v_i}{\delta v_1 \delta a_1},\ \ldots,\ \Psi','\left(\frac{\delta S_1}{\delta x_{i-1}}, a_i\right) = \frac{\delta^2 v_i}{\delta v_{i-1} \delta a_i}, \\ \Psi''(a_1) = \frac{\delta^2 v_i}{\delta a_1^2},\ \Psi','(a_1, a_2) = \frac{\delta^2 v_i}{\delta a_1 \delta a_2},\ \ldots,\ \Psi''(a_i) = \frac{\delta^2 v_i}{\delta a_i^2};\end{array}\right\} \tag{66}$$

and it remains to calculate these differential coefficients of the function v_i from those of the function Φ, or S_1, in the expression (1), or (32).

It may somewhat simplify the proceeding if we put for abridgement

$$x_1 - a_1 = u_1,\ x_2 - a_2 = u_2,\ \ldots,\ x_i - a_i = u_i, \tag{67}$$

and therefore by (32)

$$S_1 = \Phi(a_1, a_2, \ldots, a_i, u_1, u_2, \ldots, u_i). \tag{68}$$

This function is homogeneous of the first dimension (as we have seen) with respect to $u_1, u_2, \ldots u_i$; we have therefore the relation

$$S_1 = u_1 v_1 + u_2 v_2 + \ldots + u_i v_i, \tag{69}$$

because by (64) we have

$$v_1 = \frac{\delta S_1}{\delta u_1} = \Phi'(u_1),\ \ldots,\ v_i = \frac{\delta S_1}{\delta u_i} = \Phi'(u_i). \tag{70}$$

Eliminating the ratios of $u_1, u_2, \ldots, u_i$ between these last expressions, we might deduce as before a relation of the form

$$0=\Psi(\Phi'(u_1), \Phi'(u_2), \ldots, \Phi'(u_i), a_1, a_2, \ldots, a_i)=\Psi(v_1, v_2, \ldots, v_i, a_1, a_2, \ldots, a_i); \quad (71)$$

and might then deduce from this the sought partial differential coefficients $\frac{\delta v_i}{\delta v_1}, \frac{\delta v_i}{\delta v_2}, \ldots$. Without actually performing this elimination (which we cannot perform while we leave the form of Φ undetermined) we may still deduce these differential coefficients as follows:

The complete variation of S_1 is, by (68) and (70),

$$\delta S_1=v_1\delta u_1+v_2\delta u_2+\ldots+v_i\delta u_i+\Phi'(a_1)\delta a_1+\Phi'(a_2)\delta a_2+\ldots+\Phi'(a_i)\delta a_i; \quad (72)$$

and comparing this with the variation of the expression (69) we find

$$0=u_1\delta v_1+u_2\delta v_2+\ldots+u_i\delta v_i-\Phi'(a_1)\delta a_1-\Phi'(a_2)\delta a_2-\ldots-\Phi'(a_i)\delta a_i, \quad (73)$$

which gives

$$\frac{\delta v_i}{\delta v_1}=-\frac{u_1}{u_i}, \quad \frac{\delta v_i}{\delta v_2}=-\frac{u_2}{u_i}, \ldots, \frac{\delta v_i}{\delta v_{i-1}}=-\frac{u_{i-1}}{u_i}, \quad (74)$$

and

$$\frac{\delta v_i}{\delta a_1}=\frac{\Phi'(a_1)}{u_i}, \quad \frac{\delta v_i}{\delta a_2}=\frac{\Phi'(a_2)}{u_i}, \ldots, \frac{\delta v_i}{\delta a_i}=\frac{\Phi'(a_i)}{u_i}. \quad (75)$$

In this manner then the partial differential coefficients of v_i of the first order are determined.

Proceeding to the second order, how is $\frac{\delta^2 v_i}{\delta v_1^2}$ to be calculated? By supposing $\delta a_1=0, \ldots, \delta a_i=0, \delta v_2=0, \ldots, \delta v_{i-1}=0$, then taking the variation of $\frac{\delta v_i}{\delta v_1}=-\frac{u_1}{u_i}$ and dividing it by δv_1. We are therefore to put, by (70),

$$0=\Phi'{,}'(u_1, u_2)\delta u_1+\Phi''(u_2)\delta u_2+\Phi'{,}'(u_2, u_3)\delta u_3+\ldots+\Phi'{,}'(u_2, u_i)\delta u_i,$$

$$0=\Phi'{,}'(u_1, u_3)\delta u_1+\Phi'{,}'(u_2, u_3)\delta u_2+\Phi''(u_3)\delta u_3+\ldots+\Phi'{,}'(u_3, u_i)\delta u_i,$$

....

$$0=\Phi'{,}'(u_1, u_{i-1})\delta u_1+\Phi'{,}'(u_2, u_{i-1})\delta u_2+\ldots+\Phi'{,}'(u_{i-1}, u_i)\delta u_i,$$

establishing thus $i-2$ relations between $\delta u_1, \delta u_2, \ldots, \delta u_i$ or rather between their $i-1$ ratios, which leave one of these ratios undetermined. We have also

$$\delta v_1=\Phi''(u_1)\delta u_1+\Phi'{,}'(u_1, u_2)\delta u_2+\ldots+\Phi'{,}'(u_1, u_i)\delta u_i,$$

and

$$\delta v_i=\Phi'{,}'(u_1, u_i)\delta u_1+\Phi'{,}'(u_2, u_i)\delta u_2+\ldots+\Phi''(u_i)\delta u_i;$$

and hence, by elimination, we can in general express $\delta u_1-\frac{u_1}{u_i}\delta u_i$ as a linear function of $\delta v_1, \delta v_i$, also $\delta\frac{\delta v_i}{\delta v_1}=-\delta\frac{u_1}{u_i}$ and therefore finally calculate $\frac{\delta^2 v_i}{\delta v_1^2}$.

In general the i equations

$$\delta v_1=\delta\Phi'(u_1), \quad \delta v_2=\delta\Phi'(u_2), \ldots, \delta v_i=\delta\Phi'(u_i), \quad (76)$$

or any $i-1$ of them, enable us by elimination to express $\delta\frac{u_1}{u_i}, \ldots, \delta\frac{u_{i-1}}{u_i}$ and consequently

$\delta u_1 - \lambda u_1, \ldots, \delta u_i - \lambda u_i$ (where λ is any arbitrary multiplier) as linear functions of $\delta v_1, \ldots, \delta v_i$, $\delta a_1, \ldots, \delta a_i$; and then the values thus found for $\delta u_1, \ldots, \delta u_i$ are to be substituted in the following expression, which is deduced from (73) and which may be shown (by (73) and by the homogeneous forms of $\Phi'(a_1), \ldots, \Phi'(a_i)$) not to contain the arbitrary multiplier λ:

$$\delta^2 v_i = -\frac{1}{u_i}\{\delta u_1 \delta v_1 + \delta u_2 \delta v_2 + \ldots + \delta u_i \delta v_i - \delta\Phi'(a_1)\,\delta a_1 - \ldots - \delta\Phi'(a_i)\,\delta a_i\}. \qquad (77)$$

It only remains therefore to simplify and perform the elimination between the equations (76), which may be thus expanded:

$$\left.\begin{aligned}\delta v_1 &= \Phi''(u_1)\,\delta u_1 + \Phi'{,}'(u_1, u_2)\,\delta u_2 + \ldots + \Phi'{,}'(u_1, u_i)\,\delta u_i + \Phi'{,}'(u_1, a_1)\,\delta a_1\\ &\qquad + \ldots + \Phi'{,}'(u_1, a_i)\,\delta a_i,\\ \delta v_2 &= \Phi'{,}'(u_1, u_2)\,\delta u_1 + \Phi''(u_2)\,\delta u_2 + \ldots + \Phi'{,}'(u_2, u_i)\,\delta u_i + \Phi'{,}'(u_2, a_1)\,\delta a_1\\ &\qquad + \ldots + \Phi'{,}'(u_2, a_i)\,\delta a_i,\\ &\ldots\ldots\ldots\ldots\\ \delta v_i &= \Phi'{,}'(u_1, u_i)\,\delta u_1 + \Phi'{,}'(u_2, u_i)\,\delta u_2 + \ldots + \Phi''(u_i)\,\delta u_i + \Phi'{,}'(u_i, a_1)\,\delta a_1\\ &\qquad + \ldots + \Phi'{,}'(u_i, a_i)\,\delta a_i.\end{aligned}\right\} \qquad (78)$$

For this purpose we may employ the relations which result from the homogeneous form of Φ, namely,

$$\Phi = u_1\Phi'(u_1) + u_2\Phi'(u_2) + \ldots + u_i\Phi'(u_i); \qquad (79)$$

$$\left.\begin{aligned}\Phi'(a_1) &= u_1\Phi'{,}'(a_1, u_1) + u_2\Phi'{,}'(a_1, u_2) + \ldots + u_i\Phi'{,}'(a_1, u_i),\\ &\ldots\ldots\ldots\\ \Phi'(a_i) &= u_1\Phi'{,}'(a_i, u_1) + u_2\Phi'{,}'(a_i, u_2) + \ldots + u_i\Phi'{,}'(a_i, u_i);\end{aligned}\right\} \qquad (80)$$

and

$$\left.\begin{aligned}0 &= u_1\Phi''(u_1) + u_2\Phi'{,}'(u_1, u_2) + \ldots + u_i\Phi'{,}'(u_1, u_i),\\ &\ldots\ldots\ldots\\ 0 &= u_1\Phi'{,}'(u_1, u_i) + u_2\Phi'{,}'(u_2, u_i) + \ldots + u_i\Phi''(u_i).\end{aligned}\right\} \qquad (81)$$

Besides, if we take as the arbitrary multiplier λ in the expressions $\delta u_1 - \lambda u_1$, &c. the following (see (72)):

$$\lambda = \frac{1}{S_1}\{\delta S_1 - \Phi'(a_1)\,\delta a_1 - \ldots - \Phi'(a_i)\,\delta a_i\} = \frac{1}{S_1}(v_1 \delta u_1 + \ldots + v_i \delta u_i), \qquad (82)$$

we shall have, by (69), the relation

$$0 = v_1(\delta u_1 - \lambda u_1) + v_2(\delta u_2 - \lambda u_2) + \ldots + v_i(\delta u_i - \lambda u_i). \qquad (83)$$

We are therefore to determine by elimination, if we can, the i expressions $\delta u_1 - \lambda u_1, \ldots, \delta u_i - \lambda u_i$ as linear functions of $\delta v_1, \delta v_2, \ldots, \delta v_i, \delta a_1, \delta a_2, \ldots, \delta a_i$ by means of this last relation and any $i-1$, or all, of the i equations following:

$$\left.\begin{aligned}\delta v_1 &= \Phi''(u_1)(\delta u_1 - \lambda u_1) + \ldots + \Phi'{,}'(u_1, u_i)(\delta u_i - \lambda u_i) + \Phi'{,}'(a_1, u_1)\,\delta a_1\\ &\qquad + \ldots + \Phi'{,}'(a_i, u_1)\,\delta a_i,\\ &\ldots\ldots\ldots\ldots\\ \delta v_i &= \Phi'{,}'(u_1, u_i)(\delta u_1 - \lambda u_1) + \ldots + \Phi''(u_i)(\delta u_i - \lambda u_i) + \Phi'{,}'(a_1, u_i)\,\delta a_1\\ &\qquad + \ldots + \Phi'{,}'(a_i, u_i)\,\delta a_i.\end{aligned}\right\} \qquad (84)$$

If we put for abridgement

$$\delta u_1 - \lambda u_1 = \delta' u_1, \ldots, \delta u_i - \lambda u_i = \delta' u_i, \tag{85}$$

and

$$\left.\begin{array}{l} \delta v_1 - \Phi'{,}' (a_1, u_1)\,\delta a_1 - \ldots - \Phi'{,}' (a_i, u_1)\,\delta a_i = \delta' v_1, \\ \ldots\ldots\ldots\ldots\ldots \\ \delta v_i - \Phi'{,}' (a_1, u_i)\,\delta a_1 - \ldots - \Phi'{,}' (a_i, u_i)\,\delta a_i = \delta' v_i, \end{array}\right\} \tag{86}$$

The $i+1$ relations (83) and (84), equivalent only to i distinct ones, will take these simpler forms:

$$0 = v_1 \delta' u_1 + v_2 \delta' u_2 + \ldots + v_i \delta' u_i, \tag{87}$$

and

$$\left.\begin{array}{l} \delta' v_1 = \Phi'' (u_1)\,\delta' u_1 + \Phi'{,}' (u_1, u_2)\,\delta' u_2 + \ldots + \Phi'{,}' (u_1, u_i)\,\delta' u_i, \\ \ldots\ldots\ldots\ldots\ldots \\ \delta' v_i = \Phi'{,}' (u_1, u_i)\,\delta' u_1 + \Phi'{,}' (u_2, u_i)\,\delta' u_2 + \ldots + \Phi'' (u_i)\,\delta' u_i, \end{array}\right\} \tag{88}$$

in which the coefficients are connected by the conditions of homogeneity (81).

[*The case of two variables.*]

[6.] Consider first the case of only *two* variables x_1, x_2 $(i = 2)$ with two corresponding variables u_1, u_2, &c. We have now to deduce $\delta' u_1, \delta' u_2$, from the three relations following, or from any two of them:

$$0 = v_1 \delta' u_1 + v_2 \delta' u_2, \tag{89}$$

and

$$\left.\begin{array}{l} \delta' v_1 = \Phi'' (u_1)\,\delta' u_1 + \Phi'{,}' (u_1, u_2)\,\delta' u_2, \\ \delta' v_2 = \Phi'{,}' (u_1, u_2)\,\delta' u_1 + \Phi'' (u_2)\,\delta' u_2; \end{array}\right\} \tag{90}$$

and the coefficients are connected by the 2 relations

$$0 = u_1 \Phi'' (u_1) + u_2 \Phi'{,}' (u_1, u_2), \quad 0 = u_1 \Phi'{,}' (u_1, u_2) + u_2 \Phi'' (u_2); \tag{91}$$

we have also

$$u_1 v_1 + u_2 v_2 = \Phi = S_1. \tag{92}$$

By (90), we have

$$u_2 \delta' v_1 - u_1 \delta' v_2 = \{u_2 \Phi'' (u_1) - u_1 \Phi'{,}' (u_1, u_2)\}\,\delta' u_1 + \{u_2 \Phi'{,}' (u_1, u_2) - u_1 \Phi'' (u_2)\}\,\delta' u_2; \tag{93}$$

therefore, by (89), we have the following expressions for $\delta' u_1, \delta' u_2$:

$$\left.\begin{array}{l} \delta' u_1 = \dfrac{v_2 (u_2 \delta' v_1 - u_1 \delta' v_2)}{v_2 \{u_2 \Phi'' (u_1) - u_1 \Phi'{,}' (u_1, u_2)\} - v_1 \{u_2 \Phi'{,}' (u_1, u_2) - u_1 \Phi'' (u_2)\}}, \\ \delta' u_2 = \dfrac{-v_1 (u_2 \delta' v_1 - u_1 \delta' v_2)}{v_2 \{u_2 \Phi'' (u_1) - u_1 \Phi'{,}' (u_1, u_2)\} - v_1 \{u_2 \Phi'{,}' (u_1, u_2) - u_1 \Phi'' (u_2)\}}. \end{array}\right\} \tag{94}$$

In the common denominator, we have by (91)

$$\frac{\Phi'' (u_1)}{u_2^2} = -\frac{\Phi'{,}' (u_1, u_2)}{u_1 u_2} = \frac{\Phi'' (u_2)}{u_1^2} = \frac{\Phi'' (u_1) + \Phi'' (u_2)}{u_1^2 + u_2^2}; \tag{95}$$

and therefore

$$\left.\begin{array}{r} u_2 \Phi'' (u_1) - u_1 \Phi'{,}' (u_1, u_2) = u_2 \{\Phi'' (u_1) + \Phi'' (u_2)\}, \\ -u_2 \Phi'{,}' (u_1, u_2) + u_1 \Phi'' (u_2) = u_1 \{\Phi'' (u_1) + \Phi'' (u_2)\}, \end{array}\right\} \tag{96}$$

so that the common denominator is $(u_2 v_2 + u_1 v_1)\{\Phi'' (u_1) + \Phi'' (u_2)\}$, and the expressions (94) become (attending to (92))

$$\delta' u_1 = \frac{v_2}{\Phi} \cdot \frac{u_2 \delta' v_1 - u_1 \delta' v_2}{\Phi'' (u_1) + \Phi'' (u_2)}, \quad \delta' u_2 = -\frac{v_1}{\Phi} \cdot \frac{u_2 \delta' v_1 - u_1 \delta' v_2}{\Phi'' (u_1) + \Phi'' (u_2)}. \tag{97}$$

Hence

$$\left.\begin{aligned}\delta' u_1 - \frac{\delta' v_1}{\Phi''(u_1)+\Phi''(u_2)} &= -\frac{u_1}{\Phi}\cdot\frac{v_1\delta' v_1+v_2\delta' v_2}{\Phi''(u_1)+\Phi''(u_2)},\\ \delta' u_2 - \frac{\delta' v_2}{\Phi''(u_1)+\Phi''(u_2)} &= -\frac{u_2}{\Phi}\cdot\frac{v_1\delta' v_1+v_2\delta' v_2}{\Phi''(u_1)+\Phi''(u_2)};\end{aligned}\right\} \tag{98}$$

and therefore by the meanings (85) of $\delta' u_1$, $\delta' u_2$, ...

$$u_2\delta u_1 - u_1\delta u_2 = u_2\delta' u_1 - u_1\delta' u_2 = \frac{u_2\delta' v_1 - u_1\delta' v_2}{\Phi''(u_1)+\Phi''(u_2)}; \tag{99}$$

an expression which might also have been deduced more immediately from (94). Hence, by the meanings (86) of $\delta' v_1$, $\delta' v_2$,

$$u_2\delta u_1 - u_1\delta u_2 = \frac{u_2\delta v_1 - u_1\delta v_2}{\Phi''(u_1)+\Phi''(u_2)} - \frac{u_2\Phi'{,}'(a_1, u_1) - u_1\Phi'{,}'(a_1, u_2)}{\Phi''(u_1)+\Phi''(u_2)}\delta a_1 - \frac{u_2\Phi'{,}'(a_2, u_1) - u_1\Phi'{,}'(a_2, u_2)}{\Phi''(u_1)+\Phi''(u_2)}\delta a_2. \tag{100}$$

In general the equation (77) may be put under the form

$$\delta^2 v_i = -\frac{1}{u_i}(\delta u_1\delta' v_1 + \delta u_2\delta' v_2 + \ldots + \delta u_i\delta' v_i) + \frac{1}{u_i}\delta'^2\Phi, \tag{101}$$

δ' referring only to the variations of $a_1, a_2, \ldots, a_i$; so that, since

$$0 = u_1\delta' v_1 + u_2\delta' v_2 + \ldots + u_i\delta' v_i, \tag{102}$$

we have

$$\delta^2 v_i = -\frac{1}{u_i}(\delta' u_1\delta' v_1 + \delta' u_2\delta' v_2 + \ldots + \delta' u_i\delta' v_i - \delta'^2\Phi), \tag{103}$$

in which we may, by (102), introduce or suppress any set of terms in $\delta' u_1, \delta' u_2, \ldots, \delta' u_i$, which are proportional to $u_1, u_2, \ldots, u_i$.

In the particular case $i=2$, we have therefore by (98)

$$\delta^2 v_2 = -\frac{1}{u_2}\frac{\delta' v_1^2 + \delta' v_2^2}{\Phi''(u_1)+\Phi''(u_2)} + \frac{1}{u_2}\delta'^2\Phi, \tag{104}$$

in which

$$\left.\begin{aligned}\delta' v_1 &= \delta v_1 - \Phi'{,}'(a_1, u_1)\delta a_1 - \Phi'{,}'(a_2, u_1)\delta a_2,\\ \delta' v_2 &= \delta v_2 - \Phi'{,}'(a_1, u_2)\delta a_1 - \Phi'{,}'(a_2, u_2)\delta a_2;\end{aligned}\right\} \tag{105}$$

also

$$u_1\delta' v_1 + u_2\delta' v_2 = 0. \tag{106}$$

If we do not choose to suppose $\delta^2 v_1 = 0$, then instead of (104) we have the more symmetrical relation

$$0 = u_1\delta^2 v_1 + u_2\delta^2 v_2 + \frac{\delta' v_1^2 + \delta' v_2^2}{\Phi''(u_1)+\Phi''(u_2)} - \delta'^2\Phi. \tag{107}$$

Comparing these two last equations (106), (107), of which the former may be thus written

$$0 = u_1\delta v_1 + u_2\delta v_2 - \Phi'(a_1)\delta a_1 - \Phi'(a_2)\delta a_2, \tag{108}$$

with the two following:

$$0 = \delta\Psi(v_1, v_2, a_1, a_2) = \Psi'(v_1)\delta v_1 + \Psi'(v_2)\delta v_2 + \Psi'(a_1)\delta a_1 + \Psi'(a_2)\delta a_2, \tag{109}$$

and

$$0 = \delta^2\Psi = \Psi'(v_1)\,\delta^2 v_1 + \Psi'(v_2)\,\delta^2 v_2 + \Psi''(v_1)\,\delta v_1^2 + 2\Psi'{,}'(v_1, v_2)\,\delta v_1 \delta v_2 + \Psi''(v_2)\,\delta v_2^2$$
$$+ 2\Psi'{,}'(v_1, a_1)\,\delta v_1 \delta a_1 + 2\Psi'{,}'(v_1, a_2)\,\delta v_1 \delta a_2 + 2\Psi'{,}'(v_2, a_1)\,\delta v_2 \delta a_1 + 2\Psi'{,}'(v_2, a_2)\,\delta v_2 \delta a_2$$
$$+ \Psi''(a_1)\,\delta a_1^2 + 2\Psi'{,}'(a_1, a_2)\,\delta a_1 \delta a_2 + \Psi''(a_2)\,\delta a_2^2, \tag{110}$$

we find that

$$\frac{\Psi'(v_1)}{u_1} = \frac{\Psi'(v_2)}{u_2} = -\frac{\Psi'(a_1)}{\Phi'(a_1)} = -\frac{\Psi'(a_2)}{\Phi'(a_2)} = \lambda, \tag{111}$$

$$\delta^2\Psi = \lambda\,(u_1 \delta^2 v_1 + u_2 \delta^2 v_2) + \frac{\delta' v_1^2 + \delta' v_2^2}{\Phi''(u_1) + \Phi''(u_2)} + 2\,(V_1 \delta v_1 + V_2 \delta v_2 + A_1 \delta a_1 + A_2 \delta a_2)$$
$$\times \{u_1 \delta v_1 + u_2 \delta v_2 - \Phi'(a_1)\,\delta a_1 - \Phi'(a_2)\,\delta a_2\}, \tag{112}$$

λ having the same meaning as in (111), and V_1, V_2, A_1, A_2 being multipliers to be determined by the condition that this last equation shall hold good independently of the variations $\delta v_1, \delta v_2, \delta a_1, \delta a_2, \delta^2 v_1, \delta^2 v_2$. Taking therefore the four partial differential coefficients of the equation (112) with respect to $\delta v_1, \delta v_2, \delta a_1, \delta a_2$, we find

$$\delta\frac{\delta\Psi}{\delta v_1} = \frac{\lambda\,\delta' v_1}{\Phi''(u_1) + \Phi''(u_2)} + u_1\,(V_1 \delta v_1 + V_2 \delta v_2 + A_1 \delta a_1 + A_2 \delta a_2)$$
$$+ V_1 \{u_1 \delta v_1 + u_2 \delta v_2 - \Phi'(a_1)\,\delta a_1 - \Phi'(a_2)\,\delta a_2\}, \tag{113}$$

$$\delta\frac{\delta\Psi}{\delta v_2} = \frac{\lambda\,\delta' v_2}{\Phi''(u_1) + \Phi''(u_2)} + u_2\,(.........) + V_2\{.........\}, \tag{114}$$

$$\delta\frac{\delta\Psi}{\delta a_1} = -\frac{\lambda}{\Phi''(u_1) + \Phi''(u_2)}\{\Phi'{,}'(a_1, u_1)\,\delta' v_1 + \Phi'{,}'(a_1, u_2)\,\delta' v_2\} - \delta'\frac{\delta\Phi}{\delta a_1} - \Phi'(a_1)\,(V_1 \delta v_1 + \ldots)$$
$$+ A_1\{u_1 \delta v_1 + \ldots\}, \tag{115}$$

$$\delta\frac{\delta\Psi}{\delta a_2} = -\frac{\lambda}{\Phi''(u_1) + \Phi''(u_2)}\{\Phi'{,}'(a_2, u_1)\,\delta' v_1 + \Phi'{,}'(a_2, u_2)\,\delta' v_2\} - \delta'\frac{\delta\Phi}{\delta a_2} - \Phi'(a_2)\,(.........)$$
$$+ A_2\{.........\}. \tag{116}$$

We could thus express the partial differential coefficients of the first and second orders of Ψ by means of those of Φ, the expressions of these differential coefficients of Ψ involving also the 5 arbitrary multipliers $\lambda, V_1, V_2, A_1, A_2$, which cannot be determined without assuming some new condition, such as that contained in the form (61). But without making such assumption we can transform the two equations of the form (30) and (31), namely,

$$0 = \Psi'(v_1)\frac{\delta S_2}{\delta u_1} + \Psi'(v_2)\frac{\delta S_2}{\delta u_2} + \Psi'(a_1)\,u_1 + \Psi'(a_2)\,u_2, \tag{117}$$

$$0 = \Psi'(v_1)\frac{\delta S_3}{\delta u_1} + \Psi'(v_2)\frac{\delta S_3}{\delta u_2} + \tfrac{1}{2}\Psi''(v_1)\left(\frac{\delta S_2}{\delta u_1}\right)^2 + \Psi'{,}'(v_1, v_2)\frac{\delta S_2}{\delta u_1}\frac{\delta S_2}{\delta u_2} + \tfrac{1}{2}\Psi''(v_2)\left(\frac{\delta S_2}{\delta u_2}\right)^2$$
$$+ \Psi'{,}'(v_1, a_1)\frac{\delta S_2}{\delta u_1}u_1 + \Psi'{,}'(v_1, a_2)\frac{\delta S_2}{\delta u_1}u_2 + \Psi'{,}'(v_2, a_1)\frac{\delta S_2}{\delta u_2}u_1 + \Psi'{,}'(v_2, a_2)\frac{\delta S_2}{\delta u_2}u_2$$
$$+ \tfrac{1}{2}\Psi''(a_1)\,u_1^2 + \Psi'{,}'(a_1, a_2)\,u_1 u_2 + \tfrac{1}{2}\Psi''(a_2)\,u_2^2, \tag{118}$$

so as to eliminate the differential coefficients of Ψ and introduce those of Φ in their stead.

For it is evident that the equation (117) may be formed from the equation

$$\delta\Psi = 0 \tag{119}$$

by merely changing δv_1, δv_2, δa_1, δa_2 to $\frac{\delta S_2}{\delta u_1}$, $\frac{\delta S_2}{\delta u_2}$, u_1, u_2 respectively, and that the equation (118) may be formed from

$$\delta^2\Psi = 0 \tag{120}$$

by making the changes just mentioned and changing also $\delta^2 v_1$, $\delta^2 v_2$ to $2\frac{\delta S_3}{\delta u_1}$, $2\frac{\delta S_3}{\delta u_2}$; since then, by (111), we have

$$\delta\Psi = \lambda\{u_1 \delta v_1 + u_2 \delta v_2 - \Phi'(a_1)\,\delta a_1 - \Phi'(a_2)\,\delta a_2\}, \tag{121}$$

the equation (117) gives independently of λ

$$0 = u_1 \frac{\delta S_2}{\delta u_1} + u_2 \frac{\delta S_2}{\delta u_2} - u_1 \Phi'(a_1) - u_2 \Phi'(a_2), \tag{122}$$

that is, on account of the homogeneous form of S_2,

$$S_2 = \tfrac{1}{2}\{u_1 \Phi'(a_1) + u_2 \Phi'(a_2)\}, \tag{123}$$

a result agreeing with (48); and, in like manner, (118) gives, by (112),

$$0 = 2\left(u_1 \frac{\delta S_3}{\delta u_1} + u_2 \frac{\delta S_3}{\delta u_2}\right) - \Delta'^2\Phi + \frac{\Delta' v_1^2 + \Delta' v_2^2}{\Phi''(u_1) + \Phi''(u_2)} + \frac{2}{\lambda}\left(V_1 \frac{\delta S_2}{\delta u_1} + V_2 \frac{\delta S_2}{\delta u_2} + A_1 u_1 + A_2 u_2\right)$$
$$\times \left\{u_1 \frac{\delta S_2}{\delta u_1} + u_2 \frac{\delta S_2}{\delta u_2} - u_1 \Phi'(a_1) - u_2 \Phi'(a_2)\right\}, \tag{124}$$

in which the part involving the arbitrary multiplier vanishes by (122), and in which

$$\Delta' v_1 = \frac{\delta S_2}{\delta u_1} - u_1 \Phi'{}^{,\prime}(a_1, u_1) - u_2 \Phi'{}^{,\prime}(a_2, u_1), \quad \Delta' v_2 = \frac{\delta S_2}{\delta u_2} - u_1 \Phi'{}^{,\prime}(a_1, u_2) - u_2 \Phi'{}^{,\prime}(a_2, u_2). \tag{125}$$

The expression (123) gives

$$\left.\begin{aligned} \frac{\delta S_2}{\delta u_1} &= \tfrac{1}{2}\Phi'(a_1) + \tfrac{1}{2}u_1 \Phi'{}^{,\prime}(a_1, u_1) + \tfrac{1}{2}u_2 \Phi'{}^{,\prime}(a_2, u_1), \\ \frac{\delta S_2}{\delta u_2} &= \tfrac{1}{2}\Phi'(a_2) + \tfrac{1}{2}u_1 \Phi'{}^{,\prime}(a_1, u_2) + \tfrac{1}{2}u_2 \Phi'{}^{,\prime}(a_2, u_2). \end{aligned}\right\} \tag{126}$$

Therefore

$$\left.\begin{aligned} \Delta' v_1 &= \tfrac{1}{2}\{\Phi'(a_1) - u_1 \Phi'{}^{,\prime}(a_1, u_1) - u_2 \Phi'{}^{,\prime}(a_2, u_1)\}, \\ \Delta' v_2 &= \tfrac{1}{2}\{\Phi'(a_2) - u_1 \Phi'{}^{,\prime}(a_1, u_2) - u_2 \Phi'{}^{,\prime}(a_2, u_2)\}, \end{aligned}\right\} \tag{127}$$

in which, by (80),

$$\Phi'(a_1) = u_1 \Phi'{}^{,\prime}(a_1, u_1) + u_2 \Phi'{}^{,\prime}(a_1, u_2), \quad \Phi'(a_2) = u_1 \Phi'{}^{,\prime}(a_2, u_1) + u_2 \Phi'{}^{,\prime}(a_2, u_2); \tag{128}$$

therefore

$$\Delta' v_1 = \tfrac{1}{2}u_2\{\Phi'{}^{,\prime}(a_1, u_2) - \Phi'{}^{,\prime}(a_2, u_1)\}, \quad \Delta' v_2 = -\tfrac{1}{2}u_1\{\Phi'{}^{,\prime}(a_1, u_2) - \Phi'{}^{,\prime}(a_2, u_1)\}; \tag{129}$$

so that, on account of the homogeneous form and dimension (= 3) of S_3, the equation (124) gives

$$S_3 = -\frac{1}{24}\frac{u_1^2 + u_2^2}{\Phi''(u_1) + \Phi''(u_2)}\{\Phi'{}^{,\prime}(a_1, u_2) - \Phi'{}^{,\prime}(a_2, u_1)\}^2$$
$$+ \tfrac{1}{6}\{u_1^2 \Phi''(a_1) + 2u_1 u_2 \Phi'{}^{,\prime}(a_1, a_2) + u_2^2 \Phi''(a_2)\}, \tag{130}$$

because in (124) we are to make

$$\Delta'^2\Phi = u_1^2\Phi''(a_1) + 2u_1u_2\Phi','(a_1, a_2) + u_2^2\Phi''(a_2). \qquad (131)$$

We may substitute, if we choose, for the factor $\frac{u_1^2+u_2^2}{\Phi''(u_1)+\Phi''(u_2)}$ any one of the other forms (95) which are more simple but less symmetric, except indeed the form $-\frac{u_1u_2}{\Phi','(u_1, u_2)}$ which might be substituted with advantage.

We have then, to the accuracy of the 3rd order inclusive, for the case $i=2$, this expression for the principal function S:

$$\begin{aligned} S = \int_{(x_1=a_1,\ x_2=a_2)}^{(x_1=a_1+u_1,\ x_2=a_2+u_2)} \Phi(x_1, x_2, dx_1, dx_2) &= \Phi(a_1, a_2, u_1, u_2) + \tfrac{1}{2}\{u_1\Phi'(a_1) + u_2\Phi'(a_2)\} \\ &+ \tfrac{1}{6}\{u_1^2\Phi''(a_1) + 2u_1u_2\Phi','(a_1, a_2) + u_2^2\Phi''(a_2)\} \\ &- \frac{1}{24}\frac{u_1^2+u_2^2}{\Phi''(u_1)+\Phi''(u_2)}\{\Phi','(a_1, u_2) - \Phi','(a_2, u_1)\}^2, \end{aligned} \qquad (132)$$

in which $-\frac{u_1^2+u_2^2}{\Phi''(u_1)+\Phi''(u_2)} = \frac{u_1u_2}{\Phi','(u_1, u_2)}$.

[*Examples.*]

[7.] For example, if* $\Phi(x_1, x_2, dx_1, dx_2) = \frac{dx_1^2}{2dx_2} + f(x_1)\,dx_2, \qquad (133)$

then

$$\Phi(a_1, a_2, u_1, u_2) = \frac{u_1^2}{2u_2} + f(a_1)\,u_2 \qquad (134)$$

and consequently

$$\left.\begin{aligned} &\Phi'(a_1) = u_2 f'(a_1), \quad \Phi'(a_2) = 0, \\ &\Phi''(a_1) = u_2 f''(a_1), \quad \Phi','(a_1, a_2) = 0, \quad \Phi''(a_2) = 0, \\ &\Phi','(a_1, u_2) = f'(a_1), \quad \Phi','(a_2, u_1) = 0, \\ &\Phi''(u_1) = \frac{1}{u_2}, \quad \Phi''(u_2) = \frac{u_1^2}{u_2^3}; \end{aligned}\right\} \qquad (135)$$

therefore the general approximate expression (132), for the case $i=2$, gives here

$$\begin{aligned} S &= \int_{(x_1=a_1,\ x_2=a_2)}^{(x_1=a_1+u_1,\ x_2=a_2+u_2)} \left\{\frac{dx_1^2}{2dx_2} + f(x_1)\,dx_2\right\} \\ &= \frac{u_1^2}{2u_2} + u_2 f(a_1) + \tfrac{1}{2}u_1u_2 f'(a_1) + \tfrac{1}{6}u_1^2u_2 f''(a_1) - \tfrac{1}{24}u_2^3\{f'(a_1)\}^2. \end{aligned} \qquad (136)$$

In this example the general differential equations (7) become

$$d\frac{dx_1}{dx_2} = f'(x_1)\,dx_2, \quad d\left\{\frac{dx_1^2}{2dx_2^2} - f(x_1)\right\} = 0; \qquad (137)$$

equations which are obviously compatible with each other, and which concur in giving

$$\frac{dx_1^2}{dx_2^2} - 2f(x_1) = b_1^2 - 2f(a_1), \qquad (138)$$

* [This is the dynamical problem of the linear motion of a particle of unit mass whose coordinate is x_1 at time x_2, the force potential being $-f(x_1)$.]

b_1 denoting, as in page 333, the initial value of $\frac{\delta dS}{\delta dx_1}$, which is here $\frac{dx_1}{dx_2}$; hence if we suppose $\frac{dx_1}{dx_2}>0$, we shall have

$$dx_2=\frac{dx_1}{\sqrt{2f(x_1)-2f(a_1)+b_1^2}}, \tag{139}$$

and

$$S=\int_{a_1}^{a_1+u_1}\frac{2f(x_1)-f(a_1)+\frac{1}{2}b_1^2}{\sqrt{2f(x_1)-2f(a_1)+b_1^2}}dx_1 \tag{140}$$

rigorously. Change here x_1 to a_1+u_1 and we get *approximately*

$$f(x_1)=f(a_1+u_1)=f(a_1)+u_1f'(a_1)+\tfrac{1}{2}u_1^2f''(a_1) \tag{141}$$

and therefore

$$\{2f(x_1)-2f(a_1)+b_1^2\}^{-\frac{1}{2}}=b_1^{-1}\left\{1+2u_1\frac{f'(a_1)}{b_1^2}+u_1^2\frac{f''(a_1)}{b_1^2}\right\}^{-\frac{1}{2}}$$
$$=b_1^{-1}-b_1^{-3}\{u_1f'(a_1)+\tfrac{1}{2}u_1^2f''(a_1)\}+\tfrac{3}{2}b_1^{-5}u_1^2\{f'(a_1)\}^2, \tag{142}$$

therefore, by (139),

$$u_2=\int_{a_1}^{x_1}\{2f(x_1)-2f(a_1)+b_1^2\}^{-\frac{1}{2}}dx_1=\int_0^{u_1}\{2f(a_1+u_1)-2f(a_1)+b_1^2\}^{-\frac{1}{2}}du_1$$
$$=b_1^{-1}u_1-b_1^{-3}\{\tfrac{1}{2}u_1^2f'(a_1)+\tfrac{1}{6}u_1^3f''(a_1)\}+\tfrac{1}{2}b_1^{-5}u_1^3\{f'(a_1)\}^2, \tag{143}$$

$$b_1=\frac{u_1}{u_2}-\tfrac{1}{2}b_1^{-2}u_2^{-1}u_1^2f'(a_1)-\tfrac{1}{6}b_1^{-2}u_2^{-1}u_1^3f''(a_1)+\tfrac{1}{2}b_1^{-4}u_2^{-1}u_1^3\{f'(a_1)\}^2; \tag{144}$$

hence as a first approximation

$$b_1=\frac{u_1}{u_2}; \tag{145}$$

as a second approximation

$$b_1=\frac{u_1}{u_2}-\tfrac{1}{2}u_2f'(a_1); \tag{146}$$

and as a third approximation

$$b_1=\frac{u_1}{u_2}-\tfrac{1}{2}u_2f'(a_1)\left\{1+\frac{u_2^2}{u_1}f'(a_1)\right\}-\tfrac{1}{6}u_1u_2f''(a_1)+\frac{1}{2}\frac{u_2^3}{u_1}\{f'(a_1)\}^2$$
$$=\frac{u_1}{u_2}-\tfrac{1}{2}u_2f'(a_1)-\tfrac{1}{6}u_1u_2f''(a_1); \tag{147}$$

also

$$2f(a_1+u_1)-f(a_1)+\tfrac{1}{2}b_1^2=\tfrac{1}{2}b_1^2+f(a_1)+2u_1f'(a_1)+u_1^2f''(a_1), \tag{148}$$

therefore

$$\frac{2f(a_1+u_1)-f(a_1)+\frac{1}{2}b_1^2}{\sqrt{2f(x_1)-2f(a_1)+b_1^2}}=\tfrac{1}{2}b_1+\frac{f(a_1)}{b_1}+u_1\frac{f'(a_1)}{b_1}\left\{2-\frac{1}{2}-\frac{f(a_1)}{b_1^2}\right\}$$
$$+\frac{u_1^2}{b_1}\left[f''(a_1)-2\left(\frac{f'(a_1)}{b_1}\right)^2+\{\tfrac{1}{2}b_1^2+f(a_1)\}\left\{-\frac{f''(a_1)}{2b_1^2}+\frac{3}{2}\left(\frac{f'(a_1)}{b_1^2}\right)^2\right\}\right]$$
$$=\tfrac{1}{2}b_1+\frac{f(a_1)}{b_1}+u_1\frac{f'(a_1)}{b_1}\left\{\frac{3}{2}-\frac{f(a_1)}{b_1^2}\right\}+\frac{u_1^2}{b_1}\left\{\tfrac{3}{4}f''(a_1)-\frac{5}{4}\left(\frac{f'(a_1)}{b_1}\right)^2\right.$$
$$\left.-\frac{f(a_1)f''(a_1)}{2b_1^2}+\tfrac{3}{2}f(a_1)\left(\frac{f'(a_1)}{b_1^2}\right)^2\right\}; \tag{149}$$

hence

$$S=\int_0^{u_1}\frac{2f(a_1+u_1)-f(a_1)+\frac{1}{2}b_1^2}{\sqrt{2f(x_1)-2f(a_1)+b_1^2}}du_1=\left(\frac{b_1}{2}+\frac{f(a_1)}{b_1}\right)u_1+f'(a_1)\left\{\frac{3}{2b_1}-\frac{f(a_1)}{b_1^3}\right\}\frac{u_1^2}{2}$$
$$+\left\{\frac{3}{4}\frac{f''(a_1)}{b_1}-\frac{5}{4b_1}\left(\frac{f'(a_1)}{b_1}\right)^2-\frac{f(a_1)f''(a_1)}{2b_1^3}+\frac{3}{2}\frac{f(a_1)}{b_1}\left(\frac{f'(a_1)}{b_1^2}\right)^2\right\}\frac{u_1^3}{3};\quad (150)$$

in which, by (147),

$$\frac{b_1u_1}{2}=\frac{u_1^2}{2u_2}-\frac{u_1u_2}{4}f'(a_1)-\frac{u_1^2u_2}{12}f''(a_1),\quad (151)$$

$$\frac{f(a_1)}{b_1}u_1=u_2f(a_1)\left\{1-\frac{u_2^2}{2u_1}f'(a_1)-\frac{u_2^2}{6}f''(a_1)\right\}^{-1}$$
$$=u_2f(a_1)\left\{1+\frac{u_2^2}{2u_1}f'(a_1)+\frac{u_2^2}{6}f''(a_1)+\frac{u_2^4}{4u_1^2}\{f'(a_1)\}^2\right\},\quad (152)$$

$$\frac{3f'(a_1)}{4b_1}u_1^2=\tfrac{3}{4}u_1u_2f'(a_1)\left\{1+\frac{u_2^2}{2u_1}f'(a_1)\right\},\quad (153)$$

$$-\frac{f(a_1)f'(a_1)}{2b_1^3}u_1^2=-\frac{u_2^3}{2u_1}f(a_1)f'(a_1)\left\{1+\frac{3u_2^2}{2u_1}f'(a_1)\right\},\quad (154)$$

$$\left\{\frac{f''(a_1)}{4b_1}-\frac{5\{f'(a_1)\}^2}{12b_1^3}-\frac{f(a_1)f''(a_1)}{6b_1^3}+\frac{f(a_1)\{f'(a_1)\}^2}{2b_1^5}\right\}u_1^3$$
$$=\tfrac{1}{4}u_1^2u_2f''(a_1)-\tfrac{1}{12}u_2^3[5\{f'(a_1)\}^2+2f(a_1)f''(a_1)]+\tfrac{1}{2}u_2^5u_1^{-2}f(a_1)\{f'(a_1)\}^2;\quad (155)$$

therefore, adding these last five expressions,

$$S=\frac{u_1^2}{2u_2}+u_2f(a_1)$$
$$-\frac{u_1u_2}{4}f'(a_1)+\frac{u_2^3u_1^{-1}}{2}f(a_1)f'(a_1)+\frac{3u_1u_2}{4}f'(a_1)-\frac{u_2^3u_1^{-1}}{2}f(a_1)f'(a_1)$$
$$-\frac{u_1^2u_2}{12}f''(a_1)+\frac{u_2^3}{6}f(a_1)f''(a_1)+\frac{u_2^5u_1^{-2}}{4}f(a_1)\{f'(a_1)\}^2+\frac{3u_2^3}{8}\{f'(a_1)\}^2-\frac{3u_2^5u_1^{-2}}{4}f(a_1)\{f'(a_1)\}^2$$
$$+\frac{u_1^2u_2}{4}f''(a_1)-\frac{u_2^3}{6}f(a_1)f''(a_1)-\frac{5u_2^3}{12}\{f'(a_1)\}^2+\frac{u_2^5u_1^{-2}}{2}f(a_1)\{f'(a_1)\}^2$$
$$=\frac{u_1^2}{2u_2}+u_2f(a_1)+\frac{u_1u_2}{2}f'(a_1)+\frac{u_1^2u_2}{6}f''(a_1)-\frac{u_2^3}{24}\{f'(a_1)\}^2,\quad (156)$$

as in (136).

This has been a complicated process: its most essential part, after the deduction of the rigorous intermediate integral equation (138), has been the approximate elimination of b_1 between the two rigorous expressions

$$u_2=\int_0^{u_1}\frac{du_1}{\sqrt{b_1^2+2f(a_1+u_1)-2f(a_1)}}\quad (143)$$

and

$$S=\int_0^{u_1}\frac{\frac{1}{2}b_1^2+2f(a_1+u_1)-f(a_1)}{\sqrt{b_1^2+2f(a_1+u_1)-2f(a_1)}}du_1,\quad (150)$$

giving the approximate result (136) through the medium of the approximate expression (147), deduced from (143).

In the present example we have, by (133),

$$y_1=\frac{\delta dS}{\delta dx_1}=\frac{dx_1}{dx_2};\quad y_2=\frac{\delta dS}{\delta dx_2}=-\frac{1}{2}\left(\frac{dx_1}{dx_2}\right)^2+f(x_1); \tag{157}$$

so that the general equation (14), $0=\Psi(y_1,\ldots,y_i,x_1,\ldots,x_i)$, may here be put under the form

$$0=\tfrac{1}{2}y_1^2+y_2-f(x_1), \tag{158}$$

and the general partial differential equation (16), which may always be thus written

$$0=\Psi\left(\frac{\delta S}{\delta u_1},\ldots,\frac{\delta S}{\delta u_i},a_1+u_1,\ldots,a_i+u_i\right), \tag{159}$$

becomes in the present example

$$0=\frac{1}{2}\left(\frac{\delta S}{\delta u_1}\right)^2+\frac{\delta S}{\delta u_2}-f(a_1+u_1), \tag{160}$$

and gives

$$\frac{\delta S}{\delta u_1}=\pm\sqrt{2f(a_1+u_1)-2\frac{\delta S}{\delta u_2}}. \tag{161}$$

If we take the upper sign, the complete and general integral of this partial differential equation (161) is given by the following equations:

$$S=\int_0^{u_1}\sqrt{2f(a_1+u_1)-2b_2}\,du_1+b_2u_2+\phi(b_2),\quad 0=u_2-\int_0^{u_1}\frac{du_1}{\sqrt{2f(a_1+u_1)-2b_2}}+\phi'(b_2), \tag{162}$$

$\phi(b_2)$ being an arbitrary function of b_2 and $\phi'(b_2)$ being its derived function, but b_2 being treated as constant in effecting the two definite integrations; and in the present question this arbitrary function $\phi(b_2)$ and therefore also $\phi'(b_2)$ must be supposed to be identically equal to zero, because S vanishes with u_1 and u_2 independently of the auxiliary quantity b_2, which may easily be shown to be equal to $\frac{\delta S}{\delta u_2}$ and to be constant in the progression of $u_1\,u_2\,S$; we may then rigorously determine the form of the principal function S by eliminating b_2 between the two equations

$$u_2=\int_0^{u_1}\frac{du_1}{\sqrt{2f(a_1+u_1)-2b_2}},\quad S=\int_0^{u_1}\frac{2f(a_1+u_1)-b_2}{\sqrt{2f(a_1+u_1)-2b_2}}du_1, \tag{163}$$

which may easily be seen to coincide with the equations (143) and (150).

(Jan. 23rd, 1836.)

As another example,* let

$$\Phi(x_1,x_2,dx_1,dx_2)=e^{2hx_1}\left(h\frac{dx_1^2}{dx_2}+g\,dx_2\right), \tag{164}$$

h and g being any arbitrary constants and e being the napierian base. Then

$$\Phi(a_1,a_2,u_1,u_2)=e^{2ha_1}\left(h\frac{u_1^2}{u_2}+gu_2\right); \tag{165}$$

$$\Phi'(a_1)=2h\Phi,\quad \Phi'(a_2)=0,\quad \Phi'(u_1)=2h\frac{u_1}{u_2}e^{2ha_1},\quad \Phi'(u_2)=e^{2ha_1}\left(g-h\frac{u_1^2}{u_2^2}\right); \tag{166}$$

* [Problem of the fall of a heavy body in a medium resisting as the square of the velocity.]

$$\left.\begin{aligned}&\Phi''(a_1)=4h^2\Phi,\quad \Phi'{,}'(a_1,a_2)=0,\quad \Phi''(a_2)=0,\\ &\Phi'{,}'(a_1,u_1)=4h^2\frac{u_1}{u_2}e^{2ha_1},\quad \Phi'{,}'(a_1,u_2)=2h\left(g-h\frac{u_1^2}{u_2^2}\right)e^{2ha_1},\quad \Phi'{,}'(a_2,u_1)=0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Phi'{,}'(a_2,u_2)=0,\\ &\Phi''(u_1)=\frac{2h}{u_2}e^{2ha_1},\quad \Phi'{,}'(u_1,u_2)=-\frac{2hu_1}{u_2^2}e^{2ha_1},\quad \Phi''(u_2)=\frac{2hu_1^2}{u_2^3}e^{2ha_1};\end{aligned}\right\}\tag{167}$$

$$\frac{u_1^2+u_2^2}{\Phi''(u_1)+\Phi''(u_2)}=-\frac{u_1u_2}{\Phi'{,}'(u_1,u_2)}=\frac{u_2^3}{2h}e^{-2ha_1},\tag{168}$$

and the general approximate expression (132) becomes

$$S=\int_{x_1=a_1,\,x_2=a_2}^{x_1=a_1+u_1,\,x_2=a_2+u_2} e^{2hx_1}\left(h\frac{dx_1^2}{dx_2}+g\,dx_2\right)$$
$$=(1+hu_1+\tfrac{2}{3}h^2u_1^2)\,e^{2ha_1}\left(h\frac{u_1^2}{u_2}+gu_2\right)-\frac{(gu_2^2-hu_1^2)^2}{12u_2}he^{2ha_1}.\tag{169}$$

In this example the general differential equations (7) become

$$e^{2hx_1}\left(h\frac{dx_1^2}{dx_2}+g\,dx_2\right)=d\,.\,e^{2hx_1}\frac{dx_1}{dx_2},\quad 0=d\,.\,e^{2hx_1}\left(g-h\frac{dx_1^2}{dx_2^2}\right),\tag{170}$$

and both agree in giving

$$d\frac{dx_1}{dx_2}-g\,dx_2+h\frac{dx_1^2}{dx_2}=0\tag{171}$$

as the ordinary differential equation of the second order between x_1 and x_2. The second equation (170) gives, as an intermediate integral,

$$e^{2hx_1}\left(g-h\frac{dx_1^2}{dx_2^2}\right)=b_2=\text{const.},\tag{172}$$

b_2 denoting as usual the initial value of $\Phi'(dx_2)$ or of $\dfrac{\delta dS}{\delta dx_2}$; therefore

$$\frac{dx_2}{dx_1}=\pm\sqrt{\frac{h}{g-b_2e^{-hx_1}}},\tag{173}$$

and hence, taking the upper sign,

$$u_2=\sqrt{h}\int_0^{u_1}\frac{du_1}{\sqrt{g-b_2e^{-2h(a_1+u_1)}}}.\tag{174}$$

Also

$$\frac{dS}{dx_2}=e^{2hx_1}\left(g+h\frac{dx_1^2}{dx_2^2}\right)=2ge^{2hx_1}-b_2,\tag{175}$$

therefore

$$S=-b_2u_2+2g\sqrt{h}\int_0^{u_1}\frac{e^{2h(a_1+u_1)}\,du_1}{\sqrt{g-b_2e^{-2h(a_1+u_1)}}},\tag{176}$$

that is, by the expression for u_2,

$$S=\sqrt{h}\int_0^{u_1}\frac{2ge^{2h(a_1+u_1)}-b_2}{\sqrt{g-b_2e^{-2h(a_1+u_1)}}}du_1.\tag{177}$$

The equations (174), (177) are rigorous and the approximate elimination of b_2 between them ought to conduct to the expression (169).

To effect this approximate elimination, we shall first develope the reciprocal of the radical. We have

$$g-b_2e^{-2h(a_1+u_1)}=g-b_2e^{-2ha_1}e^{-2hu_1}=g-b_2e^{-2ha_1}(1-2hu_1+2h^2u_1^2), \quad (178)$$

therefore

$$\begin{aligned}\{g-b_2e^{-2h(a_1+u_1)}\}^{-\frac{1}{2}} &= e^{ha_1}\{ge^{2ha_1}-b_2+2b_2hu_1-2b_2h^2u_1^2\}^{-\frac{1}{2}}\\ &= e^{ha_1}(ge^{2ha_1}-b_2)^{-\frac{1}{2}}\left\{1+\frac{2b_2hu_1}{ge^{2ha_1}-b_2}(1-h_1u_1)\right\}^{-\frac{1}{2}}\\ &= e^{ha_1}(ge^{2ha_1}-b_2)^{-\frac{1}{2}}\left\{1-\frac{b_2hu_1(1-hu_1)}{ge^{2ha_1}-b_2}+\frac{\frac{3}{2}b_2^2h^2u_1^2}{(ge^{2ha_1}-b_2)^2}\right\}\\ &= (g-b_2e^{-2ha_1})^{-\frac{1}{2}}-u_1e^{ha_1}b_2h(ge^{2ha_1}-b_2)^{-\frac{3}{2}}\\ &\quad +u_1^2e^{ha_1}b_2h^2(ge^{2ha_1}+\tfrac{1}{2}b_2)(ge^{2ha_1}-b_2)^{-\frac{5}{2}}; \quad (179)\end{aligned}$$

therefore

$$u_2=\frac{u_1\sqrt{h}}{\sqrt{g-b_2e^{-2ha_1}}}-\frac{u_1^2}{2}e^{-2ha_1}b_2\left(\frac{he^{2ha_1}}{ge^{2ha_1}-b_2}\right)^{\frac{3}{2}}+\frac{u_1^3}{3}e^{-4ha_1}b_2(ge^{2ha_1}+\tfrac{1}{2}b_2)\left(\frac{he^{2ha_1}}{ge^{2ha_1}-b_2}\right)^{\frac{5}{2}}; \quad (180)$$

hence, as a first approximation,

$$\sqrt{\frac{g-b_2e^{-2ha_1}}{h}}=\frac{u_1}{u_2} \quad (181)$$

and so (cf. (172))

$$b_2=\left(g-h\frac{u_1^2}{u_2^2}\right)e^{2ha_1}; \quad (182)$$

and since in this approximation

$$\left(\frac{he^{2ha_1}}{ge^{2ha_1}-b_2}\right)^{\frac{1}{2}}=\frac{u_2}{u_1}, \quad (183)$$

we have as a second approximation

$$u_1\sqrt{\frac{h}{g-b_2e^{-2ha_1}}}=u_2+\frac{(gu_2^2-hu_1^2)u_2}{2u_1}, \quad (184)$$

that is,

$$\left(\frac{u_2}{u_1}\right)^2\frac{g-b_2e^{-2ha_1}}{h}=1+hu_1-\frac{gu_2^2}{u_1}$$

or

$$b_2e^{-2ha_1}=g-h\frac{u_1^2}{u_2^2}\left\{1+hu_1-\frac{gu_2^2}{u_1}\right\}=\left(g-h\frac{u_1^2}{u_2^2}\right)(1+hu_1). \quad (185)$$

Consequently, as a third approximation,

$$\begin{aligned}\frac{u_1}{u_2}\sqrt{\frac{h}{g-b_2e^{-2ha_1}}} &= 1+\frac{gu_2^2-hu_1^2}{2u_1}\left\{1+hu_1+\frac{3}{2u_1}(gu_2^2-hu_1^2)\right\}-\frac{1}{6u_1^2}(gu_2^2-hu_1^2)(3gu_2^2-hu_1^2)\\ &= 1+\frac{gu_2^2-hu_1^2}{2u_1}\left\{1-\frac{hu_1}{6}+\frac{gu_2^2}{2u_1}\right\}, \quad (186)\end{aligned}$$

$$\begin{aligned}\frac{u_2^2}{u_1^2}\frac{g-b_2e^{-2ha_1}}{h} &= 1+\frac{hu_1^2-gu_2^2}{u_1}\left\{1+hu_1+\frac{3}{2u_1}(gu_2^2-hu_1^2)\right\}+\frac{3}{4u_1^2}(gu_2^2-hu_1^2)^2\\ &\quad +\frac{1}{3u_1^2}(gu_2^2-hu_1^2)(3gu_2^2-hu_1^2), \quad (187)\end{aligned}$$

$$\frac{b_2e^{-2ha_1}u_2^2}{gu_2^2-hu_1^2}=1+hu_1+h^2u_1^2+\tfrac{3}{4}h(gu_2^2-hu_1^2)-\frac{h}{3}(3gu_2^2-hu_1^2)=1+hu_1+\tfrac{7}{12}h^2u_1^2-\tfrac{1}{4}ghu_2^2, \quad (188)$$

and
$$b_2 = e^{2ha_1}\left(g - h\frac{u_1^2}{u_2^2}\right)\{1 + hu_1 + \tfrac{7}{12}h^2u_1^2 - \tfrac{1}{4}ghu_2^2\}. \tag{189}$$

Again, we have by what precedes

$$\sqrt{\frac{h}{g - b_2e^{-2h(a_1+u_1)}}} = \sqrt{\frac{h}{g - b_2e^{-2ha_1}}} - u_1b_2e^{-2ha_1}\left(\frac{h}{g - b_2e^{-2ha_1}}\right)^{\frac{3}{2}}$$
$$+ u_1^2b_2e^{-2ha_1}(g + \tfrac{1}{2}b_2e^{-2ha_1})\left(\frac{h}{g - b_2e^{-2ha_1}}\right)^{\frac{5}{2}}; \tag{190}$$

also
$$2ge^{2hu_1} - b_2e^{-2ha_1} = 2g - b_2e^{-2ha_1} + 4ghu_1 + 4gh^2u_1^2; \tag{191}$$

therefore, by (177) and (180),
$$Se^{-2ha_1} = u_2(2g - b_2e^{-2ha_1}) + (2ghu_1^2 + \tfrac{4}{3}gh^2u_1^3)\left(\frac{h}{g - b_2e^{-2ha_1}}\right)^{\frac{1}{2}}$$
$$-\tfrac{4}{3}ghu_1^3b_2e^{-2ha_1}\left(\frac{h}{g - b_2e^{-2ha_1}}\right)^{\frac{3}{2}}, \tag{192}$$

in which
$$u_2(2g - b_2e^{-2ha_1}) = gu_2 + h\frac{u_1^2}{u_2} + \left(\frac{hu_1^2}{u_2} - gu_2\right)(hu_1 + \tfrac{7}{12}h^2u_1^2 - \tfrac{1}{4}ghu_2^2), \tag{193}$$

$$2ghu_1^2(1 + \tfrac{2}{3}hu_1)\sqrt{\frac{h}{g - b_2e^{-2ha_1}}} = 2ghu_1u_2\left(1 + \tfrac{1}{6}hu_1 + \tfrac{1}{2}g\frac{u_2^2}{u_1}\right), \tag{194}$$

and
$$-\tfrac{4}{3}ghu_1^3b_2e^{-2ha_1}\left(\frac{h}{g - b_2e^{-2ha_1}}\right)^{\frac{3}{2}} = -\tfrac{4}{3}ghu_2(gu_2^2 - hu_1^2). \tag{195}$$

Therefore, adding the three last expressions, we find
$$Se^{-2ha_1} = \left(\frac{hu_1^2}{u_2} + gu_2\right)(1 + hu_1) + \frac{7}{12}\frac{h^3u_1^4}{u_2} + \tfrac{5}{6}gh^2u_1^2u_2 - \tfrac{1}{12}g^2hu_2^3, \tag{196}$$

agreeing with (169).

It would however have been simpler, in this example, to have put the differential equation of the second order (171) under the form:
$$x_1'' = g - hx_1'^2, \tag{197}$$

and then to have deduced from it by differentiation
$$x_1''' = -2hx_1'x_1'' = -2hx_1'(g - hx_1'^2), \tag{198}$$

and therefore
$$a_1'' = g - ha_1'^2, \tag{199}$$
$$a_1''' = -2ha_1'(g - ha_1'^2), \tag{200}$$

x_1', x_1'', x_1''' being differential coefficients of x_1 considered as a function of x_2, and a_1', a_1'', a_1''' being their values when $x_2 = a_2$, $x_1 = a_1$. For thus we should obtain, by Taylor's theorem,
$$u_1 = a_1'u_2 + \tfrac{1}{2}a_1''u_2^2 + \tfrac{1}{6}a_1'''u_2^3, \tag{201}$$

that is,
$$u_1 = a_1'u_2 + \tfrac{1}{2}u_2^2(g - ha_1'^2)(1 - \tfrac{2}{3}ha_1'u_2), \tag{202}$$

which gives as a first approximation
$$a_1' = \frac{u_1}{u_2}; \tag{203}$$

as a second approximation

$$a_1' = \frac{u_1}{u_2} - \frac{u_2}{2}\left(g - h\frac{u_1^2}{u_2^2}\right); \tag{204}$$

therefore

$$g - ha_1'^2 = \left(g - h\frac{u_1^2}{u_2^2}\right)(1 + hu_1), \tag{205}$$

and

$$(1 + hu_1)(1 - \tfrac{2}{3}ha_1'u_2) = 1 + \tfrac{1}{3}hu_1. \tag{206}$$

As a third approximation

$$a_1' = \frac{u_1}{u_2} - \frac{u_2}{2}\left(g - h\frac{u_1^2}{u_2^2}\right)(1 + \tfrac{1}{3}hu_1). \tag{207}$$

Also

$$S = e^{2ha_1}\int_0^{u_2} e^{2hu_1}(hx_1'^2 + g)\,du_2, \tag{208}$$

in which

$$x_1' = a_1' + a_1''u_2 + \tfrac{1}{2}a_1'''u_2^2 \tag{209}$$

and

$$e^{2hu_1} = 1 + 2hu_1 + 2h^2u_1^2 = 1 + 2ha_1'u_2 + (2h^2a_1'^2 + ha_1'')u_2^2; \tag{210}$$

therefore

$$e^{2hu_1}(hx_1'^2 + g) = (ha_1'^2 + g)\{1 + 2ha_1'u_2 + (2h^2a_1'^2 + ha_1'')u_2^2\} + 2ha_1'a_1''\{u_2 + 2ha_1'u_2^2\} + \\ + h(a_1''^2 + a_1'a_1''')u_2^2, \tag{211}$$

and

$$\begin{aligned} Se^{-2ha_1} &= (ha_1'^2 + g)\left\{u_2 + ha_1'u_2^2 + \frac{h}{3}(2ha_1'^2 + a_1'')u_2^3\right\} + ha_1'a_1''\{u_2^2 + \tfrac{4}{3}ha_1'u_2^3\} + \frac{h}{3}(a_1''^2 + a_1'a_1''')u_2^3 \\ &= (ha_1'^2 + g)u_2 + 2gha_1'u_2^2 + \tfrac{1}{3}hu_2^3\{(g + ha_1'^2)^2 + g^2 - h^2a_1'^4\} \\ &= (ha_1'^2 + g)u_2(1 + \tfrac{2}{3}ghu_2^2) + 2gha_1'u_2^2, \end{aligned} \tag{212}$$

in which, by (207),

$$\left.\begin{aligned} (ha_1'^2 + g)u_2 &= \frac{hu_1^2}{u_2} + gu_2 - hu_1u_2\left(g - h\frac{u_1^2}{u_2^2}\right)(1 + \tfrac{1}{3}hu_1) + \frac{hu_2^3}{4}\left(g - h\frac{u_1^2}{u_2^2}\right)^2, \\ \tfrac{2}{3}ghu_2^3(ha_1'^2 + g) &= \tfrac{2}{3}ghu_2(hu_1^2 + gu_2^2), \\ 2gha_1'u_2^2 &= 2ghu_1u_2 - ghu_2^3\left(g - h\frac{u_1^2}{u_2^2}\right); \end{aligned}\right\} \tag{213}$$

therefore, adding these three last expressions, we find for the principal function S this expression, agreeing with (196) and with (169):

$$S = \left\{\left(\frac{hu_1^2}{u_2} + gu_2\right)(1 + hu_1) + \frac{7}{12}\frac{h^3u_1^4}{u_2} + \tfrac{5}{6}gh^2u_1^2u_2 - \tfrac{1}{12}g^2hu_2^3\right\}e^{2ha_1}. \tag{214}$$

It is worth observing that the differential equation of the second order (197) is that of the fall of a heavy body in a medium which resists as the square of the velocity; so that the integral of this equation can be rigorously expressed by the method of the principal function.

As a third example* put

$$dS = \Phi(x_1, x_2, dx_1, dx_2) = \frac{dx_1^2}{2dx_2} + gx_1dx_2, \tag{215}$$

g being any arbitrary constant. Then

$$\Phi(a_1, a_2, u_1, u_2) = \frac{u_1^2}{2u_2} + ga_1u_2; \tag{216}$$

* [Particular case of example on page 348.]

therefore

$$\Phi'(a_1)=gu_2,\quad \Phi'(a_2)=0,\quad \Phi'(u_1)=\frac{u_1}{u_2},\quad \Phi'(u_2)=ga_1-\frac{u_1^2}{2u_2^2}; \tag{217}$$

and

$$\left.\begin{aligned}\Phi''(a_1)=0,\quad \Phi','(a_1,a_2)=0,\quad \Phi''(a_2)=0,\\ \Phi','(a_1,u_1)=0,\quad \Phi','(a_1,u_2)=g,\quad \Phi','(a_2,u_1)=0,\quad \Phi','(a_2,u_2)=0,\\ \Phi''(u_1)=\frac{1}{u_2},\quad \Phi','(u_1,u_2)=-\frac{u_1}{u_2^2},\quad \Phi''(u_2)=\frac{u_1^2}{u_2^3};\end{aligned}\right\} \tag{218}$$

hence

$$\frac{u_1^2+u_2^2}{\Phi''(u_1)+\Phi''(u_2)}=-\frac{u_1u_2}{\Phi','(u_1,u_2)}=u_2^3, \tag{219}$$

so that the general approximate expression (132) becomes in this example

$$S=\int_{x_1=a_1,\,x_2=a_2}^{x_1=a_1+u_1,\,x_2=a_2+u_2}\frac{dx_1^2}{2dx_2}+gx_1dx_2=\frac{u_1^2}{2u_2}+ga_1u_2+\tfrac{1}{2}gu_1u_2-\tfrac{1}{24}g^2u_2^3. \tag{220}$$

In this particular example,

$$y_1=\frac{\delta dS}{\delta dx_1}=\frac{dx_1}{dx_2}, \tag{221}$$

$$y_2=\frac{\delta dS}{\delta dx_2}=gx_1-\frac{dx_1^2}{2dx_2^2}, \tag{222}$$

also

$$\frac{\delta dS}{\delta x_1}=gdx_2, \tag{223}$$

and

$$\frac{\delta dS}{\delta x_2}=0; \tag{224}$$

thus the differential equations (7) become here

$$d\frac{dx_1}{dx_2}=gdx_2,\quad d\left(gx_1-\frac{1}{2}\frac{dx_1^2}{dx_2^2}\right)=0, \tag{225}$$

and they concur in giving as the complete integral with two arbitrary constants:

$$x_1=a_1+a_1'(x_2-a_2)+\tfrac{1}{2}g(x_2-a_2)^2, \tag{226}$$

that is,

$$u_1=a_1'u_2+\tfrac{1}{2}gu_2^2. \tag{227}$$

Hence, rigorously,

$$a_1'=\frac{u_1}{u_2}-\tfrac{1}{2}gu_2. \tag{228}$$

Also

$$\frac{dx_1}{dx_2}=a_1'+gu_2 \tag{229}$$

and

$$\frac{1}{2}\left(\frac{dx_1}{dx_2}\right)^2+gx_1=ga_1+\tfrac{1}{2}a_1'^2+2ga_1'u_2+g^2u_2^2; \tag{230}$$

therefore

$$S=\int_0^{u_2}\left\{\frac{1}{2}\left(\frac{dx_1}{dx_2}\right)^2+gx_1\right\}du_2=(ga_1+\tfrac{1}{2}a_1'^2)\,u_2+ga_1'u_2^2+\tfrac{1}{3}g^2u_2^3 \tag{231}$$

rigorously, or

$$\begin{aligned} S&=ga_1u_2+\tfrac{1}{2}u_2(a_1'+gu_2)^2-\tfrac{1}{6}g^2u_2^3\\ &=ga_1u_2+\tfrac{1}{2}u_2\left(\frac{u_1}{u_2}+\tfrac{1}{2}gu_2\right)^2-\tfrac{1}{6}g^2u_2^3\\ &=ga_1u_2+\frac{u_1^2}{2u_2}+\tfrac{1}{2}gu_1u_2-\tfrac{1}{24}g^2u_2^3 \end{aligned} \tag{232}$$

rigorously, as deduced in (220) from the generally approximate expression (132), which in this example is more than approximate.