VI.

THEORY OF THE MOON*

[1837.]

[Note Book 48.]

Dunraven Castle, Sept[r] 25, 1837.

Let x, y, z be the Moon's geocentric rectangular coordinates; m Moon's mass; $x_,, y_,, z_,$ those of the Sun; $m_,$ Sun's mass; $\mu =$ sum of masses of Moon and Earth. Then

$$(1)\quad \begin{cases} 0 = x'' + \mu x (x^2+y^2+z^2)^{-\frac{3}{2}} + m_, x_, (x_,^2+y_,^2+z_,^2)^{-\frac{3}{2}} + m_, (x - x_,) \\ \qquad\qquad \times \{(x-x_,)^2 + (y-y_,)^2 + (z-z_,)^2\}^{-\frac{3}{2}}; \\ 0 = y'' + \\ 0 = z'' + \end{cases}$$

and

$$(2)\quad \begin{cases} 0 = x_,'' + (\mu - m + m_,) x_, (x_,^2+y_,^2+z_,^2)^{-\frac{3}{2}} + mx (x^2+y^2+z^2)^{-\frac{3}{2}} + m (x_, - x) \\ \qquad\qquad \times \{(x-x_,)^2 + (y-y_,)^2 + (z-z_,)^2\}^{-\frac{3}{2}}; \\ 0 = y_,'' + \\ 0 = z_,'' + \end{cases}$$

(Hence

$$0 = mx'' + m_, x_,'' + (\mu + m_,) \{mx (x^2+y^2+z^2)^{-\frac{3}{2}} + m_, x_, (x_,^2+y_,^2+z_,^2)\}^{-\frac{3}{2}}, \qquad , \qquad .)$$

If m (= Moon's mass) be extremely small, we shall have, very nearly, for the Sun's motion, the equations of elliptic motion,

$$(3)\qquad 0 = x_,'' + (\mu + m_,) x_, (x_,^2+y_,^2+z_,^2)^{-\frac{3}{2}}, \quad 0 = \qquad , \quad 0 = \qquad ;$$

& may conceive $x_,, y_,, z_,$ expressed thereby as functions of the time t. (And as a mathematical problem we may propose to integrate the 3 equations (1), considering $x_,, y_,, z_,$ as explicit functions of t, which satisfy these 3 equations (3). Physically, this corresponds to seeking the limiting state of motion of the Moon, when the Moon's mass is considered to tend to the limiting value 0.)

By easy combination of (1) and (3) we get

$$(4)\quad 0 = x'' - \frac{m_, x_,''}{\mu + m_,} + \mu x (x^2+y^2+z^2)^{-\frac{3}{2}} + m_, (x - x_,) \{(x-x_,)^2 + (y-y_,)^2 + (z-z_,)^2\}^{-\frac{3}{2}};$$

$\because$, if $\delta t = 0$ & if

$$(5)\quad s = \int_0^t \left\{ \frac{1}{2} \left(x' - \frac{m_, x_,'}{\mu + m_,} \right)^2 + \quad + \quad + \mu (x^2+y^2+z^2)^{-\frac{1}{2}} + m_, (\overline{x - x_,}^2 + \overline{y - y_,}^2 + \overline{z - z_,}^2)^{-\frac{1}{2}} \right\} dt,$$

$$(6)\qquad \delta s = \left(x' - \frac{m_, x_,'}{\mu + m_,} \right) \delta x + \quad + \quad - \left(x_0' - \frac{m_, x_{,0}'}{\mu + m_,} \right) \delta x_0 - \quad - \quad .$$

* [See also Correspondence with J. W. Lubbock, pp. 249 *et seq.*]

This last formula of variation would have resulted equally if instead of (5) we had put

$$(7)\quad s=\int_0^t\left\{\frac{x'^2+y'^2+z'^2}{2}+\mu(x^2+y^2+z^2)^{-\frac{1}{2}}+m_,\left(-\frac{x'x'_,+y'y'_,+z'z'_,}{\mu+m_,}+(\overline{x-x_,}^2+\overline{y-y_,}^2+\overline{z-z_,}^2)^{-\frac{1}{2}}\right)\right\}dt.$$

Besides, the part

$$-\frac{m_,}{\mu+m_,}\int_0^t(x'x'_,+y'y'_,+z'z'_,)\,dt=-\frac{m_,}{\mu+m_,}(xx'_,+yy'_,+zz'_,-x_0x'_{,0}-y_0y'_{,0}-z_0z'_{,0})$$
$$+\frac{m_,}{\mu+m_,}\int_0^t(xx''_,+yy''_,+zz''_,)\,dt=-\frac{m_,}{\mu+m_,}(xx'_,+\quad+\quad-x_0x'_{,0}-\quad-\quad)$$
$$-m_,\int_0^t\frac{xx_,+yy_,+zz_,}{(x_,^2+y_,^2+z_,^2)^{\frac{3}{2}}}dt.$$

If then we put*

$$(8)\qquad R=m_,\left\{\frac{xx_,+yy_,+zz_,}{(x_,^2+y_,^2+z_,^2)^{\frac{3}{2}}}-(\overline{x-x_,}^2+\overline{y-y_,}^2+\overline{z-z_,}^2)^{-\frac{1}{2}}\right\},$$

we shall have, by (7),

$$(9)\qquad s=\int_0^t\left(\frac{x'^2+y'^2+z'^2}{2}+\mu(x^2+y^2+z^2)^{-\frac{1}{2}}-R\right)dt-\frac{m_,}{\mu+m_,}(xx'_,+yy'_,+zz'_,-x_0x'_{,0}-y_0y'_{,0}-z_0z'_{,0});$$

and hence by (6) we have

$$(\mathrm{A})\ldots\quad \delta\int\left(\frac{x'^2+y'^2+z'^2}{2}+\mu(x^2+y^2+z^2)^{-\frac{1}{2}}-R\right)dt=x'\delta x+y'\delta y+z'\delta z-x'_0\delta x_0-y'_0\delta y_0-z'_0\delta z_0.$$

The only thing neglected in this very simple formula is the mass of the Moon; by neglecting which we are able to treat the Sun's coordinates $x_,, y_,, z_,$ in R as explicit functions of the time t. Accordingly when the Moon's mass is thus neglected, & δt is made $=0$, the formula (A) results at once from the 3 known equations

$$(\mathrm{B})\ldots\qquad x''+\mu x(x^2+y^2+z^2)^{-\frac{3}{2}}+\frac{\delta R}{\delta x}=0,\qquad ,\qquad .$$

In fact if we multiply the three equations (B) by δx, δy, δz and add the products, we get

$$x''\delta x+y''\delta y+z''\delta z=\mu\delta(x^2+y^2+z^2)^{-\frac{1}{2}}-\delta R;$$

∵ because

$$x'(\delta x)'+y'(\delta y)'+z'(\delta z)'=\delta\frac{x'^2+y'^2+z'^2}{2},$$

we have

$$\Delta_{t=t_0}^{t=t}(x'\delta x+y'\delta y+z'\delta z)=\delta\int_0^t\left(\frac{x'^2+y'^2+z'^2}{2}+\mu(x^2+y^2+z^2)^{-\frac{1}{2}}-R\right)dt.$$

Let $\qquad x=r\cos\lambda,\quad y=r\sin\lambda,\quad z=0,$

$$x_,=a_,\cos n_,t,\quad y_,=a_,\sin n_,t,\quad z_,=0,\quad a_,^2n_,^3=\mu+m_,;$$

so that the Sun's orbit is now supposed to be circular and the inclination of the Moon's orbit is neglected.

Then the fundamental formula (A) becomes

$$(\mathrm{C})\ldots\qquad \delta\int_0^t\left(\frac{r'^2+r^2\lambda'^2}{2}+\frac{\mu}{r}-R\right)dt=r'\delta r+r^2\lambda'\delta\lambda-r'_0\delta r_0-r_0^2\lambda'_0\delta\lambda_0;$$

* [See p. 51.]

also

(D)... $$R = m_{,}\left\{\frac{r\cos(\lambda - n_{,}t)}{a_{,}^2} - \{a_{,}^2 - 2a_{,}r\cos(\lambda - n_{,}t) + r^2\}^{-\frac{1}{2}}\right\}.$$

At the same time the differential equations (B) become

(E)... $$r'' - r\lambda'^2 + \frac{\mu}{r^2} + \frac{\delta R}{\delta r} = 0; \quad (r^2\lambda')' + \frac{\delta R}{\delta\lambda} = 0.$$

Besides we may put, with a great degree of approximation,* $m_{,} = a_{,}^3 n_{,}^2$, neglecting Earth's mass in comparison with Sun's; and then if in general we make $(1 - 2\alpha p + \alpha^2)^{-\frac{1}{2}} = 1 + \alpha P_1 + \alpha^2 P_2 +$ &c., and put also $p = \cos\overline{\lambda - n_{,}t}$, we shall then have

$$-R = a_{,}^2 n_{,}^2 + n_{,}^2\left(r^2 P_2 + \frac{r^3}{a_{,}}P_3 + \frac{r^4}{a_{,}^2}P_4 + \text{\&c.}\right).$$

Hence,

(F)... $$\delta\int_0^t\left(\frac{r'^2 + r^2\lambda'^2}{2} + \frac{\mu}{r}\right)dt + n_{,}^2\delta\int_0^t\left(r^2P_2 + \frac{r^3}{a_{,}}P_3 + \frac{r^4}{a_{,}^2}P_4 + \text{\&c.}\right)dt$$
$$= r'\delta r + r^2\lambda'\delta\lambda - r_0'\delta r_0 - r_0^2\lambda_0'\delta\lambda_0.$$

As a first approximation we may neglect $n_{,}$; (which comes to treating the month as very small in comparison with the year;) & then we may take elliptic values for r and λ, (that is, for the moon's geocentric radius vector and geocentric longitude,) which values, if we retain only the 1st power of the excentricity e, will be

$$r = a - ae\cos(nt + \epsilon - \varpi) = a - ae\cos\xi, \quad (\text{if } \xi = nt + \epsilon - \varpi,)$$
$$\lambda = nt + \epsilon + 2e\sin(nt + \epsilon - \varpi) = nt + \epsilon + 2e\sin\xi;$$

also $\mu = a^3n^2$.

Accordingly these expressions give, when we put

$$k = e\cos(\epsilon - \varpi), \quad l = e\sin(\epsilon - \varpi),$$
$$r = a(1 - k\cos nt + l\sin nt), \quad \lambda = nt + \epsilon + 2k\sin nt + 2l\cos nt,$$
$$\lambda' = n(1 + 2k\cos nt - 2l\sin nt), \quad \tfrac{1}{2}r^2\lambda'^2 = a^2n^2(\tfrac{1}{2} + k\cos nt - l\sin nt),$$

& $\because$

$$\delta\int_0^t\left(\frac{r'^2 + r^2\lambda'^2}{2} + \frac{\mu}{r}\right)dt = \delta\{\tfrac{3}{2}a^2n^2t + 2a^2n(k\sin nt + l\,\overline{\cos nt - 1})\}$$
$$= a^2n(t\delta n + 2\delta k\sin nt + 2\delta l\,\overline{\cos nt - 1}) = a^2n\,\delta(\lambda - \lambda_0) = a^2n\,\delta\lambda - a^2n\,\delta\lambda_0$$
$$= r^2\lambda'\delta\lambda - r_0^2\lambda_0'\delta\lambda_0 = r'\delta r + r^2\lambda'\delta\lambda - r_0'\delta r_0 - r_0^2\lambda_0'\delta\lambda_0.$$

In these last equations it has been supposed that k and l vanish after the act of variation δ has been performed, but if we even retain terms of the 1st dimension with respect to k & l, we find that the differential equations

$$r'' - r\lambda'^2 + \frac{\mu}{r^2} = 0 \quad \text{and} \quad (r^2\lambda')' = 0 \text{ are satisfied.}$$

* $\left[a_{,}^3n_{,}^2 = m_{,}\left(1 + \frac{\mu}{m_{,}}\right)\right.$. We have the following approximate values: $\frac{\mu}{m_{,}} = \frac{1}{330{,}000}$; $\frac{n_{,}}{n} = \frac{1}{13}$; $e = \frac{1}{20}$; $\frac{a}{a_{,}} = \frac{1}{400}$ the tangent of the inclination of the Moon's orbit $= \frac{1}{11}$ and the excentricity of the Sun's orbit $= \frac{1}{60}$. Brown, *Lunar Theory*, pp. 42, 80.]

As a second approximation we may take the differential equations

$$0=r''-r\lambda'^2+\frac{\mu}{r^2}+\frac{\delta R}{\delta r},\quad 0=(r^2\lambda')'+\frac{\delta R}{\delta\lambda},$$

retaining in R only the term

$$-n_,^2r^2P_2=-\frac{n_,^2r^2}{4}(1+3\cos\overline{2\lambda-2n_,t}).$$

At the same time,

$$\delta\int_0^t\left(\frac{r'^2+r^2\lambda'^2}{2}+\frac{\mu}{r}\right)dt+\tfrac{1}{4}n_,^2\delta\int_0^t r^2(1+3\cos\overline{2\lambda-2n_,t})\,dt=r'\delta r+r^2\lambda'\delta\lambda-r_0'\delta r_0-r_0^2\lambda_0'\delta\lambda_0.$$

We are now to employ the expressions

$$r=a(1-k\cos nt+l\sin nt),\quad \lambda=nt+\epsilon+2k\sin nt+2l\cos nt;$$

but are no longer to consider a, n, ϵ, k, l as constant. We are however to suppose $r_0=a(1-k)$, $\lambda_0=\epsilon+2l$; but r', λ', r_0', λ_0' will now have new values, at least in the 2nd member of the formula $\delta\int$ &c. $=r'\delta r+$ &c., though in the 1st member of that formula the values of r' and λ' remain unaltered. We may ∵ make, in the 2nd member of that formula, (if we neglect $n_,^2k$ and $n_,^2l$)

$$r'=\Delta r',\quad \lambda'=n+\Delta\lambda',\quad r_0'=\Delta r_0',\quad \lambda_0'=n+\Delta\lambda_0',\quad r=r_0=a,$$

$$\delta r=\delta a-a\delta k.\cos nt+a\delta l.\sin nt,\quad \delta\lambda=t\delta n+\delta\epsilon+2\delta k.\sin nt+2\delta l.\cos nt,$$

$$\delta r_0=\delta a-a\delta k,\quad \delta\lambda_0=\delta\epsilon+2\delta l,$$

& we may suppress in it the part $a^2n(\delta\lambda-\delta\lambda_0)$, if we suppress in the 1st member the part

$$\delta\int_0^t\left(\frac{r'^2+r^2\lambda'^2}{2}+\frac{\mu}{r}\right)dt;$$

by which means it becomes*

$$\Delta r'.(\delta a-a\delta k.\cos nt+a\delta l.\sin nt)+a^2\Delta\lambda'.(t\delta n+\delta\epsilon+2\delta k.\sin nt+2\delta l.\cos nt)$$
$$-\Delta r_0'.(\delta a-a\delta k)-a^2\Delta\lambda_0'.(\delta\epsilon+2\delta l)=\tfrac{1}{4}n_,^2\delta\int_0^t r^2(1+3\cos\overline{2\lambda-2n_,t})\,dt=\delta s;$$

that is, after expressing $s=\tfrac{1}{4}n_,^2\int_0^t r^2(1+3\cos\overline{2\lambda-2n_,t})\,dt$

as a function of a, n, ϵ, k, l, t & taking its variation relatively to n, ϵ, k, l, (observing that $a=(\mu/n^2)^{-\frac{1}{3}}$ & ∵ $\delta a=-\frac{2}{3}\frac{a}{n}\delta n$,)

$$\frac{\delta s}{\delta n}=-\frac{2}{3}\frac{a}{n}(\Delta r'-\Delta r_0')+a^2t\Delta\lambda';\quad \frac{\delta s}{\delta\epsilon}=a^2(\Delta\lambda'-\Delta\lambda_0');$$

$$\frac{\delta s}{\delta k}=-a(\Delta r'.\cos nt-\Delta r_0')+2a^2\Delta\lambda'.\sin nt;\quad \frac{\delta s}{\delta l}=a\Delta r'.\sin nt+2a^2(\Delta\lambda'.\cos nt-\Delta\lambda_0').$$

These four equations will give, by elimination, expressions for $\Delta r_0'$ and $\Delta\lambda_0'$ of the forms

$$R_1\frac{\delta s}{\delta n}+R_2\frac{\delta s}{\delta\epsilon}+R_3\frac{\delta s}{\delta k}+R_4\frac{\delta s}{\delta l}\quad\text{and}\quad L_1\frac{\delta s}{\delta n}+\dots+L_4\frac{\delta s}{\delta l}.$$

R_1, ... L_4 being functions of n & t; while $\frac{\delta s}{\delta n}$, ... $\frac{\delta s}{\delta l}$ are functions of n, t and $n_,$.

* [See First Essay, p. 161, (G^8.).]

Besides, if we neglect the products of n, k, l, we may retain the expressions thus found for $\Delta r_0'$ and $\Delta\lambda_0'$ and substitute them in these new equations

$$r_0' = anl + \Delta r_0', \quad \lambda_0' = n(1+2k) + \Delta\lambda_0';$$

with which we are then to combine the 2 equations

$$r_0 = a(1-k), \quad \lambda_0 = \epsilon + 2l, \quad (\text{and } a^3 n^2 = \mu,)$$

in order to get n, ϵ, k, l (and a) as functions of $r_0, \lambda_0, r_0', \lambda_0', t$ and n,: which functions are then to be substituted in the expressions

$$r = a(1 - k\cos nt + l\sin nt), \quad \lambda = nt + \epsilon + 2k\sin nt + 2l\cos nt.$$

To effect this substitution, it is convenient* to change a and n, ϵ, k, l to $a+\Delta a$, $n+\Delta n$, $\epsilon+\Delta\epsilon$, $k+\Delta k$, $l+\Delta l$, and to establish the equations

$$r_0 = a(1-k), \quad \lambda_0 = \epsilon + 2l, \quad r_0' = anl, \quad \lambda_0' = n(1+2k), \quad a^3n^2 = \mu,$$

and

$$0 = \Delta a - a\Delta k, \quad 0 = \Delta\epsilon + 2\Delta l, \quad 0 = an\Delta l + \Delta r_0',$$

$$0 = \Delta n + 2n\Delta k + \Delta\lambda_0', \quad \Delta a = -2a\Delta n/3n;$$

after which we shall have

$$\Delta r = \Delta a - a\Delta k\,.\cos nt + a\Delta l\,.\sin nt; \quad \Delta\lambda = t\Delta n + \Delta\epsilon + 2\Delta k\,.\sin nt + 2\Delta l\,.\cos nt,$$

and finally

$$r = a(1 - k\cos nt + l\sin nt) + \Delta r, \quad \lambda = nt + \epsilon + 2k\sin nt + 2l\cos nt + \Delta\lambda;$$

in which expressions a and n, ϵ, k, l are constants independent of the time t.

In this manner we have

$$\Delta a = a\Delta k; \quad \Delta n = -\frac{3n}{2}\Delta k; \quad \Delta k = -\frac{2}{n}\Delta\lambda_0';$$

$$\because \quad \Delta n = 3\Delta\lambda_0', \quad \Delta a = -\frac{2a}{n}\Delta\lambda_0'; \quad \text{and} \quad \Delta l = -\frac{1}{an}\Delta r_0', \quad \Delta\epsilon = \frac{2}{an}\Delta r_0';$$

therefore

$$\Delta r = -\frac{2a}{n}(1-\cos nt)\Delta\lambda_0' - \frac{1}{n}\sin nt\,.\Delta r_0';$$

$$\Delta\lambda = \left(3t - \frac{4\sin nt}{n}\right)\Delta\lambda_0' + \frac{2}{an}(1-\cos nt)\Delta r_0'.$$

Besides, by the equations near the foot of page 241, we have

$$\Delta\lambda' = \Delta\lambda_0' + \frac{1}{a^2}\frac{\delta s}{\delta\epsilon}, \quad \Delta r' = \Delta r_0' + \tfrac{3}{2}ant\left(\Delta\lambda_0' + \frac{1}{a^2}\frac{\delta s}{\delta\epsilon}\right) - \frac{3n}{2a}\frac{\delta s}{\delta n};$$

* [Six variable parameters are introduced: $\Delta r_0'$, $\Delta\lambda_0'$, Δn (or Δa), $\Delta\epsilon$, Δk, Δl. It is possible therefore to prescribe *four* relations between them. The four given above are not the usual relations employed. Cf. Appendix, Note 7, p. 628.]

also, $$\sin nt\,.\,\frac{\delta s}{\delta k}+\cos nt\,.\,\frac{\delta s}{\delta l}=2a^2(\Delta\lambda'-\cos nt\,\Delta\lambda'_0)+a\,\Delta r'_0\sin nt$$

$$=2a^2(1-\cos nt)\,\Delta\lambda'_0+a\sin nt\,\Delta r'_0+2\frac{\delta s}{\delta\epsilon}=-an\,\Delta r+2\frac{\delta s}{\delta\epsilon},$$

and

$$\cos nt\,.\,\frac{\delta s}{\delta k}-\sin nt\,.\,\frac{\delta s}{\delta l}=-a\,\Delta r'+a\,\Delta r'_0\cos nt+2a^2\Delta\lambda'_0\sin nt$$

$$=-a(1-\cos nt)\,\Delta r'_0+a^2(-\tfrac{3}{2}nt+2\sin nt)\,\Delta\lambda'_0-\tfrac{3}{2}nt\frac{\delta s}{\delta\epsilon}+\frac{3n}{2}\frac{\delta s}{\delta n}$$

$$=-\frac{a^2n}{2}\Delta\lambda-\frac{3nt}{2}\frac{\delta s}{\delta\epsilon}+\frac{3n}{2}\frac{\delta s}{\delta n};$$

therefore

$$\boxed{\begin{aligned}\Delta r&=\frac{2}{an}\frac{\delta s}{\delta\epsilon}-\frac{1}{an}\left(\sin nt\frac{\delta s}{\delta k}+\cos nt\frac{\delta s}{\delta l}\right),\\ \Delta\lambda&=\frac{3}{a^2}\left(\frac{\delta s}{\delta n}-t\frac{\delta s}{\delta\epsilon}\right)-\frac{2}{a^2n}\left(\cos nt\frac{\delta s}{\delta k}-\sin nt\frac{\delta s}{\delta l}\right).\end{aligned}}$$

To calculate s, we have

$$r^2=a^2(1-2k\cos nt+2l\sin nt),$$

$$2\lambda-2n_{,}t=2(n-n_{,})t+2\epsilon+4k\sin nt+4l\cos nt;$$

$$1+3\cos(2\lambda-2n_{,}t)=1+3\cos(2nt-2n_{,}t+2\epsilon)-12(k\sin nt+l\cos nt)\sin(2nt-2n_{,}t+2\epsilon);$$

$$\because\quad\frac{4s'}{a^2n_{,}^2}=\{1+3\cos(2nt-2n_{,}t+2\epsilon)\}\{1-2k\cos nt-2l\sin nt\}$$
$$-12(k\sin nt+l\cos nt)\sin(2nt-2n_{,}t+2\epsilon)$$

$$=1+3\cos(2nt-2n_{,}t+2\epsilon)-2k\cos nt+2l\sin nt$$
$$+3k\{\cos(3nt-2n_{,}t+2\epsilon)-3\cos(nt-2n_{,}t+2\epsilon)\}$$
$$-3l\{\sin(3nt-2n_{,}t+2\epsilon)+3\sin(nt-2n_{,}t+2\epsilon)\};$$

$$\because\quad\frac{4s}{n_{,}^2}=a^2t+\frac{3a^2}{2n-2n_{,}}\{\sin(2nt-2n_{,}t+2\epsilon)-\sin 2\epsilon\}$$
$$+ka^2\left\{-\frac{2}{n}\sin nt+\frac{3}{3n-2n_{,}}\{\sin(3nt-2n_{,}t+2\epsilon)-\sin 2\epsilon\}\right.$$
$$\left.-\frac{9}{n-2n_{,}}\{\sin(nt-2n_{,}t+2\epsilon)-\sin 2\epsilon\}\right\}$$
$$+la^2\left\{\frac{2}{n}-\frac{2}{n}\cos nt+\frac{3}{3n-2n_{,}}\{\cos(3nt-2n_{,}t+2\epsilon)-\cos 2\epsilon\}\right.$$
$$\left.+\frac{9}{n-2n_{,}}\{\cos(nt-2n_{,}t+2\epsilon)-\cos 2\epsilon\}\right\};$$

$$\because\quad\frac{4n}{an_{,}^2}\Delta r=\frac{6}{n-n_{,}}(\cos\overline{2nt-2n_{,}t+2\epsilon}-\cos 2\epsilon)+\frac{2}{n}(1-\cos nt)$$
$$-\frac{3}{3n-2n_{,}}(\cos\overline{2nt-2n_{,}t+2\epsilon}-\cos\overline{nt-2\epsilon})$$
$$-\frac{9}{n-2n_{,}}(\cos\overline{2nt-2n_{,}t+2\epsilon}-\cos\overline{nt+2\epsilon});$$

and $$\frac{4n}{n_,^2}\Delta\lambda = -4\left\{t+\frac{3}{2n-2n_,}(\sin\overline{2nt-2n_,t+2\epsilon}-\sin 2\epsilon)\right\}+\frac{9nt\cos 2\epsilon}{n-n_,}$$
$$-\frac{9n}{2(n-n_,)^2}(\sin\overline{2nt-2n_,t+2\epsilon}-\sin 2\epsilon)+\frac{4}{n}\sin nt$$
$$-\frac{6}{3n-2n_,}(\sin\overline{2nt-2n_,t+2\epsilon}+\sin\overline{nt-2\epsilon})$$
$$+\frac{18}{n-2n_,}(\sin\overline{2nt-2n_,t+2\epsilon}-\sin\overline{nt+2\epsilon}).$$

(As verifications, these expressions for Δr and $\Delta\lambda$ should not only vanish themselves when $t=0$, which they evidently do, but also their differential coefficients, taken with respect to t, should vanish at the same time; we ought ∵ to have
$$0=-12+3+9 \quad \& \quad 0=-4+4+\cos 2\epsilon(-12-6+18);$$
& so we have.)

It results then from the foregoing calculations that if squares & products of $n_,^2$, k, l be neglected, the two differential equations of the 2nd order
$$0=r''-r\lambda'^2+\frac{\mu}{r^2}-\frac{n_,^2 r}{2}(1+3\cos\overline{2\lambda-2n_,t}), \qquad 0=(r^2\lambda')'+\frac{3n_,^2r^2}{2}\sin\overline{2\lambda-2n_,t},$$
(in which μ and $n_,$ are constant,) admit of having their integrals expressed as follows:
$$r=a-ak\cos nt+al\sin nt+\frac{a\,n_,^2}{2\,n^2}(1-\cos nt)$$
$$+\frac{3an_,^2}{4n}\left\{\frac{2}{n-n_,}(\cos\overline{2nt-2n_,t+2\epsilon}-\cos 2\epsilon)-\frac{1}{3n-2n_,}(\cos\overline{2nt-2n_,t+2\epsilon}-\cos\overline{nt-2\epsilon})\right.$$
$$\left.-\frac{3}{n-2n_,}(\cos\overline{2nt-2n_,t+2\epsilon}-\cos\overline{nt+2\epsilon})\right\};$$
$$\lambda=nt+\epsilon+2k\sin nt+2l\cos nt-\frac{n_,^2t}{n}\left(1-\frac{9n\cos 2\epsilon}{4(n-n_,)}\right)+\frac{n_,^2}{n^2}\sin nt$$
$$-\frac{3n_,^2}{4n}\left\{\left(\frac{2}{n-n_,}+\frac{3n}{2(n-n_,)^2}\right)(\sin\overline{2nt-2n_,t+2\epsilon}-\sin 2\epsilon)\right.$$
$$\left.+\frac{2}{3n-2n_,}(\sin\overline{2nt-2n_,t+2\epsilon}+\sin\overline{nt-2\epsilon})-\frac{6}{n-2n_,}(\sin\overline{2nt-2n_,t+2\epsilon}-\sin\overline{nt+2\epsilon})\right\};$$
in which a and n, ϵ, k, l are 5 arbitrary constants determinable by the 5 conditions
$$r_0=a-ak, \quad \lambda_0=\epsilon+2l, \quad r_0'=anl, \quad \lambda_0'=n+2nk, \quad a^3n^2=\mu.$$

These expressions may be put under the forms
$$r=a\left\{1+\frac{n_,^2}{2n^2}\left(1-\frac{3n\cos 2\epsilon}{n-n_,}\right)\right\}$$
$$-a\cos nt\left\{k+\frac{n_,^2}{2n^2}\left(1-\frac{3n}{2}\left(\frac{1}{3n-2n_,}+\frac{3}{n-2n_,}\right)\cos 2\epsilon\right)\right\}$$
$$+a\sin nt\left\{l+\frac{3n_,^2}{4n}\left(\frac{1}{3n-2n_,}-\frac{3}{n-2n_,}\right)\sin 2\epsilon\right\}$$
$$+\frac{3an_,^2}{4n}\left(\frac{2}{n-n_,}-\frac{1}{3n-2n_,}-\frac{3}{n-2n_,}\right)\cos(2nt-2n_,t+2\epsilon);$$

$$\lambda = nt\left\{1-\frac{n_{,}^2}{n^2}\left(1-\frac{9n\cos 2\epsilon}{4(n-n_{,})}\right)\right\}$$

$$+\epsilon+\frac{3n_{,}^2}{4n}\left(\frac{2}{n-n_{,}}+\frac{3n}{2(n-n_{,})^2}\right)\sin 2\epsilon$$

$$+2\sin nt\left\{k+\frac{n_{,}^2}{2n^2}\left(1-\frac{3n}{2}\left(\frac{1}{3n-2n_{,}}+\frac{3}{n-2n_{,}}\right)\cos 2\epsilon\right)\right\}$$

$$+2\cos nt\left\{l+\frac{3n_{,}^2}{4n}\left(\frac{1}{3n-2n_{,}}-\frac{3}{n-2n_{,}}\right)\sin 2\epsilon\right\}$$

$$-\frac{3n_{,}^2}{4n}\left(\frac{2}{n-n_{,}}+\frac{3n}{2(n-n_{,})^2}+\frac{2}{3n-2n_{,}}-\frac{6}{n-2n_{,}}\right)\sin(2nt-2n_{,}t+2\epsilon).$$

In the same order of approximation, if we put

$$\mathrm{n}=n-\frac{n_{,}^2}{n}\left(1-\frac{9n\cos 2\epsilon}{4(n-n_{,})}\right);\quad \mathrm{a}=\sqrt[3]{\frac{\mu}{\mathrm{n}^2}}\left\{=a+\frac{2n_{,}^2 a}{3n^2}\left(1-\frac{9n\cos 2\epsilon}{4(n-n_{,})}\right)\right\};$$

$$\mathrm{k}=k+\frac{n_{,}^2}{2n^2}\left(1-\frac{3n}{2}\left(\frac{1}{3n-2n_{,}}+\frac{3}{n-2n_{,}}\right)\cos 2\epsilon\right);\quad \mathrm{l}=l+\frac{3n_{,}^2}{4n}\left(\frac{1}{3n-2n_{,}}-\frac{3}{n-2n_{,}}\right)\sin 2\epsilon;$$

$$\mathrm{e}=\epsilon+\frac{3n_{,}^2}{2n(n-n_{,})}\left(1+\frac{3n}{4(n-n_{,})}\right)\sin 2\epsilon;$$

we shall have

$$r=\mathrm{a}\left\{1-\frac{n_{,}^2}{6\mathrm{n}^2}-\mathrm{k}\cos\mathrm{n}t+\mathrm{l}\sin\mathrm{n}t\right\}$$

$$+\frac{3\mathrm{a}n_{,}^2}{4\mathrm{n}}\left(\frac{2}{\mathrm{n}-n_{,}}-\frac{1}{3\mathrm{n}-2n_{,}}-\frac{3}{\mathrm{n}-2n_{,}}\right)\cos(2\mathrm{n}t-2n_{,}t+2\mathrm{e});$$

$$\lambda=\mathrm{n}t+\mathrm{e}+2\mathrm{k}\sin\mathrm{n}t+2\mathrm{l}\cos\mathrm{n}t$$

$$-\frac{3n_{,}^2}{4\mathrm{n}}\left(\frac{2}{\mathrm{n}-n_{,}}+\frac{3\mathrm{n}}{2(\mathrm{n}-n_{,})^2}+\frac{2}{3\mathrm{n}-2n_{,}}-\frac{6}{\mathrm{n}-2n_{,}}\right)\sin(2\mathrm{n}t-2n_{,}t+2\mathrm{e}).$$

Developing, we have

$$\frac{2}{\mathrm{n}-n_{,}}=\frac{2}{\mathrm{n}}(1+m);\quad -\frac{1}{3\mathrm{n}-2n_{,}}=-\frac{1}{3\mathrm{n}}(1+\tfrac{2}{3}m);\quad -\frac{3}{\mathrm{n}-2n_{,}}=-\frac{3}{\mathrm{n}}(1+2m);$$

if we put for abbreviation $m=\frac{n_{,}}{\mathrm{n}}$, & neglect m^2; $\because$ sum $=-\frac{4}{3\mathrm{n}}-\frac{38m}{9\mathrm{n}}$; and similarly

$$\frac{2}{\mathrm{n}-n_{,}}=\frac{2}{\mathrm{n}}(1+m);\quad \frac{3\mathrm{n}}{2(\mathrm{n}-n_{,})^2}=\frac{3}{2\mathrm{n}}(1+2m);\quad \frac{2}{3\mathrm{n}-2n_{,}}=\frac{2}{3\mathrm{n}}(1+\tfrac{2}{3}m);\quad -\frac{6}{\mathrm{n}-2n_{,}}=-\frac{6}{\mathrm{n}}(1+2m);$$

$\because$ sum $=-\frac{11}{6\mathrm{n}}-\frac{59m}{9\mathrm{n}}$; $\because$ putting for abbreviation $\tau=\mathrm{n}t-n_{,}t+\mathrm{e}$, we have

$$r=\mathrm{a}\left\{1-\frac{m^2}{6}-\mathrm{k}\cos\mathrm{n}t+\mathrm{l}\sin\mathrm{n}t-m^2\left(1+\frac{19m}{6}\right)\cos 2\tau\right\},$$

$$\lambda=\mathrm{n}t+\mathrm{e}+2(\mathrm{k}\sin\mathrm{n}t+\mathrm{l}\cos\mathrm{n}t)+m^2\left(\frac{11}{8}+\frac{59m}{12}\right)\sin 2\tau;$$

and accordingly these agree, so far as they go, with the expressions of M. Plana, as cited by Mr Lubbock,* in the Appendix to the 1st Part of his Theory of the Moon.

We might have commenced our 2nd approximation by retaining in R the two terms $-n_{,}^2 r^2 P_2$ and $-\frac{n_{,}^2}{a_{,}} r^3 P_3$; in which

$$P_3 = \frac{1}{6}\left(\frac{d}{dp}\right)^3 \left(\frac{p^2-1}{2}\right)^3 = \tfrac{1}{48}(p^6 - 3p^4 + 3p^2 - 1)''' = \frac{5p^3 - 3p}{2} = \tfrac{5}{2}\cos \overline{\lambda - n_{,}t}^3 - \tfrac{3}{2}\cos\overline{\lambda - n_{,}t}$$

$$= \frac{5\cos\overline{3\lambda - 3n_{,}t} + 3\cos\overline{\lambda - n_{,}t}}{8};$$

so that we should thus have had

$$s = \frac{n_{,}^2}{4}\int_0^t r^2 (1 + 3\cos\overline{2\lambda - 2n_{,}t})\,dt + \frac{n_{,}^2}{8a_{,}}\int_0^t r^3 (3\cos\overline{\lambda - n_{,}t} + 5\cos\overline{3\lambda - 3n_{,}t})\,dt,$$

while Δr and $\Delta\lambda$ would still have been given by the formulae of page 243,

$$\Delta r = \frac{1}{an}\left(2\frac{\delta s}{\delta\epsilon} - \sin nt\frac{\delta s}{\delta k} - \cos nt\frac{\delta s}{\delta l}\right), \quad \Delta\lambda = \frac{1}{a^2 n}\left(3n\frac{\delta s}{\delta n} - 3nt\frac{\delta s}{\delta\epsilon} - 2\cos nt\frac{\delta s}{\delta k} + 2\sin nt\frac{\delta s}{\delta l}\right).$$

If then we put

$$s' = \int_0^t r^3 (3\cos\overline{\lambda - n_{,}t} + 5\cos\overline{3\lambda - 3n_{,}t})\,dt,$$

and

$$r' = 2\frac{\delta s'}{\delta\epsilon} - \sin nt\frac{\delta s'}{\delta k} - \cos nt\frac{\delta s'}{\delta l}, \quad \lambda' = 3n\frac{\delta s'}{\delta n} - 3nt\frac{\delta s'}{\delta\epsilon} - 2\cos nt\frac{\delta s'}{\delta k} + 2\sin nt\frac{\delta s'}{\delta l},$$

we shall only have to add $\frac{n_{,}^2 r'}{8aa_{,}n}$ and $\frac{n_{,}^2\lambda'}{8a^2 a_{,}n}$ to the values already found for r and λ.

In developing s' we are to use for r and λ their 1st approximate values

$$r = a - ak\cos nt + al\sin nt, \quad \lambda = nt + \epsilon + 2k\sin nt + 2l\cos nt,$$

which give

$$r^3 = a^3(1 - 3k\cos nt + 3l\sin nt),$$

$$3\cos\overline{\lambda - n_{,}t} = 3\cos\overline{nt - n_{,}t + \epsilon} - 6(k\sin nt + l\cos nt)\sin\overline{nt - n_{,}t + \epsilon},$$

$$5\cos\overline{3\lambda - 3n_{,}t} = 5\cos\overline{3nt - 3n_{,}t + 3\epsilon} - 30(k\sin nt + l\cos nt)\sin\overline{3nt - 3n_{,}t + 3\epsilon},$$

$$\frac{2s''}{a^3} = \frac{2r^3}{a^3}(3\cos\overline{\lambda - n_{,}t} + 5\cos\overline{3\lambda - 3n_{,}t}) = 6\cos\overline{nt - n_{,}t + \epsilon} + 10\cos\overline{3nt - 3n_{,}t + 3\epsilon}$$

$$-6(k\cos nt - l\sin nt)(3\cos\overline{nt - n_{,}t + \epsilon} + 5\cos\overline{3nt - 3n_{,}t + 3\epsilon})$$

$$-12(k\sin nt + l\cos nt)(\sin\overline{nt - n_{,}t + \epsilon} + 5\sin\overline{3nt - 3n_{,}t + 3\epsilon});$$

* [Lubbock, *Theory of the Moon* (1834), Appendix, pp. i, viii [1]. Brown, *Lunar Theory*, p. 110.]

$$\because\quad s' = \frac{3a^3}{n-n_{,}}(\sin\overline{nt-n_{,}t+\epsilon}-\sin\epsilon)+\frac{5a^3}{3(n-n_{,})}(\sin\overline{3nt-3n_{,}t+3\epsilon}-\sin 3\epsilon)$$

$$-\frac{3ka^3}{2}\left\{\frac{5}{n_{,}}(\sin\overline{n_{,}t-\epsilon}+\sin\epsilon)+\frac{1}{2n-n_{,}}(\sin\overline{2nt-n_{,}t+\epsilon}-\sin\epsilon)\right.$$

$$\left.+\frac{15}{2n-3n_{,}}(\sin\overline{2nt-3n_{,}t+3\epsilon}-\sin 3\epsilon)-\frac{5}{4n-3n_{,}}(\sin\overline{4nt-3n_{,}t+3\epsilon}-\sin 3\epsilon)\right\}$$

$$-\frac{3la^3}{2}\left\{\frac{5}{n_{,}}(\cos\overline{n_{,}t-\epsilon}-\cos\epsilon)+\frac{1}{2n-n_{,}}(\cos\overline{2nt-n_{,}t+\epsilon}-\cos\epsilon)\right.$$

$$\left.-\frac{15}{2n-3n_{,}}(\cos\overline{2nt-3n_{,}t+3\epsilon}-\cos 3\epsilon)-\frac{5}{4n-3n_{,}}(\cos\overline{4nt-3n_{,}t+3\epsilon}-\cos 3\epsilon)\right\};$$

& consequently,

$$r' = \frac{6a^3}{n-n_{,}}(\cos\overline{nt-n_{,}t+\epsilon}-\cos\epsilon)+\frac{10a^3}{n-n_{,}}(\cos\overline{3nt-3n_{,}t+3\epsilon}-\cos 3\epsilon)$$

$$+\frac{3a^3}{2}\left\{\frac{5}{n_{,}}(\cos\overline{nt-n_{,}t+\epsilon}-\cos\overline{nt+\epsilon})+\frac{1}{2n-n_{,}}(\cos\overline{nt-n_{,}t+\epsilon}-\cos\overline{nt-\epsilon})\right.$$

$$\left.-\frac{15}{2n-3n_{,}}(\cos\overline{3nt-3n_{,}t+3\epsilon}-\cos\overline{nt+3\epsilon})-\frac{5}{4n-3n_{,}}(\cos\overline{3nt-3n_{,}t+3\epsilon}-\cos\overline{nt-3\epsilon})\right\}$$

and

$$\lambda' = -\frac{a^3(3n-2n_{,})}{(n-n_{,})^2}\{9(\sin\overline{nt-n_{,}t+\epsilon}-\sin\epsilon)+5(\sin\overline{3nt-3n_{,}t+3\epsilon}-\sin 3\epsilon)\}$$

$$+\frac{3a^3nt}{n-n_{,}}(3\cos\epsilon+5\cos 3\epsilon)$$

$$-3a^3\left\{\frac{5}{n_{,}}(\sin\overline{nt-n_{,}t+\epsilon}-\sin\overline{nt+\epsilon})-\frac{1}{2n-n_{,}}(\sin\overline{nt-n_{,}t+\epsilon}+\sin\overline{nt-\epsilon})\right.$$

$$\left.-\frac{15}{2n-3n_{,}}(\sin\overline{3nt-3n_{,}t+3\epsilon}-\sin\overline{nt+3\epsilon})+\frac{5}{4n-3n_{,}}(\sin\overline{3nt-3n_{,}t+3\epsilon}+\sin\overline{nt-3\epsilon})\right\}.$$

We have only to add

to k the terms

$$\frac{3a}{16a_{,}}\frac{n_{,}}{n}\left\{\left(5+\frac{n_{,}}{2n-n_{,}}\right)\cos\epsilon-5\left(\frac{3n_{,}}{2n-3n_{,}}+\frac{n_{,}}{4n-3n_{,}}\right)\cos 3\epsilon\right\};$$

to l the terms

$$\frac{3a}{16a_{,}}\frac{n_{,}}{n}\left\{\left(5-\frac{n_{,}}{2n-n_{,}}\right)\sin\epsilon-5\left(\frac{3n_{,}}{2n-3n_{,}}-\frac{n_{,}}{4n-3n_{,}}\right)\sin 3\epsilon\right\};$$

to e the terms

$$\frac{1}{8}\frac{a}{a_{,}}\frac{n_{,}^2}{n}\frac{3n-2n_{,}}{(n-n_{,})^2}(9\sin\epsilon+5\sin 3\epsilon);$$

to n the terms

$$\frac{3}{8}\frac{a}{a_{,}}\frac{n_{,}^2}{n-n_{,}}(3\cos\epsilon+5\cos 3\epsilon);$$

and consequently (because $a^3n^2=\mu$)

to a the terms

$$-\frac{1}{4}\frac{a^2}{a_{,}}\frac{n_{,}}{n}\frac{n_{,}}{n-n_{,}}(3\cos\epsilon+5\cos 3\epsilon):$$

& we shall have $\left(\text{making } \frac{n_{,}}{\mathrm{n}} = m,\ \mathrm{n}t - n_{,}t + \mathrm{e} = \tau\right)$

$$\lambda = \mathrm{n}t + \mathrm{e} + 2\mathrm{k}\sin \mathrm{n}t + 2\mathrm{l}\cos \mathrm{n}t - \frac{3m^2}{4}\left\{\frac{2}{1-m} + \frac{3}{2(1-m)^2} + \frac{2}{3-2m} - \frac{6}{1-2m}\right\}\sin 2\tau$$

$$- \frac{3m}{8}\frac{\mathrm{a}}{a_{,}}\left\{5 - \frac{m}{2-m} + \frac{3m(3-2m)}{(1-m)^2}\right\}\sin\tau + \frac{5m^2}{8}\frac{\mathrm{a}}{a_{,}}\left\{\frac{9}{2-3m} - \frac{3}{4-3m} - \frac{3-2m}{(1-m)^2}\right\}\sin 3\tau;$$

$$\frac{r}{\mathrm{a}} = 1 - \frac{m^2}{6} - \mathrm{k}\cos \mathrm{n}t + \mathrm{l}\sin \mathrm{n}t + \frac{3m^2}{4}\left\{\frac{2}{1-m} - \frac{1}{3-2m} - \frac{3}{1-2m}\right\}\cos 2\tau$$

$$+ \frac{3m}{16}\frac{\mathrm{a}}{a_{,}}\left\{5 + \frac{m}{2-m} + \frac{4m}{1-m}\right\}\cos\tau - \frac{5m^2}{16}\frac{\mathrm{a}}{a_{,}}\left\{\frac{9}{2-3m} + \frac{3}{4-3m} - \frac{4}{1-m}\right\}\cos 3\tau.$$

Developing,

$$\lambda = \mathrm{n}t + \mathrm{e} + 2\mathrm{k}\sin \mathrm{n}t + 2\mathrm{l}\cos \mathrm{n}t + m^2\left(\frac{11}{8} + \frac{59m}{12}\right)\sin 2\tau - \frac{3m\mathrm{a}}{8a_{,}}\left(5 + \frac{17m}{2} + \frac{47m^2}{4}\right)\sin\tau$$

$$+ \frac{5m^2\mathrm{a}}{32a_{,}}\left(3 + \frac{35m}{4}\right)\sin 3\tau;$$

$$\frac{r}{\mathrm{a}} = 1 - \mathrm{k}\cos \mathrm{n}t + \mathrm{l}\sin \mathrm{n}t - \frac{m^2}{6} - m^2\left(1 + \frac{19m}{6}\right)\cos 2\tau + \frac{3m\mathrm{a}}{16a_{,}}\left(5 + \frac{9m}{2} + \frac{17m^2}{4}\right)\cos\tau$$

$$- \frac{5m^2\mathrm{a}}{64a_{,}}\left(5 + \frac{53m}{4}\right)\cos 3\tau.$$

And accordingly these equations are integrals, in the present order of approximation, of the following system of differential equations of the 2nd order:

$$r'' - r\lambda'^2 + \frac{\mu}{r^2} = \frac{n_{,}^2 r}{2}\left(1 + 3\cos\overline{2\lambda - 2n_{,}t}\right) + \frac{3n_{,}^2 r^2}{8a}\left(3\cos\overline{\lambda - n_{,}t} + 5\cos\overline{3\lambda - 3n_{,}t}\right);$$

$$(r^2\lambda')' = -\frac{3n_{,}^2 r^2}{2}\sin\overline{2\lambda - 2n_{,}t} - \frac{3n_{,}^2 r^3}{8a_{,}}\left(\sin\overline{\lambda - n_{,}t} + 5\sin\overline{3\lambda - 3n_{,}t}\right).$$

Yet the coefficients of $\sin\tau$ and $\cos\tau$ do not agree except in their first terms with the expressions of Plana and Lubbock.*

* [In the remainder of this manuscript Hamilton attempts to verify that these equations for λ and r satisfy the differential equations of motion to the present order of approximation. Actually he recognised later (see page 275) that the approximations of Plana and Lubbock were correct. Lubbock, *loc. cit.* pp. vi, xviii [101]. Brown, *loc. cit.* p. 241. See also Appendix, Note 6, p. 627.]