VIII.

ON THE COMPOSITION OF FORCES

Read November 8, 1841.

[*Proc. Roy. Irish Acad.* Vol. II (1840–44), pp. 166–168.]

Two rectangular forces, x and y, being supposed to be equivalent to a single resultant force p, inclined at an angle v to the force x, it is required to determine the law of the dependence of this angle on the ratio of the two component forces x and y.

Denoting by p' any other single force, intermediate between x and y and inclined to x at an angle v', which we shall suppose to be greater than v; and denoting by x' and y' the rectangular components of this new force p' in the directions of x and y, we may, by easy decompositions and recompositions, obtain a new pair of rectangular forces, x'' and y'', which are together equivalent to p' and have for components

$$x'' = \frac{x}{p}x' + \frac{y}{p}y';$$

$$y'' = \frac{x}{p}y' - \frac{y}{p}x';$$

the direction of x'' coinciding with that of p, but the direction of y'' being perpendicular thereto. Hence

$$\frac{y''}{x''} = \frac{xy' - yx'}{xx' + yy'};$$

that is,

$$\tan^{-1}\frac{y''}{x''} = \tan^{-1}\frac{y'}{x'} - \tan^{-1}\frac{y}{x};$$

or, finally,

$$f(v'-v) = f(v') - f(v), \tag{A}$$

at least for values of v, v' and $v'-v$, which are each greater than 0, and less than $\frac{\pi}{2}$; if f be a function so chosen that the equation

$$\frac{y}{x} = \tan f(v)$$

expresses the sought law of connexion between the ratio $\frac{y}{x}$ and the angle v. The functional equation (A) gives

$$f(mv) = mf(v) = \frac{m}{n}f(nv),$$

m and n being any whole numbers; and the case of equal components gives evidently

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4};$$

hence

$$f\left(\frac{m}{n}\frac{\pi}{4}\right) = \frac{m}{n}\frac{\pi}{4},$$

and ultimately,

$$f(v) = v, \tag{B}$$

because it is evident, by the nature of the question, that while v increases from 0 to $\frac{\pi}{2}$, the function $f(v)$ increases therewith, and therefore could not be equal thereto for all values of v commensurable with $\frac{\pi}{4}$, unless it had the same property also for all intermediate incommensurable values. We find, therefore, that for all values of the component forces x and y, the equation

$$\frac{y}{x} = \tan v \tag{C}$$

holds good; that is, the resultant force coincides *in direction* with the diagonal of the rectangle constructed with lines representing x and y as sides.

The other part of the known law of the composition of forces, namely, that this resultant is represented also *in magnitude* by the same diagonal, may easily be proved by the process of the *Mécanique Céleste*, which, in the present notation, corresponds to making

$$x' = x, \quad y' = y, \quad x'' = p,$$

and therefore gives

$$p = \frac{x^2 + y^2}{p}, \quad p^2 = x^2 + y^2.$$

But the demonstration above assigned for the law of the *direction* of the resultant appears to Sir William Hamilton to be new.*

* [For an account of the history of statical proofs of the parallelogram of forces, with references to the literature of the subject, see Voss, *Encyklopädie der Mathematischen Wissenschaften*, IV. 1, pp. 43–46.]