VII.

CORRESPONDENCE WITH J. W. LUBBOCK

[1837.]

[In this correspondence Hamilton numbered his own letters I, II, III and IV. Lubbock's letters are not numbered. We mark them here A, B, C, D.

CONTENTS

A. *J. W. Lubbock* to Sir W. R. Hamilton.*

London, 24*th* *July*, 1837.

MY DEAR SIR,

You are so familiar with all questions in Physical Optics that I take the liberty of troubling you with a few remarks upon a point of some importance and I shall be much obliged to you to tell me whether you find them correct and if not if you will look at them with indulgence.†

Have you thought more about the Lunar Theory and have you pursued at all the application of your general views to this question? I want to draw up some report for the Association about the construction of Tables. Airy appears to think that Sir W. Hamilton...[part of letter missing]...the labour of getting out terms in this manner would be even greater than that of pursuing theory to the requisite extent. In the latter channel my patience is exhausted and so also I fear is that of M. Pontécoulant who is not easily frightened. But we continue ever and anon to find terms differing from M. Plana's figures and there can be no doubt that until they have all been carefully verified they are not to be implicitly relied on. No one has yet succeeded in obtaining by *the method of the variation of constants* the exact numerical coefficient of the term

* [Sir J. W. Lubbock, banker & mathematician, afterwards the third baronet, had been vice-president and treasurer of the Royal Society. He was the father of the first Lord Lubbock.]

† [There is no trace of the statement of this optical problem among the correspondence.]

www.rcin.org.pl

in R multiplied by m^3 for the angle $2gt-2ct$ Arg. 77. Can you find time to put this matter straight? I think it might throw light upon the manner of employing the equations referred to in *Comptes Rendus*.

Pray let me hear from you & in the mean time

I remain, dear Sir,
Yours very faithfully
J. W. LUBBOCK.

I. *Sir W. R. Hamilton to J. W. Lubbock.*

Letter to Mr. Lubbock on the Calculus of Principal Relations.

Observatory, *Aug*t. 29, 1837.

MY DEAR SIR,

I am desirous to write to you upon a subject which has some connexion with the theory of the Moon, though not so close an one as to make my letter an answer to yours. Having lately resumed the consideration of my Method of Principal Relations, and brought it a little nearer to a state in which it may be practically applied, I wish to give you some sort of sketch of the way in which I hope, at a future but perhaps not distant time, to make such practical application.

(1.) To take one of the very simplest instances that can be employed to illustrate my method, let us consider the ordinary differential equation of the second order

$$x_t''=\alpha x_t; \tag{1}$$

in which x_t is a function of t and x_t'' is its differential coefficient of the second order, while α is a small real constant, positive or negative. You see at once that the integral of this equation is, in finite terms,

$$x_t=\tfrac{1}{2}(e^{t\sqrt{\alpha}}+e^{-t\sqrt{\alpha}})\,x_0+\frac{1}{2\sqrt{\alpha}}(e^{t\sqrt{\alpha}}-e^{-t\sqrt{\alpha}})\,x_0', \tag{2}$$

or, when developed according to ascending powers of α,

$$x_t=x_0\left(1+\alpha\frac{t^2}{2}+\alpha^2\frac{t^4}{24}+\text{etc.}\right)+x_0'\left(t+\alpha\frac{t^3}{6}+\alpha^2\frac{t^5}{120}+\text{etc.}\right). \tag{3}$$

And if we had supposed the finite form (2) unknown and had begun by assuming the expression

$$x_t=X_t+\alpha\overset{1}{X}_t+\alpha^2\overset{2}{X}_t+\alpha^3\overset{3}{X}_t+\text{etc.}, \tag{4}$$

in which X_t, $\overset{1}{X}_t$, $\overset{2}{X}_t$, etc. are unknown functions of t, we should then have had the series of differential equations

$$X_t''=0,\quad \overset{1}{X}{}_t''=X_t,\quad \overset{2}{X}{}_t''=\overset{1}{X}_t,\quad \text{etc.}, \tag{5}$$

which would have given, successively,

$$\left.\begin{aligned} &X_t=a+bt,\\ &\overset{1}{X}_t=\overset{1}{a}+\overset{1}{b}t+\tfrac{1}{2}at^2+\tfrac{1}{6}bt^3,\\ &\overset{2}{X}_t=\overset{2}{a}+\overset{2}{b}t+\tfrac{1}{2}\overset{1}{a}t^2+\tfrac{1}{6}\overset{1}{b}t^3+\tfrac{1}{24}at^4+\tfrac{1}{120}bt^5,\\ &\text{etc.,}\end{aligned}\right\} \tag{6}$$

www.rcin.org.pl

and therefore

$$\left.\begin{aligned} x_t = a + bt &+ \alpha\,(\overset{1}{a} + \overset{1}{b}t + \tfrac{1}{2}at^2 + \tfrac{1}{6}bt^3) \\ &+ \alpha^2\,(\overset{2}{a} + \overset{2}{b}t + \tfrac{1}{2}\overset{1}{a}t^2 + \tfrac{1}{6}\overset{1}{b}t^3 + \tfrac{1}{24}at^4 + \tfrac{1}{120}bt^5) \\ &+ \text{etc.}: \end{aligned}\right\} \tag{7}$$

an expression which gives

$$\left.\begin{aligned} &x'_t = b + \alpha\,(\overset{1}{b} + at + \tfrac{1}{2}bt^2) + \alpha^2\,(\overset{2}{b} + \overset{1}{a}t + \tfrac{1}{2}\overset{1}{b}t^2 + \tfrac{1}{6}at^3 + \tfrac{1}{24}bt^4) + \text{etc.}, \\ &x'_0 = b + \alpha\overset{1}{b} + \alpha^2\overset{2}{b} + \text{etc.}, \quad x_0 = a + \alpha\overset{1}{a} + \alpha^2\overset{2}{a} + \text{etc.}, \end{aligned}\right\} \tag{8}$$

and therefore, as before,

$$x_t = x_0\left(1 + \alpha\frac{t^2}{2} + \alpha^2\frac{t^4}{24} + \text{etc.}\right) + x'_0\left(t + \alpha\frac{t^3}{6} + \alpha^2\frac{t^5}{120} + \text{etc.}\right). \tag{3}$$

All these things are perceived without the slightest difficulty, but for that reason they are only the more proper to illustrate the following new mode of proceeding.

(2.) Let s denote the definite integral

$$s = \int_0^t \frac{x_t'^2 + \alpha x_t^2}{2}\,dt; \tag{9}$$

and let its variation be taken without varying t or dt: we find, by the usual rules of the Calculus of Variations,

$$\delta s = x'_t\,\delta x_t - x'_0\,\delta x_0, \tag{10}$$

if the function x_t be such as to satisfy the differential equation (1); and, if this last condition be not satisfied, then

$$\delta s = x'_t\,\delta x_t - x'_0\,\delta x_0 + \int_0^t (\alpha x_{t,} - x''_{t,})\,\delta x_{t,}\,dt,; \tag{11}$$

$t,$ being here marked with a lower accent merely to assist us in remembering that it flows, or varies differentially, from 0 to t, which latter quantity may be viewed for the present as an arbitrary constant. If we put, for abridgment,

$$b = \frac{x_t - x_0}{t}, \tag{12}$$

it is evident that $x_0 + bt,$ is an approximate expression for the function $x_{t,}$, because, by (1), it would be an exact expression for that function if the small quantity α were to vanish; treating, therefore, α as infinitely small of the first order, we may calculate the definite integral s with only an error of the second order (in virtue of a principle contained in the equation (11)), by simply making in (9)

$$x_{t,} = x_0 + bt,, \tag{13}$$

which gives

$$x'_{t,} = b, \tag{14}$$

and finally

$$s = \int_0^t \frac{b^2 + \alpha\,(x_0 + bt,)^2}{2}\,dt,$$

$$= \tfrac{1}{2}b^2\left(t + \alpha\frac{t^3}{3}\right) + \tfrac{1}{2}\alpha b x_0 t^2 + \tfrac{1}{2}\alpha x_0^2 t. \tag{15}$$

www.rcin.org.pl

In the same order of approximation we may employ the expression (10) for the variation of this integral s; and therefore, by observing that the equation (12) gives the rigorous relation

$$x_t = x_0 + bt, \tag{16}$$

we find

$$(x'_t - x'_0)\,\delta x_0 + t x'_t \delta b = \alpha\,(x_0 t + \tfrac{1}{2}bt^2)\,\delta x_0 + \left\{\tfrac{1}{2}\alpha x_0 t^2 + b\left(t + \alpha\frac{t^3}{3}\right)\right\}\delta b, \tag{17}$$

an equation which resolves itself into the two following, on account of the independence of δx_0 and δb,

$$x'_t - x'_0 = \alpha\,(x_0 t + \tfrac{1}{2}bt^2), \quad t x'_t = b\,(t + \tfrac{1}{3}\alpha t^3) + \tfrac{1}{2}\alpha x_0 t^2; \tag{18}$$

which gives, by elimination of x'_t,

$$x'_0 = b\,(1 - \tfrac{1}{6}\alpha t^2) - \tfrac{1}{2}\alpha x_0 t, \tag{19}$$

and therefore

$$\frac{x_t - x_0}{t} = b = x'_0\,(1 + \tfrac{1}{6}\alpha t^2) + \tfrac{1}{2}\alpha x_0 t, \tag{20}$$

that is,

$$x_t = x_0\,(1 + \tfrac{1}{2}\alpha t^2) + x'_0\,(t + \tfrac{1}{6}\alpha t^3). \tag{21}$$

And, on account of the arbitrariness of t, this must be, to the accuracy of the first order inclusive, an expression for the sought function x_t, or an approximate expression for the sought integral of the proposed differential equation of the second order, namely,

$$x''_t = \alpha x_t. \tag{1}$$

Accordingly, when α^2 is neglected, the expression (21) with the two arbitrary constants x_0 and x'_0 satisfies that differential equation (1) and agrees with the expressions (2) and (3).

(3.) The process thus gone through is certainly less simple, as well as less obvious, than that which consists in passing from the first approximate expression

$$x_t = X_t = a + bt \tag{22}$$

to the improved expression

$$x_t = X_t + \alpha \overset{1}{X}_t = a + bt + \alpha\,(\overset{1}{a} + \overset{1}{b}t + \tfrac{1}{2}at^2 + \tfrac{1}{6}bt^3); \tag{23}$$

which may, because α^2 is neglected, be written thus,

$$x_t = (a + \alpha \overset{1}{a})\left(1 + \frac{\alpha}{2}t^2\right) + (b + \alpha \overset{1}{b})\left(t + \frac{\alpha}{6}t^3\right), \tag{24}$$

or finally

$$x_t = x_0\left(1 + \frac{\alpha}{2}t^2\right) + x'_0\left(t + \frac{\alpha}{6}t^3\right): \tag{21}$$

the passage from (22) to (23) being made by integrating the very simple differential equation of the 2nd order

$$\overset{1}{X}{}''_t = X_t, \tag{25}$$

namely the 2nd equation of the series (5). Yet it is important to observe that the very different process of the article (2.), which conducts to the same improved approximate expression (21), attains that end without the integration of any equation of the second order, and indeed (to speak properly) *without the integration of any differential equation whatever*, through the medium of a single *definite integral* (9) and of the expression (10) for the *variation* of that integral. Before I comment further on this new process (2.), I shall extend it so as to obtain, by a definite

www.rcin.org.pl

integral alone and without the integration of any differential equation, an improved expression for x_t, correct to the accuracy of terms proportional to the cube of α inclusive, namely,

$$x_t = x_0\left(1+\alpha\frac{t^2}{2}+\alpha^2\frac{t^4}{24}+\alpha^3\frac{t^6}{720}\right)+x'_0\left(t+\alpha\frac{t^3}{6}+\alpha^2\frac{t^5}{120}+\alpha^3\frac{t^7}{5040}\right), \tag{26}$$

which would, in the ordinary course of procedure, be obtained by integrating the two differential equations

$$\overset{2}{X}{}''_t = \overset{1}{X}_t, \quad \overset{3}{X}{}''_t = \overset{2}{X}_t, \tag{27}$$

after the form of $\overset{1}{X}_t$ had been deduced from that of the first approximate expression X_t by integration of the equation (25).

(4.) In this extension which we are about to make of the process of the article (2.), we are to start with the two equations

$$x_t = x_0(1+\tfrac{1}{2}\alpha t^2)+b(t+\tfrac{1}{6}\alpha t^3), \tag{28}$$

$$x_{t,} = x_0(1+\tfrac{1}{2}\alpha t_,^2)+b(t_,+\tfrac{1}{6}\alpha t_,^3), \tag{29}$$

of which the first is to be considered as rigorous for some assumed value of t, while the second is not rigorous but has its error very small and of the same order as α^2 for every value t intermediate between 0 and t. We are next to calculate, by means of this approximate expression (29), the definite integral (9), which we know must thus be found with only an error of the order of α^4; and in this way we obtain

$$x'_{t,} = b(1+\tfrac{1}{2}\alpha t_,^2)+\alpha x_0 t_,, \tag{30}$$

$$\begin{aligned} s = \int_0^t &\{\tfrac{1}{2}b^2(1+2\alpha t_,^2+\tfrac{7}{12}\alpha^2 t_,^4+\tfrac{1}{36}\alpha^3 t_,^6)+bx_0(2\alpha t_,+\tfrac{7}{6}\alpha^2 t_,^3+\tfrac{1}{12}\alpha^3 t_,^5) \\ &+\tfrac{1}{2}x_0^2(\alpha+2\alpha^2 t_,^2+\tfrac{1}{4}\alpha^3 t_,^4)\}\,dt, \\ = \tfrac{1}{2}&b^2(t+\tfrac{2}{3}\alpha t^3+\tfrac{7}{60}\alpha^2 t^5+\tfrac{1}{252}\alpha^3 t^7)+bx_0(\alpha t^2+\tfrac{7}{24}\alpha^2 t^4+\tfrac{1}{72}\alpha^3 t^6) \\ &+\tfrac{1}{2}x_0^2(\alpha t+\tfrac{2}{3}\alpha^2 t^3+\tfrac{1}{20}\alpha^3 t^5), \end{aligned} \tag{31}$$

$$\begin{aligned} \delta s = &\{x_0(\alpha t+\tfrac{2}{3}\alpha^2 t^3+\tfrac{1}{20}\alpha^3 t^5)+b(\alpha t^2+\tfrac{7}{24}\alpha^2 t^4+\tfrac{1}{72}\alpha^3 t^6)\}\,\delta x_0 \\ &+\{x_0(\alpha t^2+\tfrac{7}{24}\alpha^2 t^4+\tfrac{1}{72}\alpha^3 t^6)+b(t+\tfrac{2}{3}\alpha t^3+\tfrac{7}{60}\alpha^2 t^5+\tfrac{1}{252}\alpha^3 t^7)\}\,\delta b; \end{aligned} \tag{32}$$

and since this variation δs must coincide by (10) with the following,

$$x'_t\delta x_t - x'_0\delta x_0 = \{x'_t(1+\tfrac{1}{2}\alpha t^2)-x'_0\}\,\delta x_0 + x'_t(t+\tfrac{1}{6}\alpha t^3)\,\delta b, \tag{33}$$

we have, by comparing the coefficients of δx_0 and δb, the two equations following,

$$\left.\begin{aligned} x'_t(1+\tfrac{1}{2}\alpha t^2)-x'_0 &= x_0(\alpha t+\tfrac{2}{3}\alpha^2 t^3+\tfrac{1}{20}\alpha^3 t^5)+b(\alpha t^2+\tfrac{7}{24}\alpha^2 t^4+\tfrac{1}{72}\alpha^3 t^6), \\ x'_t(t+\tfrac{1}{6}\alpha t^3) &= x_0(\alpha t^2+\tfrac{7}{24}\alpha^2 t^4+\tfrac{1}{72}\alpha^3 t^6)+b(t+\tfrac{2}{3}\alpha t^3+\tfrac{7}{60}\alpha^2 t^5+\tfrac{1}{252}\alpha^3 t^7); \end{aligned}\right\} \tag{34}$$

between which and (28) we are to eliminate x'_t and b, that we may obtain an expression of the sought kind for x_t as a function of t and of the two initial data x_0 and x'_0, which shall err only by a quantity of the same order as α^4. The elimination may conveniently be conducted thus:—multiplying the approximate expression (30) for $x'_{t,}$, by the coefficient of x'_t in the second equation (34), after changing t to $t_,$, we find

$$x'_{t,}(t_,+\tfrac{1}{6}\alpha t_,^3) = x_0(\alpha t_,^2+\tfrac{1}{6}\alpha^2 t_,^4)+b(t_,+\tfrac{2}{3}\alpha t_,^3+\tfrac{1}{12}\alpha^2 t_,^5), \tag{35}$$

www.rcin.org.pl

so that this 2nd equation (34) may be transformed as follows,

$$x_t' = b\left(1+\tfrac{1}{2}\alpha t^2\right)+\alpha x_0 t+\frac{1}{t+\frac{1}{6}\alpha t^3}\{x_0\left(\tfrac{1}{8}\alpha^2t^4+\tfrac{1}{72}\alpha^3t^6\right)+b\left(\tfrac{1}{30}\alpha^2t^5+\tfrac{1}{252}\alpha^3t^7\right)\}, \qquad (36)$$

and therefore the 1st equation (34) gives

$$x_0' = b\left(1+\alpha t^2+\tfrac{1}{4}\alpha^2t^4\right)+x_0\left(\alpha t+\tfrac{1}{2}\alpha^2t^3\right)+\frac{1+\frac{1}{2}\alpha t^2}{1+\frac{1}{6}\alpha t^2}\{x_0\left(\tfrac{1}{8}\alpha^2t^3+\tfrac{1}{72}\alpha^3t^5\right)+b\left(\tfrac{1}{30}\alpha^2t^4+\tfrac{1}{252}\alpha^3t^6\right)\}$$
$$-b\left(\alpha t^2+\tfrac{7}{24}\alpha^2t^4+\tfrac{1}{72}\alpha^3t^6\right)-x_0\left(\alpha t+\tfrac{2}{3}\alpha^2t^3+\tfrac{1}{20}\alpha^3t^5\right), \qquad (37)$$

that is, neglecting α^4,

$$\begin{aligned} x_0' &= b\left(1-\tfrac{1}{24}\alpha^2t^4-\tfrac{1}{72}\alpha^3t^6\right)-x_0\left(\tfrac{1}{6}\alpha^2t^3+\tfrac{1}{20}\alpha^3t^5\right)\\ &\quad+\left(1+\tfrac{1}{3}\alpha t^2\right)\{b\left(\tfrac{1}{30}\alpha^2t^4+\tfrac{1}{252}\alpha^3t^6\right)+x_0\left(\tfrac{1}{8}\alpha^2t^3+\tfrac{1}{72}\alpha^3t^5\right)\}\\ &= b\left(1-\tfrac{1}{120}\alpha^2t^4+\tfrac{1}{840}\alpha^3t^6\right)+x_0\left(-\tfrac{1}{24}\alpha^2t^3+\tfrac{1}{180}\alpha^3t^5\right), \end{aligned} \qquad (38)$$

an equation which gives

$$b = x_0'\left(1+\tfrac{1}{120}\alpha^2t^4-\tfrac{1}{840}\alpha^3t^6\right)+x_0\left(\tfrac{1}{24}\alpha^2t^3-\tfrac{1}{180}\alpha^3t^5\right), \qquad (39)$$

and therefore, by (28), the expression (26) results, namely,

$$x_t = x_0\left(1+\tfrac{1}{2}\alpha t^2+\tfrac{1}{24}\alpha^2t^4+\tfrac{1}{720}\alpha^3t^6\right)+x_0'\left(t+\tfrac{1}{6}\alpha t^3+\tfrac{1}{120}\alpha^2t^5+\tfrac{1}{5040}\alpha^3t^7\right). \qquad (26)$$

And a repetition of the same process of correction would give an expression for x_t which should err only by quantities of the same order as α^8 and so on for ever, the accuracy being *doubled* by every successive repetition.

(5.) On reviewing the foregoing process of successive and indefinite approximation, it will be perceived to be deduced, chiefly, from these two principles: First, that if the definite integral s be rigorously calculated by the formula (9), the function x_t on which it depends being such as to satisfy rigorously the differential equation (1), then the coefficients of the variation of that definite integral will be assigned rigorously by the formula (10); and second, that if, in calculating s, we employ a function which coincides rigorously with x_t at the limits of the integration and which errs only by a quantity of the order of α^n between those limits, the resulting value of s will err only by a quantity of the order of α^{2n}. The first of these principles, generalised, has long since conducted me to finite expressions for the integrals of the most important systems of differential equations in Optics and Dynamics; and a generalisation of the second of the same principles will lead, I hope, to processes of approximation practically useful. I have indeed already exemplified the theoretical possibility of employing a principle closely connected with it, so as to obtain approximate results of which the accuracy goes on increasing in geometrical progression, (namely, in the 26th and 27th articles* of my Second Essay on a General Method in Dynamics, *Phil. Trans.* Part I for 1835, pages 122, 123, 124); but because the extreme values of the variables were, in that former process of approximation, introduced *explicitly* into the expression of the Principal Function, an introduction which it would be almost impracticable to effect in the theory of the perturbations of the Moon or Planets, it was (in a manner) necessary to adopt a more convenient mode of proceeding: and the mode which has been sketched in the present letter is one of the most convenient among those which have since occurred to me.

* [Pp. 189–192 of this volume.]

www.rcin.org.pl

(**6.**) Whatever form may be chosen of my Method of the Principal Function, when it is employed to improve an approximation, it bears a close analogy to the Method of the Variation of Parameters. Thus from the expression

$$x_t = x_0 + bt, \tag{16}$$

which is subject to an error of the 1st order when b is treated as equal to the initial datum x'_0, I passed, in the present letter, to the more exact expression

$$x_t = x_0(1 + \tfrac{1}{2}\alpha t^2) + x'_0(t + \tfrac{1}{6}\alpha t^3), \tag{21}$$

by finding that the expression (16) might be retained with only an error of the 2nd order, provided that b was treated as variable and was expressed as follows:

$$b = x'_0(1 + \tfrac{1}{6}\alpha t^2) + \tfrac{1}{2}\alpha x_0 t. \tag{20}$$

And again from the expression

$$x_t = x_0(1 + \tfrac{1}{2}\alpha t^2) + b(t + \tfrac{1}{6}\alpha t^3), \tag{28}$$

in which b was known to be nearly equal to the constant x'_0, the more exact expression (26) was deduced by showing that b was more exactly equal to the function (39). But there is this essential difference between Lagrange's Method and mine, that *he* determined his varying parameter so as to make his expressions true for two moments infinitely near, *I* for two moments separated by a finite interval. Thus, while I keep the parameter x_0 unchanged and only alter b in (16) from the constant x'_0 to the function (20), Lagrange would have altered *both* x_0 and b or would have employed the system of the two expressions

$$x_t = a + bt, \quad x'_t = b, \tag{40}$$

which give

$$a = x_t - tx'_t; \tag{41}$$

and thus he would have decomposed the differential equation of the 2nd order

$$x''_t = \alpha x_t \tag{1}$$

into the system of the two following equations of the first order,

$$b' = \alpha x_t, \quad a' = -\alpha t x_t, \tag{42}$$

that is,

$$b' = \alpha(a + bt), \quad a' = -\alpha t(a + bt); \tag{43}$$

from which system both a and b should be deduced as functions of t. His method is unquestionably elegant in the highest degree and has the very great advantage of having been developed and employed by himself and by other eminent analysts, yet I do not despair of mine also being hereafter thought worthy to be developed and applied to the chief problems of dynamics by persons able to do it justice and disposed to try what can be done by taking a new track. For instance, I have some little hope that you, who confess your patience almost exhausted in working at the Theory of the Moon as it presents itself to other methods, including probably improvements of your own, may feel some curiosity to see developed, and may give your valuable aid to develope, the general and rigorous integrals which I have assigned: provided that I do not drag you through too wide a maze of abstract speculation, but reduce the problem nearly to its least terms and depart as little as the nature of my method will permit from those processes which have already been (to so great a degree) productive of success to yourself and others. At the very least there would be some utility in employing a new method as a *verification* of the results of an old one; but my hopes aspire somewhat higher.

www.rcin.org.pl

I regret that I have so little to say on the subject of your important proposition respecting the construction of Empirical Lunar Tables. I should certainly like to see such Tables carefully constructed, and think they would be useful both practically for the calculation of Ephemerides and theoretically for comparison with formulae; but I am not prepared to join in pressing on the Association at this moment the going to the necessary expense; especially as Airy, whose opinion on such a subject must be admitted to possess great weight, appears to be against the doing so. But probably you would not be disinclined to wait another year, without pressing for a final decision; and before the meeting in 1838, if we all live, we are likely to have all formed more decided and concurrent opinions.

I regret also to have to mention that I have as yet not thought at all on the subject of your communication on Optics. Perhaps even before the approaching meeting of the Association I may find time to do so, and to put in writing the result of my thoughts.

Looking forward with great pleasure to our soon seeing each other again, I remain, my dear sir, with best remembrances to Mrs. Lubbock,

Very truly yours

William R. Hamilton.

B. *J. W. Lubbock to Sir W. R. Hamilton.*

*London, Sept*r., 1837.

My dear Sir,

There is another* great objection to your method which I forgot to mention. I apprehend if you employ it to determine the constants, the resulting equations of condition between which it will be necessary to eliminate will not be linear. Moreover if you obtain the values the trouble of the substitution required to produce the longitude or latitude will be immense. Pray write soon and say whether I may communicate to M. de Pontécoulant the example given in your last letter to me & whatever you may send me in future about the Moon. I am very anxious the question should be settled as soon as possible for no one will think of carrying forward my methods while there is any probability of their being superseded by others more feasible and the announcement at Liverpool of your intention to try to apply your method to the Theory of the Moon will make those hesitate who know your great power of analysis. I shall be very agreeably surprized if you ever get out a single term in the longitude or latitude which I have not already obtained & more recently M. de Pontécoulant.

The quantities ticked in the Appendix enclosed are right, there are others undecided where I have obtained a term differing from that of M. de Pontécoulant.

I think the method of your last letter to me might be applied to find the equations which I have found for the perturbations according to powers of the time explicitly p. 39 of my Tract on Comets, and would furnish an excellent example of your method. Pray write to me soon and if you can give a glance at the optical paper I sent you.

Meanwhile, I remain, dear Sir,

Yours very truly

J. W. Lubbock.

* [Probably Lubbock had in the meantime discussed the matter with Hamilton at the British Association Meeting in Liverpool.]

www.rcin.org.pl

II. *Sir W. R. Hamilton to J. W. Lubbock.*

(*II^d Letter to Mr Lubbock on the Theory of the Moon.*)

Observatory, Oct. 4, 1837.

My dear Sir,

I am about to write to you, a little more in detail than I have yet done, though still only in a very undeveloped way, upon the Theory of the Moon. And because my present remarks will be found to have a close connexion with those contained in the letter which I wrote to you before the Liverpool Meeting, I shall (though the expedient is not an elegant one) continue to number my equations as they come, without reference to their importance or affinities, considering them as forming one unbroken series from the beginning of the former letter, & therefore calling the first of the new equations number 44, because the last of the old ones was number 43. I shall also call this letter itself II, supposing the former to have been called I; & shall retain the same plan of numbering in others which may follow these.

Adopting your fundamental equations

$$\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}+\left(\frac{dR}{dx}\right)=0, \quad \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}+\left(\frac{dR}{dy}\right)=0, \quad \frac{d^2z}{dt^2}+\frac{\mu z}{r^3}+\left(\frac{dR}{dz}\right)=0, \tag{44}$$

which I shall write as follows

$$x''+\frac{\mu x}{r^3}+\frac{\delta R}{\delta x}=0, \quad y''+\frac{\mu y}{r^3}+\frac{\delta R}{\delta y}=0, \quad z''+\frac{\mu z}{r^3}+\frac{\delta R}{\delta z}=0, \tag{45}$$

and in which, according to your notation, x, y, z, r are the Moon's geocentric rectangular coordinates and geocentric distance, while μ is the sum of the masses of Moon and Earth, & R is the disturbing function,

$$R=m_,\left\{\frac{xx_,+yy_,+zz_,}{r_,^3}-\frac{1}{\sqrt{\{r_,^2-2(xx_,+yy_,+zz_,)+r^2\}}}\right\}, \tag{46}$$

if $m_,$ be the Sun's mass, & $x_,$, $y_,$, $z_,$, $r_,$ its geocentric coordinates and distance; I shall suppose the 4 last mentioned quantities to be explicit functions of the time t, namely those corresponding to that elliptic relative motion which the Sun would have if the mass of the Moon were to vanish. This supposition is equivalent to treating the Moon as exerting no sensible attraction on Earth or Sun, although it is attracted by them, and they by each other. How far this supposition must fall short, when fully developed, of conducting to results as accurate as those obtained by the methods of yourself and others, you are a much better judge than I can at present be; but I think that the error must be small, & am sure that it will be smaller than other errors which I propose at the outset to neglect. Yet I should be glad to be honoured with your opinion on the degree of error which may in this way be occasioned.

I add the 3 equations (45) multiplied respectively by δx, δy, δz and thus obtain

$$x''\delta x+y''\delta y+z''\delta z+\mu\frac{\delta r}{r^2}+\delta R=0; \tag{47}$$

δt being here treated as $=0$, & only x, y, z (& of course r) as subject to the variation δ. To (47), I add the identical equation

$$x'(\delta x)'+y'(\delta y)'+z'(\delta z)'=\delta\frac{x'^2+y'^2+z'^2}{2}; \tag{48}$$

& thus obtain

$$(x'\delta x+y'\delta y+z'\delta z)'=\delta\left(\frac{x'^2+y'^2+z'^2}{2}+\frac{\mu}{r}-R\right), \tag{49}$$

& consequently, by integration,

$$x'\delta x+y'\delta y+z'\delta z-x'_0\delta x_0-y'_0\delta y_0-z'_0\delta z_0=\delta\int_0^t\left(\frac{x'^2+y'^2+z'^2}{2}+\frac{\mu}{r}-R\right)dt: \tag{50}$$

the quantities marked with 0 as a lower index being here initial values. This equation would be rigorously true if the Moon's mass were rigorously null; at some future time I may discuss the correction which must be introduced when that small mass is allowed for.

In the next place, I neglect the inclination of the Moon's orbit; & thus obtain, (the ecliptic being taken for the plane of xy,)

$$x'\delta x+y'\delta y-x'_0\delta x_0-y'_0\delta y_0=\delta\int_0^t\left(\frac{x'^2+y'^2}{2}+\frac{\mu}{r}-R\right)dt; \tag{51}$$

in which

$$R=m_,\left\{\frac{xx_,+yy_,}{r_,^3}-\frac{1}{\sqrt{r_,^2-2(xx_,+yy_,)+r^2}}\right\}, \tag{52}$$

$$r=\sqrt{x^2+y^2}, \tag{53}$$

$$r_,=\sqrt{x_,^2+y_,^2}: \tag{54}$$

so that if we introduce the longitudes λ, $\lambda_,$, putting

$$x=r\cos\lambda,\quad y=r\sin\lambda, \tag{55}$$

and

$$x_,=r_,\cos\lambda_,,\quad y_,=r_,\sin\lambda_,, \tag{56}$$

we have

$$r'\delta r+r^2\lambda'\delta\lambda-r'_0\delta r_0-r_0^2\lambda'_0\delta\lambda_0=\delta\int_0^t\left(\frac{r'^2+r^2\lambda'^2}{2}+\frac{\mu}{r}-R\right)dt, \tag{57}$$

and

$$R=m_,\{rr_,^{-2}\cos\overline{\lambda-\lambda_,}-(r_,^2-2rr_,\cos\overline{\lambda-\lambda_,}+r^2)^{-\frac{1}{2}}\}. \tag{58}$$

I neglect the excentricity of the Solar orbit, and thus obtain

$$r_,=a_,,\quad \lambda_,=n_,t,\quad m_,=a_,^3n_,^2; \tag{59}$$

and consequently

$$R=-n_,^2a_,^2-n_,^2\left(r^2P_2+\frac{r^3}{a_,}P_3+\frac{r^4}{a_,^2}P_4+\text{\&c.}\right), \tag{60}$$

if we denote by P_2, P_3, P_4, &c. the coefficients of the development

$$(1-2\alpha p+\alpha^2)^{-\frac{1}{2}}=1+\alpha p+\alpha^2P_2+\alpha^3P_3+\alpha^4P_4+\text{\&c.}, \tag{61}$$

in which

$$\alpha=\frac{r}{a_,},\quad p=\cos\overline{\lambda-n_,t}. \tag{62}$$

The general expression for these coefficients being

$$P_i=\frac{1}{1.2.3.4\ldots i}\left(\frac{d}{dp}\right)^i\left(\frac{p^2-1}{2}\right)^i, \tag{63}$$

we have, in particular,

$$P_2=\frac{1}{2}\left(\frac{d}{dp}\right)^2\left(\frac{p^2-1}{2}\right)^2=\frac{3p^2-1}{2}=\tfrac{3}{4}(2p^2-1)+\tfrac{1}{4}=\tfrac{1}{4}(1+3\cos\overline{2\lambda-2n_,t}); \tag{64}$$

www.rcin.org.pl

I shall therefore put, for the present,

$$R = -\frac{n_{,}^2}{4} r^2 (1 + 3\cos \overline{2\lambda - 2n_{,}t}), \tag{65}$$

suppressing the term $-n_{,}^2 a_{,}^2$ as being constant, (and as consequently contributing nothing to the variation $\delta \int_0^t R dt$,) and neglecting the terms multiplied by $n_{,}^2 \frac{r}{a_{,}}$, on account of the smallness of that factor.

If we entirely neglected the disturbing function R, we should have elliptic expressions for the Moon's polar coordinates r & λ, namely

$$r = a - ak\cos nt + al\sin nt, \tag{66}$$

$$\lambda = nt + \epsilon + 2k\sin nt + 2l\cos nt; \tag{67}$$

in which k and l are arbitrary but small constants, of which the squares and products are here neglected; (the first power only of the excentricity of the Moon's orbit being retained;) ϵ and n are other arbitrary but not small constants; and a depends on n by the relation

$$a^3 n^2 = \mu. \tag{68}$$

With these expressions we could verify the relation (57); for, making, after the variation δ, k & l each $= 0$, we have

$$r'\delta r + r^2\lambda'\delta\lambda - r_0'\delta r_0 - r_0^2\lambda_0'\delta\lambda_0 = a^2 n\,\delta(\lambda - \lambda_0)$$
$$= a^2 n\,(t\delta n + 2\sin nt\,\delta k + 2\,\overline{\cos nt - 1}\,\delta l); \tag{69}$$

and also
$$\delta\int_0^t \left(\frac{r'^2 + r^2\lambda'^2}{2} + \frac{\mu}{r}\right) dt = \delta\int_0^t a^2 n^2 \left(\tfrac{3}{2} + 2k\cos nt - 2l\sin nt\right) dt$$
$$= \delta\left\{a^2 n\left(\frac{3nt}{2} + 2k\sin nt + 2l\,\overline{\cos nt - 1}\right)\right\}$$
$$= a^2 n\,(t\delta n + 2\sin nt\,\delta k + 2\,\overline{\cos nt - 1}\,\delta l). \tag{70}$$

But when we take account of R, we must consider the expressions (66) & (67) as being defective, & as requiring to be corrected by the addition of terms proportional to $n_{,}^2$, if n, ϵ, k, l be still considered as constant. However, by considering n, ϵ, k, l $\left(\text{\& consequently } a = \sqrt[3]{\frac{\mu}{n^2}}\right)$, as suitable functions of the time, we may still retain the equations (66) and (67), & the analogous initial equations

$$r_0 = a - ak, \tag{71}$$

$$\lambda_0 = \epsilon + 2l; \tag{72}$$

though we cannot then retain also the differentiated expressions

$$r' = an\,(k\sin nt + l\cos nt), \tag{73}$$

$$\lambda' = n\,(1 + 2k\cos nt - 2l\sin nt), \tag{74}$$

and

$$r_0' = anl, \tag{75}$$

$$\lambda_0' = n\,(1 + 2k). \tag{76}$$

And to find, to the accuracy of terms proportional to $n_{,}^3$ inclusive, (the error being proportional to $n_{,}^4 = (n_{,}^2)^2$,) the new expressions analogous to (73), (74), (75), (76), which are to be combined with (66), (67), (68), (71), (72), we are, by my general method, to develope the 2nd member of (57) as a linear function of δn, $\delta\epsilon$, δk, δl, employing for R the expression (65) & using also

www.rcin.org.pl

the expressions (66), (67), (68), (73), (74) & making, after having taken the variation δ, the products $n_{,}^{2}k$ & $n_{,}^{2}l$ vanish and indeed k & l themselves.

In this manner I obtain the formula

$$\Delta r'\delta r + a^2\Delta\lambda'\delta\lambda - \Delta r'_0\delta r_0 - a^2\Delta\lambda'_0\delta\lambda_0 = -\delta\int_0^t R\,dt \tag{77}$$

in which $\Delta r'$, $\Delta\lambda'$, $\Delta r'_0$, $\Delta\lambda'_0$ are corrections proportional to $n_{,}^{2}$ (and perhaps partly to $n_{,}^{3}$), which are to be added to the 2nd members of the equations (73), (74), (75), (76), in order to make those equations consistent with (66), (67), (68), (71), (72). And because we are to make in the 1st member of (77)

$$\delta r = \delta a - a\cos nt\,\delta k + a\sin nt\,\delta l, \tag{78}$$

$$\delta\lambda = t\delta n + \delta\epsilon + 2\sin nt\,\delta k + 2\cos nt\,\delta l, \tag{79}$$

$$\delta r_0 = \delta a - a\,\delta k, \tag{80}$$

$$\delta\lambda_0 = \delta\epsilon + 2\delta l, \tag{81}$$

and

$$\delta a = -\frac{2}{3}\frac{a}{n}\delta n, \tag{82}$$

we have, by the mutual independence of δn, $\delta\epsilon$, δk, δl, the 4 following equations

$$-\frac{\delta}{\delta n}\int_0^t R\,dt = -\frac{2}{3}\frac{a}{n}(\Delta r' - \Delta r'_0) + a^2 t\Delta\lambda'; \tag{83}$$

$$-\frac{\delta}{\delta\epsilon}\int_0^t R\,dt = a^2(\Delta\lambda' - \Delta\lambda'_0); \tag{84}$$

$$-\frac{\delta}{\delta k}\int_0^t R\,dt = -a(\cos nt\,\Delta r' - \Delta r'_0) + 2a^2\sin nt\,\Delta\lambda'; \tag{85}$$

$$-\frac{\delta}{\delta l}\int_0^t R\,dt = a\sin nt\,\Delta r' + 2a^2(\cos nt\,\Delta\lambda' - \Delta\lambda'_0). \tag{86}$$

There would be little difficulty in resolving these 4 linear equations so as to deduce $\Delta r'$, $\Delta\lambda'$, $\Delta r'_0$, $\Delta\lambda'_0$ as functions of n and t & of the 4 partial differential coefficients of $\int_0^t R\,dt$, which coefficients themselves may be regarded as functions of $n_{,}$, n and t. And after correcting the equations (75) and (76) by the values thus found for $\Delta r'_0$ & $\Delta\lambda'_0$, we might then eliminate the 5 quantities a, n, ϵ, k, l between the 7 equations (66), (67), (68), (71), (72), (75), (76), so as to get expressions for r and λ as functions of t, $n_{,}$, r_0, λ_0, r'_0, λ'_0. But because the last elimination would not be linear, it is convenient to proceed as follows.

By altering the symbols $r, \lambda, a, n, \epsilon, k, l$ to the following, $r+\Delta r$, $\lambda+\Delta\lambda$, $a+\Delta a$, $n+\Delta n$, $\epsilon+\Delta\epsilon$, $k+\Delta k$, $l+\Delta l$, we may suppose that the 7 new quantities $r, \lambda, a, n, \epsilon, k, l$ satisfy the 7 old conditions (66), (67), (68), (71), (72), (75), (76), provided that we determine the 7 corrections $\Delta r, \Delta\lambda, \Delta a, \Delta n, \Delta\epsilon, \Delta k, \Delta l$ so as to satisfy these other 7 conditions

$$\Delta r = \Delta a - a\cos nt\,\Delta k + a\sin nt\,\Delta l, \tag{87}$$

$$\Delta\lambda = t\Delta n + \Delta\epsilon + 2\sin nt\,\Delta k + 2\cos nt\,\Delta l, \tag{88}$$

$$\Delta a = -\frac{2}{3}\frac{a}{n}\Delta n, \tag{89}$$

$$0 = \Delta a - a\Delta k, \tag{90}$$

$$0 = \Delta\epsilon + 2\Delta l, \tag{91}$$

$$0 = an\Delta l + \Delta r'_0, \tag{92}$$

$$0 = \Delta n + 2n\Delta k + \Delta\lambda'_0; \tag{93}$$

www.rcin.org.pl

and thus Δr and $\Delta\lambda$ will be the perturbational parts of the Moon's distance and longitude, & will be expressed as linear functions of the 4 partial differential coefficients of $\int_0^t Rdt$. This new elimination is by no means difficult,* and it gives

$$\Delta r = -\frac{2}{an}\frac{\delta}{\delta\epsilon}\int_0^t Rdt + \frac{\sin nt}{an}\frac{\delta}{\delta k}\int_0^t Rdt + \frac{\cos nt}{an}\frac{\delta}{\delta l}\int_0^t Rdt; \tag{94}$$

$$\Delta\lambda = \frac{3t}{a^2}\frac{\delta}{\delta\epsilon}\int_0^t Rdt - \frac{3}{a^2}\frac{\delta}{\delta n}\int_0^t Rdt + \frac{2\cos nt}{a^2 n}\frac{\delta}{\delta k}\int_0^t Rdt - \frac{2\sin nt}{a^2 n}\frac{\delta}{\delta l}\int_0^t Rdt; \tag{95}$$

expressions which may be considered as remarkable for their form & for their simplicity. Besides, we have, by (65),

$$\begin{aligned}\int_0^t Rdt = &-\frac{n_,^2}{4}\left\{a^2 t + \frac{3a^2}{2n-2n_,}(\sin \overline{2nt-2n_,t+2\epsilon} - \sin 2\epsilon)\right\}\\
&+\frac{n_,^2 a^2 k}{4}\left\{\frac{2}{n}\sin nt - \frac{3}{3n-2n_,}(\sin\overline{3nt-2n_,t+2\epsilon}-\sin 2\epsilon)\right.\\
&\left.+\frac{9}{n-2n_,}(\sin\overline{nt-2n_,t+2\epsilon}-\sin 2\epsilon)\right\}\\
&-\frac{n_,^2 a^2 l}{4}\left\{\frac{2}{n}(1-\cos nt) + \frac{3}{3n-2n_,}(\cos\overline{3nt-2n_,t+2\epsilon}-\cos 2\epsilon)\right.\\
&\left.+\frac{9}{n-2n_,}(\cos\overline{nt-2n_,t+2\epsilon}-\cos 2\epsilon)\right\};\end{aligned} \tag{96}$$

so that the expressions for Δr and $\Delta\lambda$ become

$$\begin{aligned}\Delta r = &\frac{n_,^2 a}{2n}\left\{\frac{3}{n-n_,}(\cos\overline{2nt-2n_,t+2\epsilon}-\cos 2\epsilon) + \frac{1}{n}(1-\cos nt)\right\}\\
&-\frac{3n_,^2 a}{4n(3n-2n_,)}(\cos\overline{2nt-2n_,t+2\epsilon}-\cos\overline{nt-2\epsilon})\\
&-\frac{9n_,^2 a}{4n(n-2n_,)}(\cos\overline{2nt-2n_,t+2\epsilon}-\cos\overline{nt+2\epsilon});\end{aligned} \tag{97}$$

and

$$\begin{aligned}\Delta\lambda = &-\frac{n_,^2 t}{n} - \frac{3n_,^2}{2n(n-n_,)}(\sin\overline{2nt-2n_,t+2\epsilon}-\sin 2\epsilon) + \frac{9n_,^2 t\cos 2\epsilon}{4(n-n_,)}\\
&-\frac{9n_,^2}{8(n-n_,)^2}(\sin\overline{2nt-2n_,t+2\epsilon}-\sin 2\epsilon) + \frac{n_,^2}{n^2}\sin nt\\
&-\frac{3n_,^2}{2n(3n-2n_,)}(\sin\overline{2nt-2n_,t+2\epsilon}+\sin\overline{nt-2\epsilon})\\
&+\frac{9n_,^2}{2n(n-2n_,)}(\sin\overline{2nt-2n_,t+2\epsilon}-\sin\overline{nt+2\epsilon});\end{aligned} \tag{98}$$

* [See pp. 241–243, $s = -\int_0^t Rdt$.]

www.rcin.org.pl

or, in the same order of approximation,

$$\Delta r=\left(\frac{n_{,}}{n}\right)^2\frac{a}{2}\left(1-3\cos 2\epsilon-3\frac{n_{,}}{n}\cos 2\epsilon\right)$$
$$-\left(\frac{n_{,}}{n}\right)^2\frac{a}{2}\left(1-5\cos 2\epsilon-\frac{28}{3}\frac{n_{,}}{n}\cos 2\epsilon\right)\cos nt$$
$$-\left(\frac{n_{,}}{n}\right)^2 a\left(2+\frac{13}{3}\frac{n_{,}}{n}\right)\sin 2\epsilon\sin nt$$
$$-\left(\frac{n_{,}}{n}\right)^2 a\left(1+\frac{19}{6}\frac{n_{,}}{n}\right)\cos\overline{2nt-2n_{,}t+2\epsilon}; \tag{99}$$

and

$$\Delta\lambda=-\left(\frac{n_{,}}{n}\right)^2 nt\left(1-\frac{9}{4}\cos 2\epsilon-\frac{9}{4}\frac{n_{,}}{n}\cos 2\epsilon\right)+\frac{3}{2}\left(\frac{n_{,}}{n}\right)^2\left(\frac{7}{4}+\frac{5}{2}\frac{n_{,}}{n}\right)\sin 2\epsilon$$
$$+\left(\frac{n_{,}}{n}\right)^2\left(1-5\cos 2\epsilon-\frac{28}{3}\frac{n_{,}}{n}\cos 2\epsilon\right)\sin nt$$
$$-2\left(\frac{n_{,}}{n}\right)^2\left(2+\frac{13}{3}\frac{n_{,}}{n}\right)\sin 2\epsilon\cos nt$$
$$+\left(\frac{n_{,}}{n}\right)^2\left(\frac{11}{8}+\frac{59}{12}\frac{n_{,}}{n}\right)\sin\overline{2nt-2n_{,}t+2\epsilon}. \tag{100}$$

These, then, are, as results of the present method and in the present order of approximation, expressions for those parts of the perturbations of the Moon's distance and longitude which are independent of the excentricities & inclinations; and by the nature of the process they are so constructed that both themselves and their first differential coefficients vanish at the origin of the time: which is in some respects advantageous. We may however dispense with this last condition, and introduce, instead, an incorporation of certain terms of the expressions (99) & (100) with the corresponding terms of the elliptic expressions (66) & (67). With this latter view we may put

$$\mathrm{n}=n-\frac{n_{,}^2}{n}\left(1-\frac{9}{4}\cos 2\epsilon-\frac{9}{4}\frac{n_{,}}{n}\cos 2\epsilon\right); \tag{101}$$

$$\mathrm{a}=\sqrt[3]{\frac{\mu}{\mathrm{n}^2}}=a+\frac{2}{3}\left(\frac{n_{,}}{n}\right)^2 a\left(1-\frac{9}{4}\cos 2\epsilon-\frac{9}{4}\frac{n_{,}}{n}\cos 2\epsilon\right); \tag{102}$$

$$\mathrm{e}=\epsilon+\frac{3}{8}\left(\frac{n_{,}}{n}\right)^2\left(7+10\frac{n_{,}}{n}\right)\sin 2\epsilon; \tag{103}$$

$$\mathrm{k}=k+\frac{1}{2}\left(\frac{n_{,}}{n}\right)^2\left(1-5\cos 2\epsilon-\frac{28}{3}\frac{n_{,}}{n}\cos 2\epsilon\right); \tag{104}$$

$$\mathrm{l}=l-\left(\frac{n_{,}}{n}\right)^2\left(2+\frac{13}{3}\frac{n_{,}}{n}\right)\sin 2\epsilon; \tag{105}$$

and if we put, besides,

$$m=\frac{n_{,}}{\mathrm{n}}, \tag{106}$$

and

$$\tau=\mathrm{n}t-n_{,}t+\epsilon, \tag{107}$$

www.rcin.org.pl

we find, by (99), (100) and (66), (67),

$$\frac{r+\Delta r}{\mathrm{a}}=1-\frac{m^2}{6}-\mathrm{k}\cos \mathrm{n}t+\mathrm{l}\sin \mathrm{n}t-m^2\left(1+\frac{19}{6}m\right)\cos 2\tau; \tag{108}$$

and

$$\lambda+\Delta\lambda=\mathrm{n}t+\mathrm{e}+2\mathrm{k}\sin \mathrm{n}t+2\mathrm{l}\cos \mathrm{n}t+m^2\left(\frac{11}{8}+\frac{59}{12}m\right)\sin 2\tau: \tag{109}$$

results which perfectly agree with those of yourself and of Plana.

Even this slight sketch of an actual application of my method to the Theory of the Moon may make my meaning clearer than it was before, and may prepare for future & more laborious & subtle applications.

I am, my dear Sir,

very truly yours

William R. Hamilton.

J. W. Lubbock, Esq$^{\mathrm{r}}$.

C. *J. W. Lubbock to Sir W. R. Hamilton.*

29 Eaton Place.

Tues., 10th *Oct*^{r}., 1837.

My dear Sir,

I have received your highly interesting letter of the 4$^{\mathrm{th}}$, the example you have given is very instructive and is worthy of your great mathematical talent. Even if your method should prove impracticable, as an example of analysis every mathematician will wish to see it carried to some other terms besides those contained in your last letter.

I do not understand why you "make after the variation δ the products $n_{,}^2k$ and $n_{,}^2l$ vanish and indeed k and l themselves."

I suspect your method as modified by the introduction of the quantities Δa, Δn, $\Delta\epsilon$, Δk and Δl amounts only to *another* way of finding Lagrange's values of these quantities. Thus for example I see that you get

$$\Delta l=\frac{1}{an}\frac{\delta}{\delta k}\int_0^t R\,dt$$

which is only another way of writing

$$dl=\frac{1}{an}\frac{\delta R}{\delta k}dt$$

and this last I believe is Lagrange's value. I suspect therefore that if you write Lagrange's values for $\Delta a=\int da$, $\Delta k=\int dk$, &c. in your equations (87) and (88) you will get precisely the values you have written down for Δr and $\Delta\lambda$ in equations (94) and (95).

If you take the trouble to calculate in my way your equations (108) and (109), you will judge which method gives less trouble. But a great difficulty begins in your method when you introduce the perturbational terms in R in order to get a further approximation because you must either develope

$$\frac{\delta}{\delta n}\int R\,dt,\quad \frac{\delta}{\delta t}\int R\,dt,\quad \frac{\delta}{\delta k}\int R\,dt,\quad \frac{\delta}{\delta l}\int R\,dt$$

separately or you must do what will be equally troublesome.

www.rcin.org.pl

I look upon this method of the variation of the constants as quite impracticable except for the easy terms and this is also the opinion of M. de Pontécoulant.

There are very few sensible terms depending upon the Moon's mass. I have calculated* one of them as an example of my methods and found a term differing from M. Poisson, but M. Pontécoulant tells me that he has found a numerical slip in M. Poisson's work which accounts for it.

The equation to the fluid surface of the ocean being

$$(X-u')\,dx+(Y-v')\,dy+(Z-w')\,dz=0,$$

u', v', w' the complete differentials of $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$, cannot we integrate and say

$$\int\{X\,dx+Y\,dy+Z\,dz\}-\frac{dx^2}{dt^2}-\frac{dy^2}{dt^2}-\frac{dz^2}{dt^2}=\text{constant}?\dagger$$

The Theory of the Tides in the *Mécanique Céleste* is the only portion which has not been commented on by Lagrange, by Poisson & by others. I wish you had time to examine it. I think much might be done in solving the problem at least in settling what are its real difficulties.

Will you oblige me by writing out according to your method the steps which would belong to the equation

$$x''+x+\frac{dR}{dx}=0,$$

beginning from

$$x'\delta x-x_0'\delta x_0=\delta\int_0^t\left\{\frac{x'^2}{2}-\frac{x^2}{2}-R\right\}dt? \qquad (50)$$

The approximate integrals are

$$x=a\cos(t+b), \quad x'=-a\sin(t+b).$$

Lagrange's equations give

$$-\frac{dR}{da}\,dt=\left\{\frac{dx'}{db}\frac{dx}{da}-\frac{dx}{db}\frac{dx'}{da}\right\}db=-a\,db,$$

$$-\frac{dR}{db}\,dt=\left\{\frac{dx'}{da}\frac{dx}{db}-\frac{dx}{da}\frac{dx'}{db}\right\}da=a\,da.$$

M. Poisson's equations give, directly,

$$db=-\frac{1}{a}\frac{dR}{da}\,dt, \quad da=-\frac{1}{a}\frac{dR}{db}\,dt.$$

You must arrive by your methods at these equations and I should like to see how you proceed. I think you hardly appreciate yet sufficiently the difficulties which will arise even in your method from the *depression* by integration, in the theory of the Moon and in similar problems.

I shall be delighted to hear from you whenever you have time to write and it will give me the greatest pleasure to find that your methods will lead to more rapid approximations than those which have been hitherto employed, including my own.

Ever most truly yours

J. W. LUBBOCK.

* [Lubbock, *Tracts on Lunar Theory*, p. 289.]

† [This, with the obvious numerical slip corrected, is true if the $\int(u\,dx+v\,dy+w\,dz)$, taken between any two points of the liquid and moving with it, is independent of the time.]

www.rcin.org.pl

P.S. Allow me to venture a few remarks upon your method, which I do with great diffidence for I am not sure yet if I quite understand it.

For simplicity I will take the differential equation

$$x'' + x + \frac{dR}{dx} = 0.$$

The approximate integrals of this equation are

$$x = a\cos(t+b), \quad x' = -a\sin(t+b).$$

Your equation (50) is

$$x'\delta x - x'_0\delta x_0 = \delta\int_0^t \left(\frac{x'^2}{2} - \frac{x^2}{2} - R\right)dt.$$

Your equation (77) is

$$\Delta x'\,\delta x - \Delta x'_0\,\delta x_0 = -\delta\int_0^t R\,dt,$$

$$x' = f'(a,b) + \Delta x', \quad x = f(a,b) + \Delta x,$$

where

$$f'(a,b) \text{ is in this example } -a\sin(t+b),$$

$$f(a,b) \qquad\qquad a\cos(t+b).$$

If I write $\bar{x}'$ for $f'(a,b)$ and $\bar{x}$ for $f(a,b)$ for convenience merely,

$$\delta x = \frac{d\bar{x}}{da}\delta a + \frac{d\bar{x}}{db}\delta b, \tag{51}$$

$$\delta x_0 = \frac{d\bar{x}_0}{da}\delta a + \frac{d\bar{x}_0}{db}\delta b. \tag{52}$$

The condition that $f(a,b)$ represents the value of x in the disturbed motion gives $\Delta x = 0$,

$$\frac{d\bar{x}}{da}\Delta a + \frac{d\bar{x}}{db}\Delta b + \frac{1}{1.2}\frac{d^2\bar{x}}{da^2}\Delta a^2 + \frac{1}{1.2}\frac{d^2\bar{x}}{db^2}\Delta b^2 + \&c. = 0, \tag{53}$$

$$\frac{d\bar{x}_0}{da}(\Delta a)_0 + \frac{d\bar{x}_0}{db}(\Delta b)_0 + \frac{1}{1.2}\frac{d^2\bar{x}_0}{da^2}(\Delta a)_0^2 + \frac{1}{1.2}\frac{d^2\bar{x}_0}{db^2}(\Delta b)_0^2 + \&c. = 0, \tag{54}$$

$$\Delta x' = -\frac{d\bar{x}'}{da}\Delta a - \frac{d\bar{x}'}{db}\Delta b - \frac{1}{1.2}\frac{d^2\bar{x}'}{da^2}\Delta a^2 - \&c., \tag{55}$$

$$\Delta x'_0 = -\frac{d\bar{x}'_0}{da}(\Delta a)_0 - \frac{d\bar{x}'_0}{db}(\Delta b)_0 - \&c. \tag{56}$$

Equation (50) gives

$$\Delta x'\frac{d\bar{x}}{da} - \Delta x'_0\frac{d\bar{x}_0}{da} = -\int_0^t \frac{dR}{da}dt, \tag{57}$$

$$\Delta x'\frac{d\bar{x}}{db} - \Delta x'_0\frac{d\bar{x}_0}{db} = -\int_0^t \frac{dR}{db}dt. \tag{58}$$

These equations correspond to the equations (83), (84), (85), and (86) of your II[nd] Letter.

Substituting in (57) and (58) the values of $\Delta x'$ and $\Delta x'_0$ from (55) and (56) I get, *neglecting* Δa^2, Δb^2, &c.,

$$-\frac{d\bar{x}}{da}\frac{d\bar{x}'}{da}\Delta a + \frac{d\bar{x}_0}{da}\frac{d\bar{x}'_0}{da}\Delta a_0 - \frac{d\bar{x}}{da}\frac{d\bar{x}'}{db}\Delta b + \frac{d\bar{x}_0}{da}\frac{d\bar{x}'_0}{db}\Delta b_0 = -\int_0^t \frac{dR}{da}dt.$$

www.rcin.org.pl

Multiply (53) by $\frac{d\bar{x}'}{da}$ and add it to the last equation,

(54) by $\frac{d\bar{x}'_0}{da}$ and subtract it from the last equation,

and we get, the coefficients of Δa and $(\Delta a)_0$ vanishing,

$$\left\{\frac{d\bar{x}}{db}\frac{d\bar{x}'}{da}-\frac{d\bar{x}}{da}\frac{d\bar{x}'}{db}\right\}\Delta b-\left\{\frac{d\bar{x}_0}{db}\frac{d\bar{x}'_0}{da}-\frac{d\bar{x}_0}{da}\frac{d\bar{x}'_0}{db}\right\}\Delta b_0=-\int_0^t\frac{dR}{da}dt.$$

But Lagrange has shown generally that the quantity which is the coefficient of Δb is constant; hence we may write

$$\left\{\frac{d\bar{x}_0}{db}\frac{d\bar{x}'_0}{da}-\frac{d\bar{x}_0}{da}\frac{d\bar{x}'_0}{db}\right\}\{\Delta b-\Delta b_0\}=-\int_0^t\frac{dR}{da}dt$$

and, differentiating,

$$-\frac{dR}{da}dt=\left\{\frac{d\bar{x}_0}{db}\frac{d\bar{x}'_0}{da}-\frac{d\bar{x}_0}{da}\frac{d\bar{x}'_0}{db}\right\}db,$$

which is Lagrange's equation.

It would have been as easy to take a more complicated example, but the reasoning would have been exactly the same.

It seems to me that you ought to distinguish between Δa and Δa_0,* Δb and Δb_0. For the equation

$$\left\{\frac{d\bar{x}_0}{da}\frac{d\bar{x}'_0}{da}-\frac{d\bar{x}}{da}\frac{d\bar{x}'}{da}\right\}\Delta a+\left\{\frac{d\bar{x}_0}{da}\frac{d\bar{x}'_0}{db}-\frac{d\bar{x}}{da}\frac{d\bar{x}'}{db}\right\}\Delta b=-\int\frac{dR}{da}dt,$$

which your methods would lead to (?) if Δa is not distinguished from Δa_0, is incorrect.

I have written out this since I closed my letter & if I am wrong I am sure you will not be severe.

III. *Sir W. R. Hamilton to J. W. Lubbock.*

Observatory, Oct. 12*th*, 1837.

MY DEAR SIR,

Your letter of the 10th has just arrived & I hasten to make some remarks in reply to it, & in continuation of my own last letter.

The sentence respecting the making k and l vanish was certainly obscure; I felt it to be so at the time of writing it, but trusted that my meaning would be fully caught when I should have proceeded farther than I have yet done in the development & application of my method; meantime I may remind you of what you yourself foresaw, namely that I am obliged to *drop an excentricity* in passing from one approximation to another; so that starting only with expressions for the coordinates r & λ which are accurate to the 1st power of the Moon's excentricity inclusive, I can only expect to get the circular parts of the perturbations, or the parts independent of excentricity. For this reason I omitted as useless (because liable to be changed by combination with unknown terms of the same order) the terms which were multiplied by $n_{,}^2k$ and $n_{,}^2l$;—the words "and indeed k and l themselves" were added in a hurry afterwards, & though I do not

* [In the Hamilton ellipse, the initial velocity is different from the orbital velocity. See Appendix, Note 7, p. 628.]

www.rcin.org.pl

exactly admit them to be erroneous, I feel that they require more commentary & justification than I think it worth while to give, considering that I intend to resume the subject soon, in a more full & explicit manner.

You say that the equations (89), (90), (91), (92), (93) are not obvious, & this is so far true, that in my first attempt (which I showed to you at Mr Turner's house in Liverpool) I used wrong signs for $\Delta r_0'$ and $\Delta\lambda_0'$ through an error of reasoning rather than through a mere slip of the pen, and so obtained expressions for the perturbations which were themselves affected with wrong signs, though I soon found out the mistake. It may therefore be proper to develope the corrected reasoning now.—The equations (68), (71), (72) may be said to be true by definition; but if we combine with them the equations (66), (67), *defining* these also to be true & considering r & λ as the Moon's *disturbed* coordinates, we then establish meanings for a, n, ϵ, k, l in virtue of which they are no longer constants but rather are parameters made variable so that they no longer satisfy the equations (75), (76) but rather satisfy these other equations

$$r_0' = anl + \Delta r_0', \tag{110}$$

$$\lambda_0' = n(1+2k) + \Delta\lambda_0'; \tag{111}$$

r_0' and λ_0' being still constants, namely the initial rates of increase of the coordinates r & λ but $\Delta r_0'$ and $\Delta\lambda_0'$ being those (small and perturbational) functions of the time which may be found by elimination from the formulae (83), (84), (85), (86). One mode, then, of expressing the Moon's disturbed coordinates is to denote them by r and λ and to retain, as definitions, the expressions (66) & (67), but to substitute in these expressions instead of a, n, ϵ, k, l those values as functions of the time which may be obtained by elimination from the equations (68), (71), (72), (110), (111) and (83), (84), (85), (86). To facilitate this elimination, we may denote the undisturbed or constant parts of a, n, ϵ, k, l by $\grave{a}, \grave{n}, \grave{\epsilon}, \grave{k}, \grave{l}$ and the remaining or perturbational parts by $\Delta a, \Delta n, \Delta\epsilon, \Delta k, \Delta l$; so that

$$a = \grave{a} + \Delta a, \quad n = \grave{n} + \Delta n, \quad \epsilon = \grave{\epsilon} + \Delta\epsilon, \quad k = \grave{k} + \Delta k, \quad l = \grave{l} + \Delta l; \tag{112}$$

and we may neglect the squares and products of $\Delta a, \Delta n, \Delta\epsilon, \Delta k, \Delta l$. In this manner the 5 equations (68), (71), (72), (110), (111) resolve themselves into ten others, namely,

$$\left.\begin{aligned} \mu &= \grave{a}^3\grave{n}^2, \\ r_0 &= \grave{a} - \grave{a}\grave{k}, \\ \lambda_0 &= \grave{\epsilon} + 2\grave{l}, \\ r_0' &= \grave{a}\grave{n}\grave{l}, \\ \lambda_0' &= \grave{n}(1+2\grave{k}), \end{aligned}\right\} \tag{113}$$

and

$$\left.\begin{aligned} 0 &= 3\grave{n}\Delta a + 2\grave{a}\Delta n, \\ 0 &= \Delta a - \grave{a}\Delta k, \\ 0 &= \Delta\epsilon + 2\Delta l, \\ 0 &= \grave{a}\grave{n}\Delta l + \Delta r_0', \\ 0 &= \Delta n + 2\grave{n}\Delta k + \Delta\lambda_0'; \end{aligned}\right\} \tag{114}$$

which latter are, as you see, equivalent to the five equations (89)...(93), only made a little fuller & (perhaps) more clear by the introduction of the grave accents over a and n, which are here designed to distinguish elliptic from disturbed parameters. The same grave accents might have

www.rcin.org.pl

been introduced (because squares and products of perturbations were neglected) into the second members of the equations (83), (84), (85), (86); and then the two expressions (66) & (67) might have been resolved each into two parts, as follows,

$$r = r' + \Delta r, \quad \lambda = \lambda' + \Delta\lambda, \tag{115}$$

in which

$$\left.\begin{aligned} r' &= a' - a'k' \cos n't + a'l' \sin n't, \\ \lambda' &= n't + \epsilon' + 2k' \sin n't + 2l' \cos n't, \end{aligned}\right\} \tag{116}$$

and

$$\left.\begin{aligned} \Delta r &= \Delta a - a' \cos n't \Delta k + a' \sin n't \Delta l, \\ \Delta\lambda &= t\Delta n + \Delta\epsilon + 2 \sin n't \Delta k + 2 \cos n't \Delta l; \end{aligned}\right\} \tag{117}$$

the two latter equations corresponding to (87) & (88). But I grudged the expense of so many new accents & new equations; nor do I think that on consideration you will desire that I should complicate my method by continuing to use them explicitly.

You seem to think that a few words more might have been expended on the elimination by which the expressions (94), (95) were deduced from the equations (83), (84), (85), (86), (87), (88), (89), (90), (91), (92), (93). I proceeded nearly as follows:—The equations (89) ... (93) give, very easily,

$$\Delta a = -\frac{2a}{n} \Delta\lambda_0'; \quad \Delta n = 3\Delta\lambda_0'; \quad \Delta\epsilon = \frac{2}{an} \Delta r_0'; \quad \Delta k = -\frac{2}{n} \Delta\lambda_0'; \quad \Delta l = -\frac{1}{an} \Delta r_0'; \tag{118}$$

and therefore, by (87) and (88),

$$\left.\begin{aligned} \Delta r &= -\frac{2a}{n}(1 - \cos nt)\Delta\lambda_0' - \frac{1}{n} \sin nt \Delta r_0', \\ \Delta\lambda &= \left(3t - \frac{4}{n} \sin nt\right) \Delta\lambda_0' + \frac{2}{an}(1 - \cos nt)\Delta r_0'. \end{aligned}\right\} \tag{119}$$

At the same time the 2 equations (83), (84) give

$$\left.\begin{aligned} \Delta\lambda' &= \Delta\lambda_0' - \frac{1}{a^2} \frac{\delta}{\delta\epsilon} \int_0^t R\,dt, \\ \Delta r' &= \Delta r_0' + \tfrac{3}{2} ant \Delta\lambda_0' - \frac{3nt}{2a} \frac{\delta}{\delta\epsilon} \int_0^t R\,dt + \frac{3n}{2a} \frac{\delta}{\delta n} \int_0^t R\,dt; \end{aligned}\right\} \tag{120}$$

and therefore the 2 equations (85), (86) give

$$\begin{aligned} \cos nt \frac{\delta}{\delta k} \int_0^t R\,dt - \sin nt \frac{\delta}{\delta l} \int_0^t R\,dt &= a(\Delta r' - \cos nt \Delta r_0') - 2a^2 \sin nt \Delta\lambda_0' \\ &= a(1 - \cos nt)\Delta r_0' + a^2(\tfrac{3}{2} nt - 2 \sin nt)\Delta\lambda_0' - \frac{3nt}{2} \frac{\delta}{\delta\epsilon} \int_0^t R\,dt + \frac{3n}{2} \frac{\delta}{\delta n} \int_0^t R\,dt \\ &= \frac{a^2 n}{2} \Delta\lambda - \frac{3nt}{2} \frac{\delta}{\delta\epsilon} \int_0^t R\,dt + \frac{3n}{2} \frac{\delta}{\delta n} \int_0^t R\,dt, \end{aligned} \tag{121}$$

and

$$\begin{aligned} \sin nt \frac{\delta}{\delta k} \int_0^t R\,dt + \cos nt \frac{\delta}{\delta l} \int_0^t R\,dt &= -a \sin nt \Delta r_0' - 2a^2(\Delta\lambda' - \cos nt \Delta\lambda_0') \\ &= -a \sin nt \Delta r_0' - 2a^2(1 - \cos nt)\Delta\lambda_0' + 2\frac{\delta}{\delta\epsilon} \int_0^t R\,dt \\ &= an\Delta r + 2\frac{\delta}{\delta\epsilon} \int_0^t R\,dt; \end{aligned} \tag{122}$$

www.rcin.org.pl

results agreeing evidently with the expressions (94), (95). Perhaps you rather meant to enquire how I propose to extend this process of elimination to the more complicated systems of equations which must arise hereafter; and this I hope to make clear at some future stage of our correspondence.

With respect generally to the comparative facility of my own & other methods, your candour of course will lead you to think that it would be unfair to judge of this by contrasting the perfected and published works of yourself and other mathematicians with the rude sketches which these letters contain; and which are written down *very* nearly as they first present themselves to my mind, scarcely any development being suppressed and scarcely any alteration being made. I feel little or no doubt that the same *principle* and (in a large sense) the same *method* as that which is here employed will conduct me or some one else hereafter to *processes* of actual calculation, more convenient and more elegant than those which are here deduced. Even my two published Essays (on a general method in dynamics) contain results, I think, which would be useful in this way, though I prefer at present to suppose that you have not read them, & to take up the subject from the beginning. For instance, if you take the trouble to turn to the last page but two of my First Essay* (*Phil. Trans.* 1834, Part II, Page 306), you will find two systems of expressions for the variations of the parameters of a system of attracting or repelling particles, & a remark analogous to that which you have made in your last letter, respecting the possibility of deducing Lagrange's formulae & mine reciprocally from each other; and some of the theorems of my Second Essay might doubtless have been used with advantage; as probably I shall make use of them, whenever I come to write for publication. Meanwhile it is essential to remember that although the equations (94), (95) of my last letter *might* have been deduced from (66), (67) by supposing a, n, ϵ, k, l to receive the perturbational increments

$$\left.\begin{aligned} \Delta a &= -\frac{2}{an}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt, \quad \Delta n = \frac{3}{a^2}\frac{\delta}{\delta\epsilon}\int_0^t R\,dt, \quad \Delta\epsilon = -\frac{3}{a^2}\frac{\delta}{\delta n}\int_0^t R\,dt, \\ \Delta k &= -\frac{1}{a^2 n}\frac{\delta}{\delta l}\int_0^t R\,dt, \quad \Delta l = \frac{1}{a^2 n}\frac{\delta}{\delta k}\int_0^t R\,dt, \end{aligned}\right\} \qquad (123)$$

which would correspond to Lagrange's conception of a varying and tangent ellipse, yet they *were* in fact deduced by a very different set of increments (118); & that the set last mentioned corresponded to the geometrical conception of a variable ellipse† which *cuts*, not touches, the geocentric orbit of the Moon, but which has over the tangent ellipse at least one advantage in simplicity, namely that it passes through one *fixed initial point* determined by the Moon's fixed initial coordinates r_0 and λ_0. You say that Lagrange's method has been found in practice far from convenient, or rather practically useless; & *that* is doubtless a strong presumption against the practical success of my method: but I cannot give it up without a further trial and especially without being convinced of its inutility as applied to terms depending on the higher powers of the Sun's disturbing force.

I confess however, that when I wrote (in haste) my II[d] Letter of this series & made (in equal haste) the calculations contained in it, I did not remember that the expressions (94), (95) might be obtained by Lagrange's method; and I was about to verify them by submitting them to the following transformation:— R being originally a function of r and λ, let its two partial differential

* [Pp. 159, 160 of this volume.] † [See Appendix, Note 7, p. 628.]

www.rcin.org.pl

coefficients $\frac{\delta R}{\delta r}$ and $\frac{\delta R}{\delta \lambda}$ be conceived to take values denoted by R' and R'' when r and λ reduce themselves to the circular values a and $nt+\epsilon$; we shall then have, by making k and l each $=0$,

$$\left.\begin{aligned}&\frac{\delta R}{\delta n}=-\frac{2}{3}\frac{a}{n}R'+tR'', \quad \frac{\delta R}{\delta \epsilon}=R'',\\ &\frac{\delta R}{\delta k}=-aR'\cos nt+2R''\sin nt, \quad \frac{\delta R}{\delta l}=aR'\sin nt+2R''\cos nt;\end{aligned}\right\} \tag{124}$$

and therefore

$$\left.\begin{aligned}&\frac{\delta}{\delta n}\int_0^t R\,dt=-\frac{2a}{3n}\int_0^t R'\,dt+\int_0^t tR''\,dt, \quad \frac{\delta}{\delta \epsilon}\int_0^t R\,dt=\int_0^t R''\,dt,\\ &\frac{\delta}{\delta k}\int_0^t R\,dt=-a\int_0^t R'\cos nt\,dt+2\int_0^t R''\sin nt\,dt,\\ &\frac{\delta}{\delta l}\int_0^t R\,dt=a\int_0^t R'\sin nt\,dt+2\int_0^t R''\cos nt\,dt;\end{aligned}\right\} \tag{125}$$

so that the expressions (94) and (95) become

$$\begin{aligned}\Delta r=&\frac{1}{n}\left(\cos nt\int_0^t R'\sin nt\,dt-\sin nt\int_0^t R'\cos nt\,dt\right)\\ &+\frac{2}{an}\left(\sin nt\int_0^t R''\sin nt\,dt+\cos nt\int_0^t R''\cos nt\,dt-\int_0^t R''\,dt\right),\end{aligned} \tag{126}$$

and

$$\begin{aligned}\Delta\lambda=&-\frac{2}{an}\left(\sin nt\int_0^t R'\sin nt\,dt+\cos nt\int_0^t R'\cos nt\,dt-\int_0^t R'\,dt\right)\\ &+\frac{4}{a^2n}\left(\cos nt\int_0^t R''\sin nt\,dt-\sin nt\int_0^t R''\cos nt\,dt\right)\\ &+\frac{3}{a^2}\left(t\int_0^t R''\,dt-\int_0^t R''t\,dt\right).\end{aligned} \tag{127}$$

These give, by a first differentiation,

$$\begin{aligned}(\Delta r)'=&-\left(\sin nt\int_0^t R'\sin nt\,dt+\cos nt\int_0^t R'\cos nt\,dt\right)\\ &+\frac{2}{a}\left(\cos nt\int_0^t R''\sin nt\,dt-\sin nt\int_0^t R''\cos nt\,dt\right),\end{aligned} \tag{128}$$

$$\begin{aligned}(\Delta\lambda)'=&-\frac{2}{a}\left(\cos nt\int_0^t R'\sin nt\,dt-\sin nt\int_0^t R'\cos nt\,dt\right)\\ &-\frac{4}{a^2}\left(\sin nt\int_0^t R''\sin nt\,dt+\cos nt\int_0^t R''\cos nt\,dt\right)+\frac{3}{a^2}\int_0^t R''\,dt;\end{aligned} \tag{129}$$

and, by a second,

$$\begin{aligned}(\Delta r)''=&-R'-n\left(\cos nt\int_0^t R'\sin nt\,dt-\sin nt\int_0^t R'\cos nt\,dt\right)\\ &-\frac{2n}{a}\left(\sin nt\int_0^t R''\sin nt\,dt+\cos nt\int_0^t R''\cos nt\,dt\right),\end{aligned} \tag{130}$$

$$\begin{aligned}(\Delta\lambda)''=&\frac{2n}{a}\left(\sin nt\int_0^t R'\sin nt\,dt+\cos nt\int_0^t R'\cos nt\,dt\right)\\ &-\frac{4n}{a^2}\left(\cos nt\int_0^t R''\sin nt\,dt-\sin nt\int_0^t R''\cos nt\,dt\right)-\frac{R''}{a^2};\end{aligned} \tag{131}$$

www.rcin.org.pl

so that the functions Δr & $\Delta\lambda$, besides having the property that they and their first differential coefficients $(\Delta r)'$ & $(\Delta\lambda)'$ vanish when $t=0$, have also the property that they satisfy the following pair of differential equations of the 2[nd] order,

$$(\Delta r)'' - 3n^2\Delta r - 2an\,(\Delta\lambda)' + R^{\backprime} = 0, \tag{132}$$

$$a^2\,(\Delta\lambda)'' + 2an\,(\Delta r)' + R^{\backprime\backprime} = 0. \tag{133}$$

Now these last two equations are precisely the conditions necessary, in order that the functions

$$r = a + \Delta r, \quad \lambda = nt + \epsilon + \Delta\lambda, \tag{134}$$

may satisfy (when the square of the disturbing force is neglected & when a and n are connected by the relation

$$a^3n^2 = \mu) \tag{68}$$

the two differential equations, deduced from (48),

$$r'' - r\lambda'^2 + \frac{\mu}{r^2} + \frac{\delta R}{\delta r} = 0, \quad (r^2\lambda')' + \frac{\delta R}{\delta\lambda} = 0. \tag{135}$$

And if, reciprocally, without thinking of my own method, we had set about to integrate the equations (132), (133), we should have had, first, from the equation (133),

$$a^2\,(\Delta\lambda)' + 2an\,\Delta r + \int_0^t R^{\backprime\backprime}\,dt = 0, \tag{136}$$

(observing that Δr, $\Delta\lambda$ and $(\Delta r)'$, $(\Delta\lambda)'$ are supposed to vanish when $t=0$,) and, next, by (132),

$$(\Delta r)'' + n^2\Delta r + R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt = 0; \tag{137}$$

which would have given, by well-known methods,

$$\Delta r = \frac{\cos nt}{n}\int_0^t\left(R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt\right)\sin nt\,dt$$
$$-\frac{\sin nt}{n}\int_0^t\left(R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt\right)\cos nt\,dt; \tag{138}$$

and also

$$\Delta\lambda = -\frac{2n}{a}\int_0^t \Delta r\,dt - \frac{1}{a^2}\int_0^t\left(\int_0^t R^{\backprime\backprime}\,dt\right)dt, \tag{139}$$

that is

$$\Delta\lambda = -\frac{2\sin nt}{an}\int_0^t\left(R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt\right)\sin nt\,dt$$
$$-\frac{2\cos nt}{an}\int_0^t\left(R^{\backprime} + \frac{2n}{a}\int_0^t R^{\backprime\backprime}\,dt\right)\cos nt\,dt$$
$$+\frac{1}{an}\int_0^t\left(2R^{\backprime} + \frac{3n}{a}\int_0^t R^{\backprime\backprime}dt\right)dt; \tag{140}$$

in which expressions the double integrals might have been reduced by observing that

$$\left.\begin{aligned}
&\int_0^t\left(\sin nt\int_0^t R^{\backprime\backprime}dt\right)dt = -\frac{\cos nt}{n}\int_0^t R^{\backprime\backprime}dt + \frac{1}{n}\int_0^t R^{\backprime\backprime}\cos nt\,dt,\\
&\int_0^t\left(\cos nt\int_0^t R^{\backprime\backprime}dt\right)dt = \frac{\sin nt}{n}\int_0^t R^{\backprime\backprime}dt - \frac{1}{n}\int_0^t R^{\backprime\backprime}\sin nt\,dt,\\
&\int_0^t\left(\int_0^t R^{\backprime\backprime}dt\right)dt = t\int_0^t R^{\backprime\backprime}dt - \int_0^t R^{\backprime\backprime}t\,dt;
\end{aligned}\right\} \tag{141}$$

and then the expressions (126) and (127) would have resulted.

www.rcin.org.pl

I intended also to remark that, whichever of these methods we employ, if we introduce into the expression of R the additional term

$$-\frac{n_,^2 r^3}{a_,} P_3 = -\frac{n_,^2 r^3}{6a_,}\left(\frac{d}{dp}\right)^3\left(\frac{p^2-1}{2}\right)^3 = -\frac{n_,^2 r^3}{48a_,}\left(\frac{d}{dp}\right)^3(p^6-3p^4+3p^2-1)$$

$$= -\frac{n_,^2 r^3}{2a_,}(5p^3-3p) = -\frac{n_,^2 r^3}{8a_,}\{5\cos\overline{3\lambda-3n_,t}+3\cos\overline{\lambda-n_,t}\}, \tag{142}$$

we obtain, in the expressions of the Moon's disturbed coordinates, the following additional terms:

$$\begin{aligned}\Delta r = & -\frac{2}{3}\frac{a}{n}\Delta' n - a\cos nt\,\Delta' k + a\sin nt\,\Delta' l\\ & +\frac{3}{16}\frac{n_,}{n}\frac{a^2}{a_,}\left(5+\frac{n_,}{2n-n_,}+\frac{4n_,}{n-n_,}\right)\cos(nt-n_,t+\epsilon)\\ & -\frac{5}{16}\left(\frac{n_,}{n}\right)^2\frac{a^2}{a_,}\left(\frac{9n}{2n-3n_,}+\frac{3n}{4n-3n_,}-\frac{4n}{n-n_,}\right)\cos(3nt-3n_,t+3\epsilon);\end{aligned} \tag{143}$$

$$\begin{aligned}\Delta\lambda = & \,t\Delta' n+\Delta'\epsilon+2\sin nt\,\Delta' k+2\cos nt\,\Delta' l\\ & -\frac{3}{8}\frac{n_,}{n}\frac{a}{a_,}\left\{5-\frac{n_,}{2n-n_,}+\frac{3n_,(3n-2n_,)}{(n-n_,)^2}\right\}\sin(nt-n_,t+\epsilon)\\ & +\frac{5}{8}\left(\frac{n_,}{n}\right)^2\frac{a}{a_,}\left\{\frac{9n}{2n-3n_,}-\frac{3n}{4n-3n_,}-\frac{(3n-2n_,)n}{(n-n_,)^2}\right\}\sin(3nt-3n_,t+3\epsilon);\end{aligned} \tag{144}$$

in which I have put, for abridgment,

$$\left.\begin{aligned}\Delta' n &= \frac{3}{8}\frac{a}{a_,}\frac{n_,^2}{n-n_,}(3\cos\epsilon+5\cos3\epsilon);\\ \Delta'\epsilon &= \frac{1}{8}\frac{a}{a_,}\frac{n_,^2}{n}\frac{3n-2n_,}{(n-n_,)^2}(9\sin\epsilon+5\sin3\epsilon);\\ \Delta' k &= \frac{3}{16}\frac{a}{a_,}\frac{n_,}{n}\left\{\left(5+\frac{n_,}{2n-n_,}\right)\cos\epsilon-5\left(\frac{3n_,}{2n-3n_,}+\frac{n_,}{4n-3n_,}\right)\cos3\epsilon\right\};\\ \Delta' l &= \frac{3}{16}\frac{a}{a_,}\frac{n_,}{n}\left\{\left(5-\frac{n_,}{2n-n_,}\right)\sin\epsilon-5\left(\frac{3n_,}{2n-3n_,}-\frac{n_,}{4n-3n_,}\right)\sin3\epsilon\right\}.\end{aligned}\right\} \tag{145}$$

It is true that these results would add, to the second members of the equations (108) & (109) respectively, the terms

$$\frac{15a}{16a_,}\left(m+\frac{9m^2}{10}+\frac{17m^3}{20}\right)\cos\tau-\frac{25a}{64a_,}\left(m^2+\frac{53m^3}{20}\right)\cos3\tau, \tag{146}$$

$$-\frac{15a}{8a_,}\left(m+\frac{17m^2}{10}+\frac{47m^3}{20}\right)\sin\tau+\frac{15a}{32a_,}\left(m^2+\frac{35m^3}{12}\right)\sin3\tau, \tag{147}$$

and that the coefficients of

$$\frac{am^2\cos\tau}{a_,},\quad \frac{am^2\sin\tau}{a_,},\quad \frac{am^3\cos\tau}{a_,},\quad \frac{am^3\sin\tau}{a_,},\quad \frac{am^3\cos3\tau}{a_,},\quad \frac{am^3\sin3\tau}{a_,},$$

thus obtained, do not agree with those deduced by you & Plana:* I must therefore suppose that an improved approximation would correct the subordinate (though not the principal) numerical

* [See p. 248 and Appendix, Note 6, p. 627.]

www.rcin.org.pl

coefficients of the expressions last obtained by me. But, in the meantime, I incline to think that *so far as the mere ordering according to powers of m (or of $n_,$) is concerned*, (*abstraction being made of the indefinite increase which the variable t may and must receive in the course of time*,) my expressions for the Moon's coordinates r & λ are accurate, *as far as the cube of m and the* 1st *power of the reciprocal of a, inclusive*. However, even the appearance of differing from you and Plana must naturally make me cautious: and I intend to consider this point also with a more close attention hereafter. The terms proportional to $\frac{m^3}{a_,}\cos\tau$ and $\frac{m^3}{a_,}\sin\tau$ seem to me at present to be the most doubtful of those in which the difference apparently exists.

It may be useful to observe that in deducing the perturbational terms (143) & (144) from the part (142) of R in the way most immediately suggested by my method I was led to calculate the following part of the integral $\int_0^t R\,dt$:

$$-\frac{n_,^2}{8a_,}\int_0^t r^3\{5\cos\overline{3\lambda-3n_,t}+3\cos\overline{\lambda-n_,t}\}\,dt=-\frac{3}{8}\frac{a^3n_,^2}{a_,(n-n_,)}(\sin\overline{nt-n_,t+\epsilon}-\sin\epsilon)$$
$$-\frac{5a^3n_,^2}{24a_,(n-n_,)}(\sin\overline{3nt-3n_,t+3\epsilon}-\sin 3\epsilon)+\frac{3ka^3n_,^2}{16a_,}\left\{\frac{5}{n_,}(\sin\overline{n_,t-\epsilon}+\sin\epsilon)\right.$$
$$+\frac{1}{2n-n_,}(\sin\overline{2nt-n_,t+\epsilon}-\sin\epsilon)+\frac{15}{2n-3n_,}(\sin\overline{2nt-3n_,t+3\epsilon}-\sin 3\epsilon)$$
$$\left.-\frac{5}{4n-3n_,}(\sin\overline{4nt-3n_,t+3\epsilon}-\sin 3\epsilon)\right\}+\frac{3la^3n_,^2}{16a_,}\left\{\frac{5}{n_,}(\cos\overline{n_,t-\epsilon}-\cos\epsilon)\right.$$
$$+\frac{1}{2n-n_,}(\cos\overline{2nt-n_,t+\epsilon}-\cos\epsilon)-\frac{15}{2n-3n_,}(\cos\overline{2nt-3n_,t+3\epsilon}-\cos 3\epsilon)$$
$$\left.-\frac{5}{4n-3n_,}(\cos\overline{4nt-3n_,t+3\epsilon}-\cos 3\epsilon)\right\}; \qquad (148)$$

in which, for the first time, a lowering of the order of a small term occurred, so far as the powers of the small quantity $n_,$ are concerned; namely by the introduction of that small quantity $n_,$ as divisor in passing from the nearly constant (or rather slowly varying) terms

$$\frac{15a^3n_,^2}{16a_,}(k\cos\overline{n_,t-\epsilon}-l\sin\overline{n_,t-\epsilon}), \qquad (149)$$

which occur in the development of R itself, to the corresponding terms

$$\frac{15a^3n_,}{16a_,}\{k(\sin\overline{n_,t-\epsilon}+\sin\epsilon)+l(\cos\overline{n_,t-\epsilon}-\cos\epsilon)\}, \qquad (150)$$

in the integral $\int_0^t R\,dt$. Reciprocally, so far as the mere ordering according to the powers of $n_,$ is concerned, these terms (150) are really, though not apparently, of the same order as $n_,^2$, or rather as $\frac{kn_,^2}{a_,}$; and in like manner the corresponding terms in Δr and $\Delta\lambda$, namely (by (94) & (95))

$$\frac{15a^2n_,}{16a_,n}(\cos\overline{nt-n_,t+\epsilon}-\cos\overline{nt+\epsilon}), \qquad (151)$$

and

$$-\frac{15an_,}{8a_,n}(\sin\overline{nt-n_,t+\epsilon}-\sin\overline{nt+\epsilon}), \qquad (152)$$

www.rcin.org.pl

are really of the same order as $\frac{n_,^2}{a_,}$. This remark appears to me not useless because, from the form of the differential equations (135) and from the circumstance that R contains $n_,^2$ as a factor, it might have been foreseen that the perturbations Δr and $\Delta\lambda$ ought also to contain that small square $n_,^2$ or m^2 as a factor; and therefore the occurrence of such terms as

$$\frac{15\mathrm{a}^2 m}{16 a_,}\cos\tau \quad\text{and}\quad -\frac{15\mathrm{a}m}{8a_,}\sin\tau, \tag{153}$$

in the expressions of those perturbations, may seem at first paradoxical.—But though *some* depression of order, by introduction of small divisors, will certainly occur in my method, namely in the calculation of that single definite integral to which the problem is reduced, it is still not obvious how *so great* a depression can arise as in that double integration which is required by other methods; on this point also I therefore await the conviction of what may be called experiment, before I give up all hope of being useful in the new track which I am attempting to pursue. I can only write by snatches, but expect to make & to transmit some calculations soon, which will present my views in a still more definite shape. Meantime believe me to remain, my dear Sir,

very truly yours,

William R. Hamilton.

J. W. Lubbock, Esq[r].

P.S. To prevent any misunderstanding with respect to the nature of the results to which I have arrived, so far, in investigating the coordinates of the Moon by my own method, it may be right to recapitulate them thus.—I have been led to the 2 equations

$$r=\mathrm{a}\left\{1-\mathrm{k}\cos \mathrm{n}t+\mathrm{l}\sin \mathrm{n}t-\frac{m^2}{6}-m^2\left(1+\frac{19m}{6}\right)\cos 2\tau + \frac{15}{16}\frac{\mathrm{a}}{a_,}m\left(1+\frac{9m}{10}+\frac{17m^2}{20}\right)\cos\tau-\frac{25}{64}\frac{\mathrm{a}}{a_,}m^2\left(1+\frac{53m}{20}\right)\cos 3\tau\right\}, \tag{154}$$

and

$$\lambda=\mathrm{n}t+\mathrm{e}+2\mathrm{k}\sin \mathrm{n}t+2\mathrm{l}\cos \mathrm{n}t+\frac{11m^2}{8}\left(1+\frac{118m}{33}\right)\sin 2\tau -\frac{15}{8}\frac{\mathrm{a}}{a_,}m\left(1+\frac{17m}{10}+\frac{47m^2}{20}\right)\sin\tau+\frac{15}{32}\frac{\mathrm{a}}{a_,}m^2\left(1+\frac{35m}{12}\right)\sin 3\tau, \tag{155}$$

(in which a, m, & τ denote respectively $\sqrt[3]{\frac{\mu}{\mathrm{n}^2}}$, $\frac{n_,}{\mathrm{n}}$, & $\mathrm{n}t-n_,t+\mathrm{e}$, & k and l are small) as integrals, with 4 arbitrary constants n, e, k, l, of the two differential equations of the second order

$$\left.\begin{aligned} r''-r\lambda'^2+\frac{\mu}{r^2}&=\frac{n_,^2 r}{2}(1+3\cos\overline{2\lambda-2n_,t})+\frac{3n_,^2r^2}{8a_,}(3\cos\overline{\lambda-n_,t}+5\cos\overline{3\lambda-3n_,t}),\\ (r^2\lambda')'&=-\frac{3n_,^2r^2}{2}\sin\overline{2\lambda-2n_,t}-\frac{3n_,^2r^3}{8a_,}(\sin\overline{\lambda-n_,t}+5\sin\overline{3\lambda-3n_,t});\end{aligned}\right\} \tag{156}$$

in which μ, $n_,$ and $a_,$ are constants, and $n_,$ & $\frac{1}{a_,}$ are small, and the upper accents correspond to differentiations with respect to t: also the 4[th] power of $n_,$ and the square of $\frac{1}{a_,}$ are neglected;

www.rcin.org.pl

and so are the squares and product of k and l, and the products of these small quantities (k & l) with others. These expressions (154) & (155) were obtained through the medium of the formulae (94), (95), and the development of $\int_0^t R\,dt$; but I have verified them by actual differentiation, & have reobtained them, on principles much more elementary, by integrating the equations (132), (133) after assigning for R' and R'' their values. Yet the subordinate coefficients of the sine and cosine of τ & 3τ do not agree with yours and Plana's & will, therefore, I am sure, be altered by a subsequent approximation:—the ground of alteration being, I suppose, the indefinite increase of t and the consequent practical necessity of keeping that symbol under the signs of periodicity, except in the one term nt; which latter necessity has not yet shown itself in the operation of my method and indeed is not recognised in the first conception of that method—certain quantities being treated as small but none thought of as large. I mean in my *present* method of successive & indefinite *approximation* for my *rigorous and finite integrals* of the differential equations of motion of any system of attracting or repelling points extend to any intervals of time, however great.

Since writing the above I have perceived that my results can be reconciled with yours & Plana's in such a way as I conjectured; at least as far as the coefficients of $\frac{m^2a}{a_{,}}$ inclusive beyond which you have marked Plana's numbers as erroneous without stating (at least in your appendix) what your own numbers are. But I am unwilling to delay this letter or to add to its length. (Observatory, Oct. 17th, 1837.)

Sir W. R. Hamilton to J. W. Lubbock.

Oct. 18th, 1837.

MY DEAR SIR,

The enclosed letter having been delayed, I add a word or two. You suggested that I should compare my own process of obtaining my equations (108) and (109) with yours. I wish to do so: would it be quite unreasonable to ask you to take the trouble to write out just so much of your own method as is absolutely necessary for deducing those two equations? At least you might point out the pages & parts of pages most requisite for me to read with that particular view—but I fear that I am likely to postpone the reading of your printed work until I have advanced much farther in the development of my own views unless you stimulate me and set me a-going by doing some such thing as I just now ventured to request—which, after all, would not perhaps give you much trouble, & might not be uninstructive to other students hereafter. An *elementary* treatise on the theory of the Moon, going about as far as Airy's tract but drawn up on *your* plan, would doubtless be useful & acceptable to a large class of readers; and I, for one, should hail its appearance with great pleasure.

All that I am *now* doing is (you see) quite piecemeal & preparatory. My *hope* is that a nearly perfect theory of the Moon may be obtained by my method, through the medium of *two* successive developments; namely, first, the development of $\int R\,dt$ with elliptic values of the coordinates, and secondly the development of the same integral with improved expressions for those coordinates. In fact, I think that after the second of these developments a new set of expressions for the coordinates must result by differentiation and elimination, which will err

www.rcin.org.pl

only by quantities of the 8th order relatively to n, and of the 4th order relatively to $\frac{1}{a}$, abstraction being made of the indefinite increase of t. But there are, I own, two practical dangers. One is this very increase which may ultimately vitiate the expressions, or at least prevent them from being accurate to so high an order as I have named. I hope however that this will not depress the final order so much as to prevent the results from being useful. The other danger is that the second development of $\int R dt$ may be intolerably laborious. I have some hope of your co-operation in that task, & with it I do not despair.

Once more, very truly yours

W. R. H.

D. *J. W. Lubbock to Sir W. R. Hamilton.*

EATON PLACE.
Octr. 21, 1837.

MY DEAR SIR,

I was very glad to receive your letter of the 12th, in which you have calculated the terms belonging to τ to 3τ or my arguments [101] and [116]. I hope you will not calculate r (radius vector) but its reciprocal which will I apprehend be quite the same to you and by so doing your radius vector results will be directly comparable with Plana's and mine but not otherwise.

I still think that your expressions are to be considered as identical with those of Lagrange as far as our purpose is concerned (although I admit that your method of arriving at them is very remarkable) so that I think your views conduct to precisely the same calculations as those developed by M. Poisson in his paper. In the *Phil. Trans.* (1832), p. 229, I showed how these equations lead to precisely the same results generally at least when the squares of the excentricities are neglected as those given by the other method which I may call mine as I believe no one ever made use of it before. But I think Poisson in his paper about the Moon first enunciated the manner in which the equations are to be employed in further approximations and in one instance of the simplest kind applied them.

M. Poisson says, p. 50:* "La substitution des elements elliptiques et de ξ ainsi augmenté pourra se faire soit dans les formules (10) après avoir effectué les differentiations relatives à ces mêmes elements soit dans la fonction R &c." The calculations which are indicated in this paragraph are precisely those which seem to me to be practically beyond the reach of human patience. And I believe that you will not be able to evade this difficulty.† I consider your equations (94) and (95) as precisely those which would arise from making use of Lagrange's expressions for the variation of the constants and therefore although your proof of Lagrange's expressions be new yet your method as explained in your last letter is precisely, if I am not mistaken, the same as that proposed by Poisson in his "Mémoire sur le mouvement de la Lune", *Mémoires de l'Institut*, Tome XIII, and of which he gave some examples choosing however those cases in which that method is seen to the best advantage. This method has never been carried further than by myself in the *Phil. Trans.* (1832), p. 229, & by Poisson in the Memoir alluded to, so that Lagrange's method can hardly yet be said to have been found in *practice* inconvenient. After writing the

* *Mémoires de l'Institut*, Tome XIII.

† We shall be able to judge of this when you have got out some terms depending on the next approximation.

www.rcin.org.pl

paper however to which I have alluded in the *Phil. Trans.* I was able better to estimate its relative advantage, & I reflected much upon them before I finally adopted the methods which I have employed. The analytical difficulties even are not trifling as may be seen by looking at the communications from Plana & Pontécoulant in the *Comptes Rendus* about the value of R_{77}. M. Cauchy has just published in Liouville's journal a new method of deducing Lagrange's expressions, viz. from the integral of vis viva, but in other respects it has no similarity with yours. I sent you some time back a short paper by Pontécoulant, you will see what he thinks of Poisson's method (p. 22) & he has expressed himself quite as strongly in his letter to me. I shall be glad to know what you think about his controversy with Poisson. I think Poisson quite right. I think particularly that the introduction of the quantities σ and σ' (p. 53) & the interpretation assigned to them by Poisson completely obviates Pontécoulant's objections.*

However Pontécoulant is very positive and he ought to know what he is about. (I enclose Pontécoulant's pamphlet, if you find the other copy you can return it.)

The weakest point analytically about the theory of the Moon seems to me to be that I mentioned to you about the form which has been given to the acceleration of the mean motion & I hope you will consider this difficult question.

The reason why your term in the longitude argument [101] or τ does not agree with mine is I think because your de† and $d\varpi$ contain argument [102] or $\tau-\xi$ which will always be lowered by integration, & this by combination with ξ gives the term sought in the longitude argument [101]. The same applies to argument [116].

I make

$$\frac{2}{r}=\left\{-\frac{15m}{16}-\frac{81m^2}{16}-\frac{3021m^3}{128}\right\}\frac{a}{a_,}\cos\tau \qquad [101], \text{p. } 276‡$$

$$+\left\{\frac{25m^2}{64}-\frac{115m^3}{128}\right\}\frac{a}{a_,}\cos 3\tau, \qquad [116], \text{p. } 274$$

$$\lambda=\left\{-\frac{15m}{8}-\frac{93m^2}{8}-\frac{889m^3}{16}\right\}\frac{a}{a_,}\sin\tau \qquad [101], \text{p. } 277$$

$$+\left\{\frac{15m^2}{32}-\frac{5m^3}{8}\right\}\frac{a}{a_,}\sin 3\tau. \qquad [116]$$

Unless you write your results in this form we shall have much trouble hereafter in comparing them with mine & Plana's. I mean without putting your first coefficient *outside* the bracket. By employing other constants than mine such as l and k you will give yourself an immense deal of unnecessary labour because you will not be able to employ my development of R (p. 30) which would otherwise serve your purpose. You can hardly think how much trouble I have spent upon this development in order to get it free from error and with a few exceptions all the terms have been calculated by two distinct and independent methods.§ If I tried to apply the method of variation of constants I think I should use the expressions which I have given at the foot of p. 75. I now close this long letter hoping to hear from you again soon,

& remain, yours very faithfully

J. W. Lubbock.

* [For a general account of the different methods, see Brown, *Lunar Theory*, Cambridge University Press, Chap. XII.]

† dl and dk come to precisely the same.

‡ [These page references are to Lubbock's *Theory of the Moon and the Perturbations of the Planets*, Part II (1836).]

§ And have been verified by Pontécoulant by another method.

www.rcin.org.pl

IV. *Sir W. R. Hamilton to J. W. Lubbock.*

[Four pages of this letter are missing. At the top of page 5 is "written subsequently to the meeting of the British Association at Liverpool in 1837."

It deals with the integration of the equation

$$x'' = -\frac{\delta R}{\delta x},$$

where R is a perturbing function having as multiplier a small quantity α. Neglecting R we have

$$x = a + bt. \tag{23}$$

If the actual initial and final values of x' are denoted by x'_0 and x'_t, we have then by Hamilton's method

$$x'_t \delta x_t - x'_0 \delta x_0 = \delta \int_0^t (\tfrac{1}{2} x_t'^2 - R_{x_t})\, dt, \tag{160}$$

which gives

$$(x'_t - b)\,\delta x_t - (x'_0 - b)\,\delta x_0 = -\delta \int R dt \tag{161}$$

or

$$(x'_t - b)(\delta a + t\delta b) - (x'_0 - b)\,\delta a = -\delta \int R dt, \tag{162}$$

that is,

$$x'_t - x'_0 = -\frac{\delta}{\delta a}\int R dt, \quad x'_t - b = -\frac{\delta}{t\delta b}\int R dt{:}]$$

therefore, by elimination of x'_t,

$$x'_0 = b + \frac{\delta}{\delta a}\int_0^t R dt - \frac{1}{t}\frac{\delta}{\delta b}\int R dt, \tag{163}$$

by which equation, combined with the following

$$x_0 = a, \tag{164}$$

we are to eliminate the parameters a & b from the expression (23), & so obtain x_t as a function of x_0, x'_0 and t, involving also the small constant α.

In the present example the elimination is easy and gives

$$x_t = x_0 + t x'_0 + \frac{\delta}{\delta x'_0}\int_0^t R dt - t\frac{\delta}{\delta x_0}\int_0^t R dt, \tag{165}$$

R being treated as that function of x_0, x'_0 and t, which is obtained by changing x_t in it to $x_0 + t x'_0$. But to illustrate the less simple process which it is in general convenient to employ, we may make

$$x'_t = b + \Delta x'_t, \quad x'_0 = b + \Delta x'_0, \tag{166}$$

& then the equations (162) become

$$\Delta x'_t - \Delta x'_0 = -\frac{\delta}{\delta a}\int_0^t R dt; \quad t\Delta x'_t = -\frac{\delta}{\delta b}\int_0^t R dt; \tag{167}$$

we may also make

$$a = a^{\backprime} + \Delta a, \quad b = b^{\backprime} + \Delta b, \quad x_t = x_t^{\backprime} + \Delta x_t, \tag{168}$$

$$a^{\backprime} = x_0, \quad b^{\backprime} = x'_0, \quad x_t^{\backprime} = a^{\backprime} + b^{\backprime} t, \tag{169}$$

$$0 = \Delta a, \quad 0 = \Delta b + \Delta x'_0, \quad \Delta x_t = \Delta a + t\Delta b; \tag{170}$$

and then we find, for what may be called the perturbation of x_t,

$$\Delta x_t = -t\Delta x'_0 = \frac{\delta}{\delta b^{\backprime}}\int_0^t R dt - \frac{\delta}{\delta a^{\backprime}}\int_0^t R dt, \tag{171}$$

www.rcin.org.pl

R being treated here as a function of $a` + b`t$; a result which entirely agrees with the expression (165).

Now it is certain that this very expression (165) can be found also by Lagrange's method of varying the parameter. For if, with (23), we had combined not (164) but the following,

$$x'_t = b, \tag{172}$$

as the second equation of definition to determine the varying quantities a & b, we should have had, rigorously,

$$\left.\begin{aligned} b' = x''_t = -\frac{\delta R}{\delta x_t} = -\frac{\delta R}{\delta a}, \\ a' = -tb'_t = t\frac{\delta R}{\delta x_t} = \frac{\delta R}{\delta b}, \end{aligned}\right\} \tag{173}$$

and therefore approximately

$$\left.\begin{aligned} a = a_0 + \int_0^t \frac{\delta R}{\delta b_0} dt = x_0 + \int_0^t \frac{\delta R}{\delta x'_0} dt, \\ b = b_0 - \int_0^t \frac{\delta R}{\delta a_0} dt = x'_0 - \int_0^t \frac{\delta R}{\delta x_0} dt, \end{aligned}\right\} \tag{174}$$

so that, by (23),

$$x_t = x_0 + tx'_0 + \int_0^t \frac{\delta R}{\delta x'_0} dt - t\int_0^t \frac{\delta R}{\delta x_0} dt, \tag{175}$$

or finally

$$x_t = x_0 + tx'_0 + \frac{\delta}{\delta x'_0}\int_0^t R\,dt - t\frac{\delta}{\delta x_0}\int_0^t R\,dt. \tag{165}$$

So far, I certainly can pretend to no practical advantage of any sort for my method over that of Lagrange, or rather they both conduct to the very same system of integration & partial differentiation. And I am inclined to admit (what doubtless it would be easy to decide & what I should probably find decided already in my own published essays if I had leisure at this moment to consult them) that however far the approximation might be carried with respect to excentricities & inclinations, the equations analogous to (94) & (95) of II,* obtained by my own method, would agree entirely with those to which the method of Lagrange conducts. We see, indeed, that the results in these two methods are obtained from very different conceptions; in such a manner that if the differential equation (157) be considered as determining the varied motion of a point P along a line, while a and b represent respectively the *epoch* and *velocity* of the uniform motion of another point P' in the same line which coincides in position with P at the time t, my P' coincides also with P at the time 0, but Lagrange's at the time $t + dt$; and algebraically my a is fixed $(= x_0)$, while my b is determined (approximately) by the equation (163), or by the following,

$$b = x'_0 - \frac{\delta}{\delta x_0}\int_0^t R\,dt + \frac{1}{t}\frac{\delta}{\delta x_0}\int_0^t R\,dt, \tag{176}$$

but Lagrange's a and b both vary & are determined approximately by the equations (174) or rigorously by the equation (175). But the practical question is whether in passing to a higher order of approximation, & introducing the square & cube of the small constant α, it is easier to develope a & b from Lagrange's equations

$$b' = -\frac{\delta R}{\delta a}, \quad a' = \frac{\delta R}{\delta b}, \tag{177}$$

* [Second letter to Lubbock, p. 261.]

www.rcin.org.pl

or to affect the analogous development with my different system of conceptions & of formulae.

Confining ourselves to the square of α, I suppose that Poisson's mode of applying Lagrange's method to the Theory of the Moon would correspond to the following treatment of the equations (173) or (177). Introducing the approximate values (174) for a & b into the partial differential coefficients $-\frac{\delta R}{\delta a}$ and $\frac{\delta R}{\delta b}$, we get, as an improved system of approximations,

$$\left.\begin{aligned} b &= b_0 - \int_0^t \frac{\delta R}{\delta a_0} dt - \int_0^t \left(\frac{\delta^2 R}{\delta a_0^2} \int_0^t \frac{\delta R}{\delta b_0} dt\right) dt + \int_0^t \left(\frac{\delta^2 R}{\delta a_0 \delta b_0} \int_0^t \frac{\delta R}{\delta a_0} dt\right) dt, \\ a &= a_0 + \int_0^t \frac{\delta R}{\delta b_0} dt + \int_0^t \left(\frac{\delta^2 R}{\delta a_0 \delta b_0} \int_0^t \frac{\delta R}{\delta b_0} dt\right) dt - \int_0^t \left(\frac{\delta^2 R}{\delta b_0^2} \int_0^t \frac{\delta R}{\delta a_0} dt\right) dt; \end{aligned}\right\} \tag{178}$$

and consequently (23), being now true by definition, we have to the order of accuracy sought

$$\begin{aligned} x_t = a_0 + tb_0 &+ \int_0^t \frac{\delta R}{\delta b_0} dt - t\int_0^t \frac{\delta R}{\delta a_0} dt \\ &+ \int_0^t \left(\frac{\delta^2 R}{\delta a_0 \delta b_0} \int_0^t \frac{\delta R}{\delta b_0} dt\right) dt - \int_0^t \left(\frac{\delta^2 R}{\delta b_0^2} \int_0^t \frac{\delta R}{\delta a_0} dt\right) dt \\ &+ t\int_0^t \left(\frac{\delta^2 R}{\delta a_0 \delta b_0} \int_0^t \frac{\delta R}{\delta a_0} dt\right) dt - t\int_0^t \left(\frac{\delta^2 R}{\delta a_0^2} \int_0^t \frac{\delta R}{\delta b_0} dt\right) dt; \end{aligned} \tag{179}$$

R being treated as a function of a_0, b_0 and t, obtained by changing x_t in it to $a_0 + tb_0$. The lower indices 0 may be suppressed, & we may write, more simply,

$$\begin{aligned} x_t = a + bt &+ \int_0^t \frac{\delta R}{\delta b} dt - t\int_0^t \frac{\delta R}{\delta a} dt \\ &+ \int_0^t \left(\frac{\delta^2 R}{\delta a \delta b} \int_0^t \frac{\delta R}{\delta b} dt\right) dt - \int_0^t \left(\frac{\delta^2 R}{\delta b^2} \int_0^t \frac{\delta R}{\delta a} dt\right) dt \\ &+ t\int_0^t \left(\frac{\delta^2 R}{\delta a \delta b} \int_0^t \frac{\delta R}{\delta a} dt\right) dt - t\int_0^t \left(\frac{\delta^2 R}{\delta a^2} \int_0^t \frac{\delta R}{\delta b} dt\right) dt, \end{aligned} \tag{180}$$

if we here agree to treat a and b as denoting not the variable parameters which were introduced into the expression (23), when that expression was extended by definition so as to be true for disturbed motion, but the fixed initial constants a_0 & b_0, or x_0 & x'_0.

Consider now my method of improving the approximate expression (165), so as to take account of α^2, that is, of the square of the disturbing force. I put the expression (165) under the form

$$x_t = a + bt + \left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right) \int_0^t R dt, \tag{181}$$

a and b being quantities which would be constant if α^2 or R^2 were neglected, & R being a function of them and of t, obtained by changing in it x_t to $a + bt$. This expression if it were rigorous would give by differentiation, $\left(\text{because } \frac{\delta R}{\delta b} = t\frac{\delta R}{\delta a}\right)$,

$$x'_t = b - \frac{\delta}{\delta a} \int_0^t R dt; \tag{182}$$

www.rcin.org.pl

it would therefore give, for the function under the integral sign in (160), the value

$$\tfrac{1}{2}x_t'^2 - R_{x_t} = \tfrac{1}{2}b^2 - b\frac{\delta}{\delta a}\int_0^t R\,dt - R_{a+bt} - R^{(2)}, \tag{183}$$

R in the first member being a function of the disturbed variable x_t, but being in the second member the same function of the undisturbed part $a+bt$, so that R_{a+bt} is only a first approximate value of R_{x_t}; while $R^{(2)}$ has been written for abridgment instead of the following part, which is of the order of the square of R,

$$R^{(2)} = -\frac{1}{2}\left(\frac{\delta}{\delta a}\int_0^t R\,dt\right)^2 + \frac{\delta R}{\delta a}\left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R\,dt. \tag{184}$$

Thus $-\delta\int_0^t R^{(2)}dt$ becomes the part of the second order in the second member of the rigorous formula (160), and is to be equated to the part of the same order in the first member of that formula. We have therefore*

$$-\delta\int_0^t R^{(2)}dt = \Delta x_t'(\delta a + t\,\delta b) - \Delta x_0'.\,\delta a - \frac{\delta}{\delta a}\int_0^t R\,dt.\,\delta\left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R\,dt, \tag{185}$$

if we make

$$x_t' = b - \frac{\delta}{\delta a}\int_0^t R\,dt + \Delta x_t', \quad x_0' = b + \Delta x_0', \tag{186}$$

(so that $\Delta x_t'$ and $\Delta x_0'$ are here of the second order) defining a & b to be such as to satisfy exactly the equation (181) and the following

$$x_0 = a. \tag{164}$$

The formula (185) resolves itself into the 2 following

$$\left.\begin{aligned} -\frac{\delta}{\delta a}\int_0^t R^{(2)}dt &= \Delta x_t' - \Delta x_0' - \frac{\delta}{\delta a}\int_0^t R\,dt.\left(\frac{\delta^2}{\delta a\,\delta b} - t\frac{\delta^2}{\delta a^2}\right)\int_0^t R\,dt,\\ -\frac{\delta}{\delta b}\int_0^t R^{(2)}dt &= t\,\Delta x_t' - \frac{\delta}{\delta a}\int_0^t R\,dt.\left(\frac{\delta^2}{\delta b^2} - t\frac{\delta^2}{\delta a\,\delta b}\right)\int_0^t R\,dt.\end{aligned}\right\} \tag{187}$$

At the same time, the new part or part of the second order in x_t, which will result on substituting for b its value $x_0' - \Delta x_0'$, must evidently be, by (181),

$$\Delta x_t = -t\,\Delta x_0'. \tag{188}$$

We have therefore, by an easy elimination,

$$\Delta x_t = -\left(t\frac{\delta}{\delta a} - \frac{\delta}{\delta b}\right)\int_0^t R^{(2)}dt - \frac{\delta}{\delta a}\int_0^t R\,dt.\left(t\frac{\delta}{\delta a} - \frac{\delta}{\delta b}\right)^2\int_0^t R\,dt; \tag{189}$$

the notation being such that we have the symbolic equation

$$\left(t\frac{\delta}{\delta a} - \frac{\delta}{\delta b}\right)^2 = t^2\frac{\delta^2}{\delta a^2} - 2t\frac{\delta^2}{\delta a\,\delta b} + \frac{\delta^2}{\delta b^2}. \tag{190}$$

This correction (189) is therefore that which must be added to the expression (181), if we desire to treat a & b as equal to the constants x_0 and x_0'; & consequently, with this last condition, we have

$$x_t = a + bt + \left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R\,dt + \left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R^{(2)}dt - \frac{\delta}{\delta a}\int_0^t R\,dt.\left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)^2\int_0^t R\,dt; \tag{191}$$

* [The last term arises from (181), which gives $\delta x_t = \delta a + t\,\delta b + \delta\left(\frac{\delta}{\delta b} - t\frac{\delta}{\delta a}\right)\int_0^t R\,dt$.]

www.rcin.org.pl

a and b being here, as was said, the constants x_0 and x_0', while R is the old function formed from them by changing x_t to $a+bt$, and $R^{(2)}$ again is formed from R according to the rule (184).

The perturbation of the 2nd order expressed by the second line of my formula (191) is so unlike in appearance to that expressed by the sum of the last two lines of the formula (180) deduced from Lagrange's method that I thought it useful to confirm their agreement by submitting both of them to suitable reductions,* depending on the circumstance that R in each is a function of $a+bt$ alone. And thus I found from each this common transformed expression for that part of the perturbation which is of the order of the square of the disturbing force,

$$\Delta x_t = \frac{1}{b}\int_0^t \frac{\delta R}{\delta a}\,dt \,.\int_0^t\left(\int_0^t \frac{\delta R}{\delta a}\,dt\right)dt - \frac{3}{2b}\int_0^t\left(\int_0^t \frac{\delta R}{\delta a}\,dt\right)^2 dt; \qquad (192)$$

[Four pages are missing here. From a fragment it is evident that they contained a verification of these formulae for the case $x'' = \alpha\,(a+bt)$.]

.... I believe however that I need take no pains to prove to you the existence of this inconvenience in the application of Lagrange's method, for I suspect that it was to it, & to what you thought its probable occurrence in the corresponding stage of the application of my own method also, that you attended in your answer to my IInd Letter, when you said: "But a great difficulty begins in your method when you introduce the perturbational terms in R in order to get a further approximation, because you must either develope $\frac{\delta}{\delta n}\int R\,dt$, $\frac{\delta}{\delta \epsilon}\int R\,dt$, $\frac{\delta}{\delta k}\int R\,dt$, $\frac{\delta}{\delta l}\int R\,dt$ *separately* or you must do what will be equally troublesome." I did not on first reading understand this passage, because I had reflected much less on Lagrange's method than on my own; I supposed, as I still do, that you would not reckon as a serious addition to the other difficulties of the problem the more explicit *differentiation of the development of a simple function* (such as my $\int R^{(2)}dt$) even with respect to several different elements, if that one development were once effected.

On the other hand, while I think that I escape from the important difficulty of having, at each new stage of the approximation, to *develope separately as many different and independent functions as there are varying elements*, (which is the difficulty you chiefly feared from the experience of Lagrange's method,) I will not be too sanguine that some new & equally formidable difficulty of *some other kind* may not arise in applying my method, though I see no reason why it should. And it is right to state, because in a kind of appendix to my IIId Letter (dated Oct. 18th) I may seem to have slurred over the point, that in passing to a second correction & calculating that new function which answers to the $R^{(2)}$ of the present letter, it will be necessary to develope the terms which depend upon the square of the disturbing force not only in $-R$ but also in $\frac{r'^2+r^2\lambda'^2}{2}+\frac{\mu}{r}$ which accompanies it under the integral sign in the expression of my principal function, or in the formula (57).

* [If R_u denotes the value of R with u inserted for t we get

$$\int_0^t R^{(2)}\,dt = \frac{1}{2}\int_0^t \frac{(R_u-R_0)^2}{b^2}\,du - \frac{R_t-R_0}{b^2}\int_0^t (R_u-R_0)\,du,$$

$$\left(\frac{\delta}{\delta b}-t\frac{\delta}{\delta a}\right)\frac{1}{2}\int_0^t \frac{(R_u-R_0)^2}{b^2}\,du = -\frac{3}{2b^2}\int_0^t (R_u-R_0)^2\,du + \frac{t}{b^2}\frac{\delta R_0}{\delta a}\int_0^t (R_v-R_0)\,du.$$

The equation (192) follows easily.]

www.rcin.org.pl

It is now the 7th of November, this letter having been begun a week ago, but continued only at intervals since. Next week I begin to lecture (as I do every year) on Astronomy in the University of Dublin; and though I propose to introduce more of dynamics this time than I have ever done before, my lectures will still, I foresee, be, as they have always been, a very serious distraction from all private studies & researches. It is likely therefore that I may not be able to write again upon the Theory of the Moon till Christmas. But at that time perhaps I may have leisure and courage to enter on the subject again. In the mean while, I shall esteem it a great favour if you will inform me of the result of your reflexions on what I have already written; and will tell me, among other things, whether I have rightly divined the sort of process by which Poisson proposes to allow for the square of the disturbing force in applying Lagrange's method. No doubt it will be my duty to read Poisson's memoir myself, & I can procure it from the R.I. Academy—but still I wish to have your opinion too. As to foreign books in general, you can hardly conceive the difficulty I find in getting at them. For instance, our Academy, though subscribing to the *Comptes Rendus*, has not yet received a single number for 1837. Perhaps that difficulty might be a little diminished by employing for the aforesaid Academy a better London Bookseller—& you would confer a great favour on it & me, by suggesting one who would take some trouble about the matter. The present man is Boone (of the Strand, I think), very respectable I dare say and so forth, but of almost no use to us. And in the meantime, for the reason that I mentioned, I should be very glad to have a sight of what you said Jacobi has written in connexion with my speculations, if his paper could be fairly franked. I return one of the two copies of Pontécoulant's pamphlet, which is still among my *legenda*. I send also a copy of my second Essay on Dynamics & beg you to accept it. I must send another to Poisson, who in the paper that we saw at Liverpool spoke only of my first Essay & seemed, as I said, to have read only the first few pages even of that; but of course I must consider it as a compliment that he even troubled himself to read so much. Perhaps you can name some other persons, who ought to receive separate copies of my Second Essay, for the copies only reached me as I was leaving Liverpool; Pontécoulant of course ought to have one: would you take the trouble of transmitting any? And can you also send the enclosed order for a set of Italian Memoirs to the proper place in London, namely the bookseller Molini, with a request that he will send the books to me at the Royal Irish Academy, Dublin?

You will easily conceive that if I venture to attack the Theory of the Moon with the bold hope of adding anything to what Plana & you have done I shall avail myself of whatever facility may be derived from starting with a movable ellipse; & shall endeavour so to manage my formulae as to keep the time under the signs of periodicity. No doubt it will be prudent to conform in all things possible to your notation, & your choice of coordinates & of elements such as $\frac{1}{r}$ & e instead of r & k. But I dare not pledge myself to voyage out of sight of land upon so vast a sea of labour.

With best regards to Mrs Lubbock, I am, my dear Sir,

very sincerely yours

William R. Hamilton.

[The correspondence closes with a letter from J. W. Lubbock dated 12th Nov., 1837, the extant parts of which are of no mathematical interest.]

www.rcin.org.pl