V.

[ON NEARLY CIRCULAR ORBITS]

[1836.]

[Note Book 42.]

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[Jan. 6, 1836.]

[1. *The principal function for an elliptic orbit.*]

For the undisturbed motion of a planet, with Newton's law of attraction, let*

$$s = \int_0^t \left\{ \frac{\xi_t'^2 + \eta_t'^2}{2} + \frac{\mu}{r_t} \right\} dt,$$

ξ_t, η_t, r_t being the two rectangular heliocentric coordinates and the heliocentric distance of a planet, and μ being the sum of the two masses; so that the differential equations of motion of the second order are

$$\xi_t'' = -\mu \xi_t r_t^{-3}, \quad \eta_t'' = -\mu \eta_t r_t^{-3},$$

and, by my general theorem of dynamics,†

$$\delta s_t = \xi_t' \delta \xi_t - \xi_0' \delta \xi_0 + \eta_t' \delta \eta_t - \eta_0' \delta \eta_0 + \left(\frac{\mu}{r_t} - \frac{\xi_t'^2 + \eta_t'^2}{2} \right) \delta t,$$

in which the coefficient of δt is constant,

$$\frac{\mu}{r_t} - \tfrac{1}{2}(\xi_t'^2 + \eta_t'^2) = \frac{\mu}{r_0} - \tfrac{1}{2}(\xi_0'^2 + \eta_0'^2).$$

The function s will therefore satisfy the two partial differential equations following:†

$$\boxed{\frac{\delta s}{\delta t} + \frac{1}{2}\left(\frac{\delta s}{\delta \xi_t}\right)^2 + \frac{1}{2}\left(\frac{\delta s}{\delta \eta_t}\right)^2 = \frac{\mu}{r_t}, \quad \frac{\delta s}{\delta t} + \frac{1}{2}\left(\frac{\delta s}{\delta \xi_0}\right)^2 + \frac{1}{2}\left(\frac{\delta s}{\delta \eta_0}\right)^2 = \frac{\mu}{r_0};}$$

s being considered as dependent on the 5 independent variables ξ_0, η_0, ξ_t, η_t and t, besides the constant μ.

* [See Appendix, Note 5, p. 624.]

† [Cf. pp. 160, 166.]

Let $r_t = \alpha + \beta$, $r_0 = \alpha - \beta$, $(\xi_t - \xi_0)^2 + (\eta_t - \eta_0)^2 = (r + r_0)^2 \sin\lambda^2$. I have found,* first, that s depends only on the three variables α, λ, t, being independent of β; and, secondly, that when λ is given, $s/\sqrt{\mu\alpha}$ depends only on $t\sqrt{\mu/\alpha^3}$; so that if we put

$$s = \kappa\sqrt{\mu\alpha}, \quad \frac{t}{2}\sqrt{\frac{\mu}{\alpha^3}} = \nu,$$

κ will be a certain function, of which the form is to be discovered, of the *two* variables λ and ν. And if the motion is *nearly circular*, these two variables λ, ν will be *nearly equal* to each other. (In some loose sheets of investigations begun in May, 1834, I have used the symbol ν to denote $t\sqrt{\frac{\mu}{\alpha^3}}$.)

Taking the variation of the expression $s = \kappa\sqrt{\mu\alpha}$, we find

$$\delta s = \sqrt{\mu\alpha}\left(\frac{\delta\kappa}{\delta\lambda}\delta\lambda + \frac{\delta\kappa}{\delta\nu}\delta\nu\right) + \frac{\kappa}{2}\sqrt{\frac{\mu}{\alpha}}\delta\alpha;$$

in which

$$\delta\alpha = \tfrac{1}{2}(\delta r_t + \delta r_0) = \frac{\xi_t\delta\xi_t + \eta_t\delta\eta_t}{2r_t} + \frac{\xi_0\delta\xi_0 + \eta_0\delta\eta_0}{2r_0},$$

$$\delta\nu = \frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\delta t - \frac{3t}{4}\sqrt{\frac{\mu}{\alpha^5}}\left(\frac{\xi_t\delta\xi_t + \eta_t\delta\eta_t}{2r_t} + \frac{\xi_0\delta\xi_0 + \eta_0\delta\eta_0}{2r_0}\right),$$

and

$$\delta\lambda = \frac{(\xi_t - \xi_0)(\delta\xi_t - \delta\xi_0) + (\eta_t - \eta_0)(\delta\eta_t - \delta\eta_0) - (r + r_0)\sin\lambda^2(\delta r + \delta r_0)}{(r + r_0)^2\sin\lambda\cos\lambda}.$$

Hence

$$\delta s = \left(\frac{\kappa}{2}\sqrt{\frac{\mu}{\alpha}} - \frac{3t}{4}\frac{\mu}{\alpha^2}\frac{\delta\kappa}{\delta\nu} - \tan\lambda\sqrt{\frac{\mu}{\alpha}}\frac{\delta\kappa}{\delta\lambda}\right)\left(\frac{\xi_t\delta\xi_t + \eta_t\delta\eta_t}{2r_t} + \frac{\xi_0\delta\xi_0 + \eta_0\delta\eta_0}{2r_0}\right)$$
$$+ \frac{\operatorname{cosec} 2\lambda}{2}\sqrt{\frac{\mu}{\alpha^3}}\frac{\delta\kappa}{\delta\lambda}\{(\xi_t - \xi_0)(\delta\xi_t - \delta\xi_0) + (\eta_t - \eta_0)(\delta\eta_t - \delta\eta_0)\} + \frac{\mu}{2\alpha}\frac{\delta\kappa}{\delta\nu}\delta t;$$

therefore

$$\frac{\delta s}{\delta\xi_t} = A\frac{\xi_t}{r_t} + B(\xi_t - \xi_0), \quad \frac{\delta s}{\delta\eta_t} = A\frac{\eta_t}{r_t} + B(\eta_t - \eta_0),$$
$$\frac{\delta s}{\delta\xi_0} = A\frac{\xi_0}{r_0} - B(\xi_t - \xi_0), \quad \frac{\delta s}{\delta\eta_0} = A\frac{\eta_0}{r_0} - B(\eta_t - \eta_0), \quad \frac{\delta s}{\delta t} = C,$$

if we put, for abridgment,

$$A = \left(\frac{\kappa}{4} - \frac{\tan\lambda}{2}\frac{\delta\kappa}{\delta\lambda} - \frac{3\nu}{4}\frac{\delta\kappa}{\delta\nu}\right)\sqrt{\frac{\mu}{\alpha}}, \quad B = \frac{\operatorname{cosec} 2\lambda}{2\alpha}\frac{\delta\kappa}{\delta\lambda}\sqrt{\frac{\mu}{\alpha}}, \quad C = \frac{\mu}{2\alpha}\frac{\delta\kappa}{\delta\nu};$$

and since

$$\left(\frac{\xi_t}{r_t}\right)^2 + \left(\frac{\eta_t}{r_t}\right)^2 = 1, \quad \left(\frac{\xi_0}{r_0}\right)^2 + \left(\frac{\eta_0}{r_0}\right)^2 = 1, \quad (\xi_t - \xi_0)^2 + (\eta_t - \eta_0)^2 = 4\alpha^2\sin\lambda^2,$$

$$\frac{2}{r_t}\{\xi_t(\xi_t - \xi_0) + \eta_t(\eta_t - \eta_0)\} = \frac{r_t^2 - r_0^2 + 4\alpha^2\sin\lambda^2}{r_t},$$

$$-\frac{2}{r_0}\{\xi_0(\xi_t - \xi_0) + \eta_0(\eta_t - \eta_0)\} = \frac{r_0^2 - r_t^2 + 4\alpha^2\sin\lambda^2}{r_0},$$

* [See Appendix, Note 5, p. 624.]

the two partial differential equations become, when multiplied by r_t, r_0, and then added and subtracted, and divided by 2α, 2β,

$$\frac{\mu}{\alpha} = C + \tfrac{1}{2}A^2 + 2B^2\alpha^2 \sin\lambda^2 + 2AB\alpha\sin\lambda^2,$$

$$0 = C + \tfrac{1}{2}A^2 + 2B^2\alpha^2\sin\lambda^2 + 2AB\alpha.$$

Resolving these two equations for the two quantities $C + \frac{1}{2}A^2 + 2B^2\alpha^2\sin\lambda^2$ and $2AB\alpha$, we find

$$\frac{\alpha}{\mu}(2C + A^2 + 4B^2\alpha^2\sin\lambda^2) = -\frac{4AB}{\mu}\alpha^2 = 2\sec\lambda^2;$$

therefore

$$(A + 2B\alpha\sin\lambda)^2 = 2\left(\frac{\mu}{\alpha}\sec\lambda^2 - C - \frac{\mu}{\alpha}\sin\lambda\sec\lambda^2\right),$$

$$(A - 2B\alpha\sin\lambda)^2 = 2\left(\frac{\mu}{\alpha}\sec\lambda^2 - C + \frac{\mu}{\alpha}\sin\lambda\sec\lambda^2\right);$$

and, substituting for A, B, C their values, the equations become

$$\boxed{\begin{aligned}&\left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)^2 + 4\left(\frac{\delta\kappa}{\delta\lambda}\right)^2\sec\lambda^2 + 16\frac{\delta\kappa}{\delta\nu} = 32\sec\lambda^2,\\ &\left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)\frac{\delta\kappa}{\delta\lambda} + 8\tan\lambda = 0.\end{aligned}}$$

[Hence

$$\left(\kappa - 3\nu\frac{\delta\kappa}{\delta\nu}\right)^2 + 4\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 + 16\left(\frac{\delta\kappa}{\delta\nu} - 2\right) = 0.]$$

These two equations ought to be compatible; that is, they ought to give equal values of $\dfrac{\delta^2\kappa}{\delta\lambda\,\delta\nu}$, whether we calculate by elimination $\dfrac{\delta\kappa}{\delta\lambda}$ and differentiate it with respect to ν, or calculate by elimination $\dfrac{\delta\kappa}{\delta\nu}$ and differentiate it with respect to λ. Without this previous elimination, we may differentiate the two partial differential equations separately and successively with respect to λ and ν, and then eliminate, between the four resulting equations, the three partial differential coefficients of the second order $\dfrac{\delta^2\kappa}{\delta\lambda^2}$, $\dfrac{\delta^2\kappa}{\delta\lambda\,\delta\nu}$, $\dfrac{\delta^2\kappa}{\delta\nu^2}$; the resulting equation ought to be true either identically or at least in virtue of the two partial differential equations of the first order. After some reduction it is easy to see that this condition of compatibility is

$$0 = -4\frac{\delta\kappa}{\delta\nu}\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 + 16\tan\lambda^2(4\sec\lambda^2 - 1) + 2\left\{\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 - 8\right\}\tan\lambda^2\sec\lambda^2\left\{\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 + 4\right\}$$
$$- \sec\lambda^2\left(\frac{\delta\kappa}{\delta\lambda}\right)^2\left\{(2\sec\lambda^2 - 1)\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 - 8\sec\lambda^2\right\},$$

in which, by the first partial differential equation,

$$-4\frac{\delta\kappa}{\delta\nu}\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 = 16\tan\lambda^2 + \left(\frac{\delta\kappa}{\delta\lambda}\right)^4\sec\lambda^2 - 8\left(\frac{\delta\kappa}{\delta\lambda}\right)^2\sec\lambda^2,$$

so that the condition is in fact satisfied, and the two partial differential equations are compatible.

[2. *Approximations to the principal function in the case of nearly circular orbits.*]

Of these two partial differential equations, the second, namely

$$\left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)\frac{\delta\kappa}{\delta\lambda} + 8\tan\lambda = 0,$$

seems well adapted for determining the coefficients of a series such as the following:

$$\boxed{\kappa = \kappa_0 + \kappa_1(\nu-\lambda) + \kappa_2\frac{(\nu-\lambda)^2}{2} + \kappa_3\frac{(\nu-\lambda)^3}{2\,.\,3} + \text{etc.},}$$

if it is possible, as it seems likely that it is, to represent κ by such a series for the case of nearly circular orbits.* Assuming this series, we have

$$\frac{\delta\kappa}{\delta\lambda} = \kappa_0' - \kappa_1 + (\nu-\lambda)(\kappa_1' - \kappa_2) + \frac{(\nu-\lambda)^2}{2}(\kappa_2' - \kappa_3) + \frac{(\nu-\lambda)^3}{2\,.\,3}(\kappa_3' - \kappa_4) + \text{etc.};$$

$$\frac{\delta\kappa}{\delta\nu} = \kappa_1 + (\nu-\lambda)\kappa_2 + \frac{(\nu-\lambda)^2}{2}\kappa_3 + \frac{(\nu-\lambda)^3}{2\,.\,3}\kappa_4 + \text{etc.};$$

κ_i being a function of λ only, and κ_i' being its first derivative with respect to λ. Thus, comparing the terms independent of $\nu-\lambda$, we find the equation

$$0 = \{\kappa_0 - 2\tan\lambda(\kappa_0' - \kappa_1) - 3\lambda\kappa_1\}(\kappa_0' - \kappa_1) + 8\tan\lambda;$$

in which, from the theory of circular motion, $\kappa_0 = 3\lambda$, and therefore $\kappa_0' = 3$, so that the equation becomes

$$0 = (\kappa_1 - 3)\{\kappa_1(3\lambda - 2\tan\lambda) + 3(2\tan\lambda - \lambda)\} + 8\tan\lambda,$$

that is,

$$0 = \kappa_1^2(3\lambda - 2\tan\lambda) + 12\kappa_1(\tan\lambda - \lambda) + 9\lambda - 10\tan\lambda$$
$$= (\kappa_1 - 1)\{\kappa_1(3\lambda - 2\tan\lambda) - (9\lambda - 10\tan\lambda)\};$$

therefore either $\kappa_1 = 1$, or else

$$\kappa_1 = \frac{9\lambda - 10\tan\lambda}{3\lambda - 2\tan\lambda};$$

and in order to decide which of these two roots we are to take, it seems necessary, in the present method, to employ the other partial differential equation, namely,

$$0 = \left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)^2 + 4\left(\frac{\delta\kappa}{\delta\lambda}\right)^2\sec\lambda^2 + 16\left(\frac{\delta\kappa}{\delta\nu} - 2\sec\lambda^2\right).$$

This equation gives, by making $\nu = \lambda$,

$$0 = \{\kappa_0 - 3\lambda\kappa_1 + 2\tan\lambda(\kappa_1 - \kappa_0')\}^2 + 4(\kappa_1 - \kappa_0')^2\sec\lambda^2 + 16(\kappa_1 - 2\sec\lambda^2);$$

therefore, since $\kappa_0 = 3\lambda$, $\kappa_0' = 3$,

$$0 = \{(3\lambda - 2\tan\lambda)\kappa_1 + 3(2\tan\lambda - \lambda)\}^2 + 4(\kappa_1 - 3)^2\sec\lambda^2 + 16(\kappa_1 - 2\sec\lambda^2)$$
$$= \kappa_1^2\{(3\lambda - 2\tan\lambda)^2 + 4\sec\lambda^2\} + 2\kappa_1\{3(3\lambda - 2\tan\lambda)(2\tan\lambda - \lambda) - 12\sec\lambda^2 + 8\}$$
$$+ 9(2\tan\lambda - \lambda)^2 + 4\sec\lambda^2$$
$$= (\kappa_1 - 1)\{\kappa_1(9\lambda^2 - 12\lambda\tan\lambda + 8\tan\lambda^2 + 4) - (9\lambda^2 - 36\lambda\tan\lambda + 40\tan\lambda^2 + 4)\};$$

* [$\iota = \nu - \lambda$ is small when the orbit is nearly circular. See Appendix, Note 5, p. 626.]

therefore by this equation we have either $\kappa_1 = 1$, or

$$\kappa_1 = \frac{9(2\tan\lambda - \lambda)^2 + 4\sec\lambda^2}{(3\lambda - 2\tan\lambda)^2 + 4\sec\lambda^2};$$

and, since, by the former equation, we had either $\kappa_1 = 1$ or $\kappa_1 = \dfrac{9\lambda - 10\tan\lambda}{3\lambda - 2\tan\lambda}$, we must conclude that $\kappa_1 = 1$. If we then neglect $(\nu - \lambda)^2$, we have simply

$$\boxed{\kappa = \kappa_0 + \kappa_1(\nu - \lambda) = 2\lambda + \nu,}$$

because $\kappa_0 = 3\lambda$, $\kappa_1 = 1$.

Since the two partial differential equations separately gave, as we just now said, ambiguous results, but such that the ambiguity was removed when the results of the one equation were compared with those of the other, it seems evident that some combination of the two equations must be capable of giving unambiguous results, at least for the coefficient κ_1, and probably for the other coefficients. Accordingly, if we put $\dfrac{\delta\kappa}{\delta\lambda}$ under the form $\dfrac{\delta\kappa}{\delta\lambda} + \dfrac{\delta\kappa}{\delta\nu} - \dfrac{\delta\kappa}{\delta\nu}$, and then eliminate the square of that $\dfrac{\delta\kappa}{\delta\nu}$ which is not added to $\dfrac{\delta\kappa}{\delta\lambda}$, we shall accomplish the purpose proposed; and the same object may be accomplished still more simply by introducing a new variable ι, which shall be equal to $\nu - \lambda$, and shall therefore be small for nearly circular orbits, and thus changing $\dfrac{\delta\kappa}{\delta\nu}$ to $\dfrac{\delta\kappa}{\delta\iota}$ and $\dfrac{\delta\kappa}{\delta\lambda}$ to $\dfrac{\delta\kappa}{\delta\lambda} - \dfrac{\delta\kappa}{\delta\iota}$ in the two partial differential equations, and then eliminating $\left(\dfrac{\delta\kappa}{\delta\iota}\right)^2$ between them.

In this manner we have

$$\kappa = \kappa_0 + \kappa_1\iota + \kappa_2\frac{\iota^2}{2} + \kappa_3\frac{\iota^3}{2\,.\,3} + \text{etc.},$$

$$\frac{\delta\kappa}{\delta\lambda} = \kappa_0' + \kappa_1'\iota + \kappa_2'\frac{\iota^2}{2} + \kappa_3'\frac{\iota^3}{2\,.\,3} + \text{etc.},$$

$$\frac{\delta\kappa}{\delta\iota} = \kappa_1 + \kappa_2\iota + \kappa_3\frac{\iota^2}{2} + \kappa_4\frac{\iota^3}{2\,.\,3} + \text{etc.},$$

κ_i being some function of λ, and $\kappa_i' = \dfrac{\delta\kappa_i}{\delta\lambda}$; also the two partial differential equations become

$$\boxed{\begin{aligned} 0 &= \left\{\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + (2\tan\lambda - 3\lambda)\frac{\delta\kappa}{\delta\iota} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\}^2 + 4\left(\frac{\delta\kappa}{\delta\lambda} - \frac{\delta\kappa}{\delta\iota}\right)^2\sec\lambda^2 + 16\left(\frac{\delta\kappa}{\delta\iota} - 2\sec\lambda^2\right), \\ 0 &= \left\{\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + (2\tan\lambda - 3\lambda)\frac{\delta\kappa}{\delta\iota} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\}\left(\frac{\delta\kappa}{\delta\lambda} - \frac{\delta\kappa}{\delta\iota}\right) + 8\tan\lambda. \end{aligned}}$$

The coefficient of $\left(\dfrac{\delta\kappa}{\delta\iota}\right)^2$ in the first is $(2\tan\lambda - 3\lambda - 3\iota)^2 + 4\sec\lambda^2$, and in the second it is $-(2\tan\lambda - 3\lambda - 3\iota)$; and since the terms of the form const. $\times\,\iota\left(\dfrac{\delta\kappa}{\delta\iota}\right)^2$ cannot cause ambiguity, it is sufficient for our present purpose to multiply the first equation by $2\tan\lambda - 3\lambda$ and the second by $(2\tan\lambda - 3\lambda)^2 + 4\sec\lambda^2$, and to add the products together. In this manner we get

$$0 = \left\{\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + (2\tan\lambda - 3\lambda)\frac{\delta\kappa}{\delta\iota} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\}(2\tan\lambda - 3\lambda)\left\{\kappa - 3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\}$$
$$+ 4\left(\frac{\delta\kappa}{\delta\lambda} - \frac{\delta\kappa}{\delta\iota}\right)\sec\lambda^2\left\{\kappa - 3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right\} + 16\left(\frac{\delta\kappa}{\delta\iota} - 2\sec\lambda^2\right)(2\tan\lambda - 3\lambda)$$
$$+ 8\tan\lambda\{(2\tan\lambda - 3\lambda)^2 + 4\sec\lambda^2\};$$

that is,

$$0 = \left\{\left(\kappa - 3\iota\frac{\delta\kappa}{\delta\iota}\right)(2\tan\lambda - 3\lambda) + 2(2 + 3\lambda\tan\lambda)\frac{\delta\kappa}{\delta\lambda} + (9\lambda^2 - 12\lambda\tan\lambda - 4)\frac{\delta\kappa}{\delta\iota}\right\}\left(3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right)$$
$$+ 16(2\tan\lambda - 3\lambda)\left(\frac{\delta\kappa}{\delta\iota} - 2\sec\lambda^2\right) + 8\tan\lambda\{(2\tan\lambda - 3\lambda)^2 + 4\sec\lambda^2\};$$

that is,

$$0 = \left(\kappa - 3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right)^2(2\tan\lambda - 3\lambda) + (9\lambda^2 - 12\lambda\tan\lambda - 4)\left(\kappa - 3\lambda\frac{\delta\kappa}{\delta\lambda} - 3\iota\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\delta\kappa}{\delta\iota} - \frac{\delta\kappa}{\delta\lambda}\right)$$
$$+ 16(2\tan\lambda - 3\lambda)\frac{\delta\kappa}{\delta\iota} + 8\tan\lambda(9\lambda^2 - 4) + 96\lambda.$$

Making, in this, $\kappa = \kappa_0$, $\dfrac{\delta\kappa}{\delta\lambda} = \kappa_0'$, $\dfrac{\delta\kappa}{\delta\iota} = \kappa_1$, $\iota = 0$, we get

$$0 = (\kappa_0 - 3\lambda\kappa_0')^2(2\tan\lambda - 3\lambda) + (9\lambda^2 - 12\lambda\tan\lambda - 4)(\kappa_0 - 3\lambda\kappa_0')(\kappa_1 - \kappa_0')$$
$$+ 16(2\tan\lambda - 3\lambda)\kappa_1 + 8\tan\lambda(9\lambda^2 - 4) + 96\lambda;$$

and if we further employ the values $\kappa_0 = 3\lambda$, $\kappa_0' = 3$, we find

$$0 = 36\lambda^2(2\tan\lambda - 3\lambda) - 6\lambda(9\lambda^2 - 12\lambda\tan\lambda - 4)(\kappa_1 - 3)$$
$$+ 16(2\tan\lambda - 3\lambda)\kappa_1 + 8\tan\lambda(9\lambda^2 - 4) + 96\lambda$$
$$= 2(\kappa_1 - 1)(9\lambda^2 + 4)\{4\tan\lambda - 3\lambda\};$$

therefore $\kappa_1 = 1$, as before.

In general,

$$0 = \left\{-\kappa + 3\lambda\frac{\delta\kappa}{\delta\lambda} + 3\iota\frac{\delta\kappa}{\delta\iota}\right\}^2 + \left(6\lambda + \frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda}\right)\left(\frac{\delta\kappa}{\delta\iota} - \frac{\delta\kappa}{\delta\lambda}\right)\left(-\kappa + 3\lambda\frac{\delta\kappa}{\delta\lambda} + 3\iota\frac{\delta\kappa}{\delta\iota}\right)$$
$$+ 16\frac{\delta\kappa}{\delta\iota} + 4(9\lambda^2 - 4) + 12\lambda\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda};$$

a formula which, when $\iota = 0$, $\kappa = 3\lambda$, $\dfrac{\delta\kappa}{\delta\lambda} = 3$, $\dfrac{\delta\kappa}{\delta\iota} = \kappa_1$, becomes

$$0 = 36\lambda^2 + 6\lambda\left(6\lambda + \frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda}\right)(\kappa_1 - 3) + 16\kappa_1 + 4(9\lambda^2 - 4) + 12\lambda\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda};$$

and gives $\kappa_1 = 1$. When we make $\kappa = 3\lambda + \iota$, $\dfrac{\delta\kappa}{\delta\lambda} = 3$, $\dfrac{\delta\kappa}{\delta\iota} = 1 + \kappa_2\iota$, and neglect ι^2, the formula becomes

$$0 = (6\lambda + 2\iota)^2 + \left(6\lambda + \frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda}\right)(-2 + \kappa_2\iota)(6\lambda + 2\iota) + 16(1 + \kappa_2\iota) + 4(9\lambda^2 - 4) + 12\lambda\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda};$$

that is,

$$0 = 24\lambda + (6\lambda\kappa_2 - 4)\left(6\lambda + \frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda}\right) + 16\kappa_2$$
$$= \kappa_2\left\{36\lambda^2 + 6\lambda\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda} + 16\right\} - 4\frac{9\lambda^2 + 4}{2\tan\lambda - 3\lambda};$$

that is, finally, $$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda}.$$

Thus

$$\boxed{\kappa = 3\lambda + \iota + \frac{\iota^2}{4\tan\lambda - 3\lambda},}$$

neglecting ι^3.

If we make the same substitutions, $\left(\kappa = 3\lambda + \iota, \ \frac{\delta\kappa}{\delta\lambda} = 3, \ \frac{\delta\kappa}{\delta\iota} = 1 + \kappa_2\iota\right)$, and still neglect ι^2, in the partial differential equation

$$0 = \left\{\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + (2\tan\lambda - 3\lambda - 3\iota)\frac{\delta\kappa}{\delta\iota}\right\}\left(\frac{\delta\kappa}{\delta\lambda} - \frac{\delta\kappa}{\delta\iota}\right) + 8\tan\lambda,$$

we find

$$0 = \{3\lambda + \iota - 6\tan\lambda + (2\tan\lambda - 3\lambda - 3\iota)(1 + \kappa_2\iota)\}(2 - \kappa_2\iota) + 8\tan\lambda,$$

that is,

$$0 = 2\{1 - 3 + (2\tan\lambda - 3\lambda)\kappa_2\} + 4\kappa_2\tan\lambda,$$

or

$$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda},$$

as before. But it is remarkable that this value has been deduced *without ambiguity* from the tolerably simple partial differential equation which we set aside before as likely to give only ambiguous results.

If we return to the system of variables κ, λ, ν, instead of κ, λ, ι, and therefore resume the old partial differential equation

$$\boxed{0 = \left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)\frac{\delta\kappa}{\delta\lambda} + 8\tan\lambda,}$$

together with the old development

$$\kappa = \kappa_0 + \kappa_1(\nu - \lambda) + \kappa_2\frac{(\nu - \lambda)^2}{2} + \kappa_3\frac{(\nu - \lambda)^3}{2\,.\,3} + \text{etc.},$$

we may propose to determine the several coefficients $\kappa_0, \kappa_1, \kappa_2, \ldots$ as the values of the partial differential coefficients

$$\frac{\delta^0\kappa}{\delta\nu^0}, \quad \frac{\delta^1\kappa}{\delta\nu^1}, \quad \frac{\delta^2\kappa}{\delta\nu^2}, \quad \ldots$$

when ν becomes equal to λ. Thus, differentiating the equation with respect to ν once, we find

$$0 = -\frac{\delta}{\delta\nu}\left(\kappa\frac{\delta\kappa}{\delta\lambda}\right) + 2\tan\lambda\frac{\delta}{\delta\nu}\left(\frac{\delta\kappa}{\delta\lambda}\right)^2 + 3\frac{\delta\kappa}{\delta\lambda}\frac{\delta\kappa}{\delta\nu} + 3\nu\frac{\delta}{\delta\nu}\left(\frac{\delta\kappa}{\delta\lambda}\frac{\delta\kappa}{\delta\nu}\right)$$

$$= 3\nu\left(\frac{\delta\kappa}{\delta\lambda}\frac{\delta^2\kappa}{\delta\nu^2} + \frac{\delta\kappa}{\delta\nu}\frac{\delta^2\kappa}{\delta\lambda\delta\nu}\right) + 2\frac{\delta\kappa}{\delta\lambda}\frac{\delta\kappa}{\delta\nu} - \kappa\frac{\delta^2\kappa}{\delta\lambda\delta\nu} + 4\tan\lambda\frac{\delta\kappa}{\delta\lambda}\frac{\delta^2\kappa}{\delta\lambda\delta\nu};$$

which, if we make $\nu = \lambda$, $\kappa = \kappa_0$, $\frac{\delta\kappa}{\delta\lambda} = \kappa_0' - \kappa_1$, $\frac{\delta\kappa}{\delta\nu} = \kappa_1$, $\frac{\delta^2\kappa}{\delta\lambda\delta\nu} = \kappa_1' - \kappa_2$, $\frac{\delta^2\kappa}{\delta\nu^2} = \kappa_2$, becomes

$$0 = 3\lambda\{(\kappa_0' - \kappa_1)\kappa_2 + \kappa_1(\kappa_1' - \kappa_2)\} + 2\kappa_1(\kappa_0' - \kappa_1) - \kappa_0(\kappa_1' - \kappa_2) + 4\tan\lambda(\kappa_0' - \kappa_1)(\kappa_1' - \kappa_2),$$

that is, $$0=\{(3\lambda-4\tan\lambda)(\kappa_0'-\kappa_1)-(3\lambda\kappa_1-\kappa_0)\}(\kappa_2-\kappa_1')+(3\lambda\kappa_1'+2\kappa_1)(\kappa_0'-\kappa_1);$$
and this, when we make $\kappa_0=3\lambda$, $\kappa_1=1$, $\kappa_0'=3$, $\kappa_1'=0$, becomes
$$\kappa_2=\frac{2}{4\tan\lambda-3\lambda},$$
as before.

Accurately, for elliptic orbits, the relation between κ, λ, ι may be found by eliminating the auxiliary variables e and v between the three following equations:*

$$\boxed{\kappa=\frac{3v+e\sin v}{(1-e\cos v)^{\frac{1}{2}}};\quad \tan\frac{\lambda}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{v}{2};\quad \lambda+\iota=\frac{v-e\sin v}{(1-e\cos v)^{\frac{3}{2}}},}$$

and if the orbit be nearly circular, then ι and e will be small and of one common order.

Moreover, if a be the semiaxis major of the elliptic orbit and α (as before) the semi-sum of the two given distances r and r_0, then
$$\frac{\alpha}{\mathrm{a}}=\frac{\delta\kappa}{\delta\nu}=\frac{\delta\kappa}{\delta\iota},$$
λ being treated as a constant. Hence, in series,
$$\frac{\alpha}{\mathrm{a}}=\kappa_1+\kappa_2\iota+\kappa_3\frac{\iota^2}{2}+\kappa_4\frac{\iota^3}{2.3}+\text{etc.};$$
and hence we might have easily foreseen that κ_1 must be equal to 1, because it is the value of α/a for circular motion. The finding of this value $\kappa_1=1$ was almost the only difficulty in the use of the partial differential equation
$$0=\left(\kappa-2\tan\lambda\frac{\delta\kappa}{\delta\lambda}-3\nu\frac{\delta\kappa}{\delta\nu}\right)\frac{\delta\kappa}{\delta\lambda}+8\tan\lambda;$$
and if we assume, as we now see that we may,
$$\kappa=2\lambda+\nu+\kappa_2\frac{(\nu-\lambda)^2}{2}+\kappa_3\frac{(\nu-\lambda)^3}{2.3}+\text{etc.},$$
the labour of calculating κ_2, κ_3 will be but trifling and the operation will become one of elementary facility. In fact, to resume, we have, neglecting ι^3,
$$\frac{\delta\kappa}{\delta\lambda}=2-\kappa_2\iota+(\kappa_2'-\kappa_3)\frac{\iota^2}{2},$$
$$\frac{\delta\kappa}{\delta\nu}=1+\kappa_2\iota+\kappa_3\frac{\iota^2}{2},$$
$$\kappa=3\lambda+\iota+\kappa_2\frac{\iota^2}{2},\quad \nu=\lambda+\iota;$$
and, if we substitute these values in the partial differential equation, we find
$$\kappa-2\tan\lambda\frac{\delta\kappa}{\delta\lambda}-3\nu\frac{\delta\kappa}{\delta\nu}=-4\tan\lambda+\iota\{\kappa_2(2\tan\lambda-3\lambda)-2\}$$
$$+\frac{\iota^2}{2}\{\kappa_3(2\tan\lambda-3\lambda)-5\kappa_2-2\kappa_2'\tan\lambda\}.$$

* [Cf. Appendix, Note 5, p. 625.]

Therefore, multiplying this by $\frac{\delta\kappa}{\delta\lambda} = 2 - \kappa_2\iota - \frac{\iota^2}{2}(\kappa_3 - \kappa_2')$, adding $8\tan\lambda$, and equating to zero the coefficients of ι and ι^2, we have

$$0 = \kappa_2(4\tan\lambda - 3\lambda) - 2, \quad 0 = \kappa_3(4\tan\lambda - 3\lambda) - \kappa_2^2(2\tan\lambda - 3\lambda) - 3\kappa_2 - 4\kappa_2'\tan\lambda;$$

the first equation gives

$$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda},$$

as before, and therefore

$$\kappa_2' = \frac{-2(1 + 4\tan\lambda^2)}{(4\tan\lambda - 3\lambda)^2},$$

and then the second equation gives

$$\kappa_3(4\tan\lambda - 3\lambda)^3 = 4(2\tan\lambda - 3\lambda) + 6(4\tan\lambda - 3\lambda) - 8(1 + 4\tan\lambda^2)\tan\lambda$$
$$= -30\lambda + 24\tan\lambda - 32\tan\lambda^3,$$

that is,

$$\kappa_3 = -\frac{32\tan\lambda^3 - 24\tan\lambda + 30\lambda}{(4\tan\lambda - 3\lambda)^3}.$$

Hence, neglecting ι^4,

$$\boxed{\kappa = 3\lambda + \iota + \frac{\iota^2}{4\tan\lambda - 3\lambda} - \frac{\iota^3}{3}\frac{16\tan\lambda^3 - 12\tan\lambda + 15\lambda}{(4\tan\lambda - 3\lambda)^3}.}$$

It is extremely remarkable how little laborious this process is, as compared with any other likely to occur to a mathematician, for the elimination of e and v between the three equations

$$\kappa = \frac{3v + e\sin v}{(1 - e\cos v)^{\frac{1}{2}}}, \quad \tan\frac{\lambda}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{v}{2}, \quad \nu = \lambda + \iota = \frac{v - e\sin v}{(1 - e\cos v)^{\frac{3}{2}}}.$$

[3. *The general expansion of* κ *in terms of* ι.]

To calculate now the general expression for κ_4 from the partial differential equation

$$0 = \left(\kappa - 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} - 3\nu\frac{\delta\kappa}{\delta\nu}\right)\frac{\delta\kappa}{\delta\lambda} + 8\tan\lambda.$$

We have*

$$\kappa = \Sigma_{(i)0}^{\infty}\kappa_i[0]^{-i}\iota^i;$$

$$\frac{\delta\kappa}{\delta\lambda} = \Sigma_{(i)0}^{\infty}\kappa_i'[0]^{-i}\iota^i - \Sigma_{(i)1}^{\infty}\kappa_i[0]^{-(i-1)}\iota^{i-1} = \Sigma_{(i)0}^{\infty}(\kappa_i' - \kappa_{i+1})[0]^{-i}\iota^i = -\Sigma_{(i)0}^{\infty}a_i[0]^{-i}\iota^i;$$

$$\nu\frac{\delta\kappa}{\delta\nu} = (\lambda + \iota)\frac{\delta\kappa}{\delta\iota} = \Sigma_{(i)1}^{\infty}(\lambda + \iota)\kappa_i[0]^{-(i-1)}\iota^{(i-1)} = \Sigma_{(i)0}^{\infty}(\lambda + \iota)\kappa_{i+1}[0]^{-i}\iota^i$$
$$= \Sigma_{(i)0}^{\infty}(\lambda\kappa_{i+1} + i\kappa_i)[0]^{-i}\iota^i;$$

therefore,

$$-\kappa + 2\tan\lambda\frac{\delta\kappa}{\delta\lambda} + 3\nu\frac{\delta\kappa}{\delta\nu} = \Sigma_{(i)0}^{\infty}[0]^{-i}\iota^i\{-\kappa_i + 2\tan\lambda(\kappa_i' - \kappa_{i+1}) + 3\lambda\kappa_{i+1} + 3i\kappa_i\} = \Sigma_{(i)0}^{\infty}b_i[0]^{-i}\iota^i.$$

* [$[0]^{-i} = 1/i!$ See *Mathematical Papers*, Vol. I, Appendix, Note 4, p. 468.]

The partial differential equation gives

$$\{\Sigma_{(i)0}^{\infty} a_i [0]^{-i} \iota^i\} \times \{\Sigma_{(i)0}^{\infty} b_i [0]^{-i} \iota^i\} = -8\tan\lambda;$$

that is,

$$\boxed{a_0 b_0 = -8\tan\lambda;}$$

and when $i>0$,

$$\Sigma_{(i')0}^{i} a_{i-i'} b_{i'} [0]^{-(i-i')} [0]^{-i'} = 0,$$

a formula which may also be thus written*

$$\boxed{0=(a+b)_i.}$$

Here

$$a_i = \kappa_{i+1} - \kappa_i', \quad b_i = (3\lambda - 2\tan\lambda)\kappa_{i+1} + 2\kappa_i'\tan\lambda + (3i-1)\kappa_i;$$
$$a_0 = \kappa_1 - \kappa_0', \quad b_0 = -\kappa_0 + 2\kappa_0'\tan\lambda + (3\lambda - 2\tan\lambda)\kappa_1;$$

and, since we know by the circular values of κ and a that $\kappa_0 = 3\lambda$, $\kappa_1 = 1$, $\kappa_0' = 3$, $\kappa_1' = 0$, we have

$$a_0 = -2, \quad b_0 = 4\tan\lambda,$$

values which satisfy, accordingly, the condition $a_0 b_0 = -8\tan\lambda$. Again

$$a_1 = \kappa_2 - \kappa_1' = \kappa_2, \quad b_1 = 2\kappa_1 + 2\kappa_1'\tan\lambda + (3\lambda - 2\tan\lambda)\kappa_2 = (3\lambda - 2\tan\lambda)\kappa_2 + 2;$$

therefore the condition

$$(a+b)_1 = a_0 b_1 + a_1 b_0 = 0$$

becomes

$$0 = 2(2\tan\lambda - 3\lambda)\kappa_2 - 4 + 4\kappa_2\tan\lambda,$$

that is,

$$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda},$$

as before. Again,

$$b_2 = (3\lambda - 2\tan\lambda)\kappa_3 + 2\kappa_2'\tan\lambda + 5\kappa_2, \quad a_2 = \kappa_3 - \kappa_2',$$
$$b_3 = (3\lambda - 2\tan\lambda)\kappa_4 + 2\kappa_3'\tan\lambda + 8\kappa_3, \quad a_3 = \kappa_4 - \kappa_3';$$

therefore the condition

$$(a+b)_2 = a_2 b_0 + 2a_1 b_1 + a_0 b_2 = 0$$

becomes

$$0 = 4\tan\lambda(\kappa_3 - \kappa_2') + 2\kappa_2^2(3\lambda - 2\tan\lambda) + 4\kappa_2 + 2(2\tan\lambda - 3\lambda)\kappa_3 - 4\kappa_2'\tan\lambda - 10\kappa_2,$$

that is,

$$0 = (4\tan\lambda - 3\lambda)\kappa_3 - 4\kappa_2'\tan\lambda + \kappa_2^2(3\lambda - 2\tan\lambda) - 3\kappa_2,$$

in which

$$\kappa_2 = \frac{2}{4\tan\lambda - 3\lambda}, \quad \kappa_2' = \frac{-2(1+4\tan\lambda^2)}{(4\tan\lambda - 3\lambda)^2}.$$

Therefore

$$0 = \tfrac{1}{2}\kappa_3(4\tan\lambda - 3\lambda)^3 + 4\tan\lambda(1+4\tan\lambda^2) + 2(3\lambda - 2\tan\lambda) - 3(4\tan\lambda - 3\lambda),$$

and so

$$\kappa_3 = \frac{-2(16\tan\lambda^3 - 12\tan\lambda + 15\lambda)}{(4\tan\lambda - 3\lambda)^3},$$

as before.

* $\left[(a+b)_i = a_i b_0 + i a_{i-1} b_i + \frac{i(i-1)}{2} a_{i-2} b_2 + \ldots + a_0 b_i.\right]$

It might have been convenient to have calculated a_1, b_1 as functions of λ, and a_2, b_2 as functions of λ and κ_3, before we proceeded to calculate κ_3 from the equation $0=(a+b)_2$. In this way we should have had

$$a_0=-2, \quad b_0=4\tan\lambda;$$

$$a_1=\kappa_2=\frac{2}{4\tan\lambda-3\lambda}, \quad b_1=(3\lambda-2\tan\lambda)\,\kappa_2+2=\frac{4\tan\lambda}{4\tan\lambda-3\lambda};$$

$$a_2=\kappa_3-\kappa_2'=\kappa_3+\frac{2\,(1+4\tan\lambda^2)}{(4\tan\lambda-3\lambda)^2},$$

$$\begin{aligned} b_2&=(3\lambda-2\tan\lambda)\,\kappa_3+2\kappa_2'\tan\lambda+5\kappa_2\\ &=(3\lambda-2\tan\lambda)\,\kappa_3-\frac{16\tan\lambda^3-36\tan\lambda+30\lambda}{(4\tan\lambda-3\lambda)^2};\end{aligned}$$

so that the condition $0=(a+b)_2=a_2b_0+2a_1b_1+a_0b_2$ would have become

$$0=2\kappa_3\{4\tan\lambda-3\lambda\}+\frac{4}{(4\tan\lambda-3\lambda)^2}\{2\tan\lambda\,(1+4\tan\lambda^2)+4\tan\lambda+8\tan\lambda^3-18\tan\lambda+15\lambda\},$$

that is,

$$\kappa_3=\frac{-2\,\{16\tan\lambda^3-12\tan\lambda+15\lambda\}}{(4\tan\lambda-3\lambda)^3},$$

as before.

In like manner, substituting this value of κ_3 in a_2 and b_2, we find

$$\begin{aligned}\tfrac{1}{2}a_2\,(4\tan\lambda-3\lambda)^3&=-(16\tan\lambda^3-12\tan\lambda+15\lambda)+(4\tan\lambda^2+1)\,(4\tan\lambda-3\lambda)\\ &=16\tan\lambda-12\lambda\tan\lambda^2-18\lambda,\end{aligned}$$

therefore

$$a_2=\frac{4\,(8\tan\lambda-6\lambda\tan\lambda^2-9\lambda)}{(4\tan\lambda-3\lambda)^3};$$

$$\begin{aligned}\tfrac{1}{2}b_2\,(4\tan\lambda-3\lambda)^3&=(2\tan\lambda-3\lambda)\,(16\tan\lambda^3-12\tan\lambda+15\lambda)\\ &\qquad-(4\tan\lambda-3\lambda)\,(8\tan\lambda^3-18\tan\lambda+15\lambda)\\ &=48\tan\lambda^2-24\lambda\tan\lambda^3-48\lambda\tan\lambda,\end{aligned}$$

therefore

$$b_2=\frac{48\tan\lambda\,(2\tan\lambda-\lambda\tan\lambda^2-2\lambda)}{(4\tan\lambda-3\lambda)^3}.$$

Hence

$$3a_2b_1+3a_1b_2=\frac{48\tan\lambda\,(20\tan\lambda-12\lambda\tan\lambda^2-21\lambda)}{(4\tan\lambda-3\lambda)^4}.$$

Also

$$\begin{aligned}a_3b_0+a_0b_3&=4\tan\lambda\,(\kappa_4-\kappa_3')+2\,(2\tan\lambda-3\lambda)\,\kappa_4-4\kappa_3'\tan\lambda-16\kappa_3\\ &=2\,(4\tan\lambda-3\lambda)\,\kappa_4-8\,(\kappa_3'\tan\lambda+2\kappa_3);\end{aligned}$$

and

$$\begin{aligned}\tfrac{1}{6}\,(4\tan\lambda-3\lambda)^4\,\kappa_3'&=(4\tan\lambda^2+1)\,(16\tan\lambda^3-12\tan\lambda+15\lambda)\\ &\qquad-(4\tan\lambda-3\lambda)\,(16\tan\lambda^4+12\tan\lambda^2+1)\\ &=-80\tan\lambda^3-16\tan\lambda+48\lambda\tan\lambda^4+96\lambda\tan\lambda^2+18\lambda,\end{aligned}$$

therefore

$$\kappa_3'=\frac{12\,(-40\tan\lambda^3-8\tan\lambda+24\lambda\tan\lambda^4+48\lambda\tan\lambda^2+9\lambda)}{(4\tan\lambda-3\lambda)^4}.$$

Substituting these values in the condition

$$(a+b)_3 = a_0 b_3 + 3a_1 b_2 + 3a_2 b_1 + a_3 b_0 = 0,$$

we get

$$-\tfrac{1}{8}(4\tan\lambda - 3\lambda)^5 \kappa_4 = 3\tan\lambda\,(80\tan\lambda^3 + 36\tan\lambda - 48\lambda\tan\lambda^4 - 108\lambda\tan\lambda^2 - 39\lambda)$$
$$+ 2\,(64\tan\lambda^4 - 48\tan\lambda^2 - 48\lambda\tan\lambda^3 + 96\lambda\tan\lambda - 45\lambda^2)$$
$$= 368\tan\lambda^4 + 12\tan\lambda^2 - 144\lambda\tan\lambda^5 - 420\lambda\tan\lambda^3 + 75\lambda\tan\lambda - 90\lambda^2,$$

that is, finally,

$$\boxed{\kappa_4 = \frac{8}{(4\tan\lambda - 3\lambda)^5}\{-4\tan\lambda^2(92\tan\lambda^2+3) + 3\lambda\tan\lambda\,(48\tan\lambda^4 + 140\tan\lambda^2 - 25) + 90\lambda^2\}.}$$

Thus if we only neglect ι^5, or $(\nu-\lambda)^5$, we have

$$\boxed{\begin{aligned}\kappa = 3\lambda + \iota + \frac{\iota^2}{4\tan\lambda - 3\lambda} - \frac{\iota^3}{3}\cdot\frac{16\tan\lambda^3 - 12\tan\lambda + 15\lambda}{(4\tan\lambda - 3\lambda)^3}\\ + \frac{\iota^4}{3}\cdot\frac{-4\tan\lambda^2(92\tan\lambda^2+3) + 3\lambda\tan\lambda\,(48\tan\lambda^4 + 140\tan\lambda^2 - 25) + 90\lambda^2}{(4\tan\lambda - 3\lambda)^5}.\end{aligned}}$$

[4. *The results expressed in polar coordinates.*]

Here, to recapitulate, λ is the half of the angle $\theta_t - \theta_0$ described in the time t, *reduced* to the case of equal final and initial radii vectores r_t, r_0; so that

$$(\xi_t - \xi_0)^2 + (\eta_t - \eta_0)^2 + (\zeta_t - \zeta_0)^2 = (r_t + r_0)^2 \sin\lambda^2,$$

(ξ_t, η_t, ζ_t being the three rectangular heliocentric coordinates of a planet at the time t, and ξ_0, η_0, ζ_0 being their values at the time 0), and

$$\sin\lambda^2 = \left(\sin\frac{\theta_t - \theta_0}{2}\right)^2 + \left(\frac{r_t - r_0}{r_t + r_0}\right)^2\left(\cos\frac{\theta_t - \theta_0}{2}\right)^2;$$

ι is the following quantity, which is small for nearly circular orbits (such as here considered),

$$\iota = -\lambda + t\sqrt{\frac{2\mu}{(r_t + r_0)^3}},$$

μ being a constant such that the differential equations of relative motion are

$$\xi_t'' = -\mu\xi_t r_t^{-3}, \quad \eta_t'' = -\mu\eta_t r_t^{-3}, \quad \zeta_t'' = -\mu\zeta_t r_t^{-3};$$

and

$$\kappa = \frac{\sqrt{2}s}{\sqrt{\mu(r_t + r_0)}},$$

s being the principal function of the motion, or the definite integral

$$s = \int_0^t \left\{\frac{\xi_t'^2 + \eta_t'^2}{2} + \frac{\mu}{r_t}\right\} dt,$$

which, when considered as depending on ξ_t, η_t, ζ_t, ξ_0, η_0, ζ_0 and t, has (by my general theorem) its variation equal to

$$\xi_t'\delta\xi_t - \xi_0'\delta\xi_0 + \eta_t'\delta\eta_t - \eta_0'\delta\eta_0 + \zeta_t'\delta\zeta_t - \zeta_0'\delta\zeta_0 + \frac{\mu}{2\mathrm{a}}\delta t,$$

2a being the axis major of the ellipse.

Accurately $$\cos 2\lambda = \cos(\theta_t - \theta_0) - 2\left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \left(\cos\frac{\theta_t - \theta_0}{2}\right)^2;$$

therefore

$$\sin\{\lambda - \tfrac{1}{2}(\theta_t - \theta_0)\}\sin\{\lambda + \tfrac{1}{2}(\theta_t - \theta_0)\} = \left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \left(\cos\frac{\theta_t - \theta_0}{2}\right)^2.$$

Therefore, nearly, $\lambda = \dfrac{\theta_t - \theta_0}{2}$; more nearly

$$\lambda = \frac{\theta_t - \theta_0}{2} + \frac{1}{2}\left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \operatorname{cotan}\frac{\theta_t - \theta_0}{2};$$

more nearly still

$$\begin{aligned}\lambda &= \frac{\theta_t - \theta_0}{2} + \left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \left(\cos\frac{\theta_t - \theta_0}{2}\right)^2 \operatorname{cosec}\left\{\theta_t - \theta_0 + \frac{1}{2}\left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \operatorname{cotan}\frac{\theta_t - \theta_0}{2}\right\} \\ &= \frac{\theta_t - \theta_0}{2} + \frac{1}{2}\left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \operatorname{cotan}\frac{\theta_t - \theta_0}{2}\left\{1 - \frac{1}{2}\left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \operatorname{cotan}\frac{\theta_t - \theta_0}{2}\operatorname{cotan}(\theta_t - \theta_0)\right\} \\ &= \frac{\theta_t - \theta_0}{2} + \frac{1}{2}\left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \operatorname{cotan}\frac{\theta_t - \theta_0}{2} - \frac{1}{4}\left(\frac{r_t - r_0}{r_t + r_0}\right)^4 \left(\operatorname{cotan}\frac{\theta_t - \theta_0}{2}\right)^2 \operatorname{cotan}(\theta_t - \theta_0);\end{aligned}$$

and since, in general,

$$(\operatorname{cotan} x)^2 \operatorname{cotan} 2x = \frac{(\cos x)^2 \cos 2x}{(\sin x)^2 \sin 2x} = \tfrac{1}{2}(\operatorname{cotan} x^3 - \operatorname{cotan} x),$$

we have, neglecting $\left(\dfrac{r_t - r_0}{r_t + r_0}\right)^6$,

$$\boxed{\lambda = \frac{\theta_t - \theta_0}{2} + \frac{1}{2}\left(\frac{r_t - r_0}{r_t + r_0}\right)^2 \operatorname{cotan}\frac{\theta_t - \theta_0}{2} + \frac{1}{8}\left(\frac{r_t - r_0}{r_t + r_0}\right)^4 \left\{\operatorname{cotan}\frac{\theta_t - \theta_0}{2} - \left(\operatorname{cotan}\frac{\theta_t - \theta_0}{2}\right)^3\right\}.}$$

If we neglect the square of the eccentricity, then

$$\lambda = \frac{\theta_t - \theta_0}{2}, \quad \kappa = \theta_t - \theta_0 + t\sqrt{\frac{2\mu}{(r_t + r_0)^3}},$$

$$\boxed{s = (\theta_t - \theta_0)\sqrt{\frac{\mu(r_t + r_0)}{2}} + \frac{\mu t}{r_t + r_0}.}$$

And if, in this expression, we change r_t into $r_0 + r_t - r_0$, and still neglect the square of the eccentricity, we find more simply (but less symmetrically)

$$s = (\theta_t - \theta_0)\sqrt{\mu r_0} + \frac{\mu t}{2r_0};$$

because the coefficient of $r_t - r_0$ is equal to

$$\frac{\theta_t - \theta_0}{4}\sqrt{\frac{\mu}{r_0}} - \frac{\mu t}{4r_0^2} = \left(\theta_t - \theta_0 - t\sqrt{\frac{\mu}{r_0^3}}\right)\sqrt{\frac{\mu}{16r_0}},$$

and $\theta_t - \theta_0 - t\sqrt{\dfrac{\mu}{r_0^3}}$ is small of the order of the eccentricity.

We can carry s to a further approximation as follows. We have

$$\frac{s}{\sqrt{\mu r_0}} = \kappa\sqrt{\frac{\alpha}{r_0}} = \kappa\sqrt{\frac{r_t+r_0}{2r_0}} = \kappa\sqrt{1+\frac{r_t-r_0}{2r_0}},$$

accurately; therefore, nearly,

$$\frac{s}{\sqrt{\mu r_0}} = \kappa\left\{1+\frac{r_t-r_0}{4r_0}-\frac{1}{2}\left(\frac{r_t-r_0}{4r_0}\right)^2\right\},$$

neglecting the cube of the eccentricity. Neglecting the same cube,

$$\kappa = 3\lambda+\iota+\frac{\iota^2}{4\tan\lambda-3\lambda}; \quad \lambda=\frac{\theta_t-\theta_0}{2}+\frac{1}{2}\left(\frac{r_t-r_0}{2r_0}\right)^2\operatorname{cotan}\frac{nt}{2};$$

where we have put $n=\sqrt{\mu/r_0^3}$;

$$\iota = -\lambda+t\sqrt{\frac{2\mu}{(r_t+r_0)^3}} = -\lambda+\frac{nt}{2}\left\{1+\frac{r_t-r_0}{2r_0}\right\}^{-\frac{3}{2}} = -\lambda+\frac{nt}{2}\left\{1-\frac{3}{2}\left(\frac{r_t-r_0}{2r_0}\right)+\frac{15}{8}\left(\frac{r_t-r_0}{2r_0}\right)^2\right\};$$

therefore

$$\kappa=\frac{3nt}{2}+(\theta_t-\theta_0-nt)-\frac{3nt}{4}\left(\frac{r_t-r_0}{2r_0}\right)+\left(\frac{r_t-r_0}{2r_0}\right)^2\left\{\frac{15nt}{16}+\operatorname{cotan}\frac{nt}{2}\right\}+\frac{\frac{1}{2}\left\{\theta_t-\theta_0-nt+\frac{3nt}{2}\left(\frac{r_t-r_0}{2r_0}\right)\right\}^2}{8\tan\frac{1}{2}nt-3nt};$$

and hence

$$\boxed{\begin{aligned}\frac{s}{\sqrt{\mu r_0}} &= \frac{3nt}{2}+(\theta_t-\theta_0-nt)+\left(\frac{r_t-r_0}{2r_0}\right)^2\left\{\frac{3nt}{8}+\operatorname{cotan}\frac{nt}{2}+\frac{\frac{9}{8}n^2t^2}{8\tan\frac{1}{2}nt-3nt}\right\}\\ &\quad+\frac{4\tan\frac{nt}{2}\left(\frac{r_t-r_0}{2r_0}\right)(\theta_t-\theta_0-nt)+\frac{1}{2}(\theta_t-\theta_0-nt)^2}{8\tan\frac{1}{2}nt-3nt}.\end{aligned}}$$

[5. *The principal function in the three-body problem.*]

Also, if we consider the system of the sun, M, and 2 planets, $m, m_,$, the Principal Function S of the relative motion of this system must satisfy the two partial differential equations following:*

$$\boxed{\begin{aligned}&\frac{\delta S}{\delta t}+\frac{1}{2}\left(\frac{1}{m}+\frac{1}{M}\right)\left\{\left(\frac{\delta S}{\delta\xi_t}\right)^2+\left(\frac{\delta S}{\delta\eta_t}\right)^2+\left(\frac{\delta S}{\delta\zeta_t}\right)^2\right\}+\frac{1}{2}\left(\frac{1}{m_,}+\frac{1}{M}\right)\left\{\left(\frac{\delta S}{\delta\xi_{,t}}\right)^2+\left(\frac{\delta S}{\delta\eta_{,t}}\right)^2+\left(\frac{\delta S}{\delta\zeta_{,t}}\right)^2\right\}\\ &\quad+\frac{1}{M}\left(\frac{\delta S}{\delta\xi_t}\frac{\delta S}{\delta\xi_{,t}}+\frac{\delta S}{\delta\eta_t}\frac{\delta S}{\delta\eta_{,t}}+\frac{\delta S}{\delta\zeta_t}\frac{\delta S}{\delta\zeta_{,t}}\right)=\frac{mM}{\sqrt{r_t^2+\zeta_t^2}}+\frac{m_,M}{\sqrt{r_{,t}^2+\zeta_{,t}^2}}+\frac{mm_,}{\sqrt{\{(\xi_{,t}-\xi_t)^2+(\eta_{,t}-\eta_t)^2+(\zeta_{,t}-\zeta_t)^2\}}};\\ &\frac{\delta S}{\delta t}+\frac{1}{2}\left(\frac{1}{m}+\frac{1}{M}\right)\left\{\left(\frac{\delta S}{\delta\xi_0}\right)^2+\left(\frac{\delta S}{\delta\eta_0}\right)^2+\left(\frac{\delta S}{\delta\zeta_0}\right)^2\right\}+\frac{1}{2}\left(\frac{1}{m_,}+\frac{1}{M}\right)\left\{\left(\frac{\delta S}{\delta\xi_{,0}}\right)^2+\left(\frac{\delta S}{\delta\eta_{,0}}\right)^2+\left(\frac{\delta S}{\delta\zeta_{,0}}\right)^2\right\}\\ &\quad+\frac{1}{M}\left(\frac{\delta S}{\delta\xi_0}\frac{\delta S}{\delta\xi_{,0}}+\frac{\delta S}{\delta\eta_0}\frac{\delta S}{\delta\eta_{,0}}+\frac{\delta S}{\delta\zeta_0}\frac{\delta S}{\delta\zeta_{,0}}\right)=\frac{mM}{\sqrt{r_0^2+\zeta_0^2}}+\frac{m_,M}{\sqrt{r_{,0}^2+\zeta_{,0}^2}}+\frac{mm_,}{\sqrt{\{(\xi_{,0}-\xi_0)^2+(\eta_{,0}-\eta_0)^2+(\zeta_{,0}-\zeta_0)^2\}}}.\end{aligned}}$$

* [These equations follow easily from (D⁶.), p. 153.]

If we put

$$\xi_t = r_t \cos\theta_t,\quad \eta_t = r_t \sin\theta_t,\quad \xi_0 = r_0\cos\theta_0,\quad \eta_0 = r_0 \sin\theta_0,$$

and similarly

$$\xi_{,t} = r_{,t}\cos\theta_{,t},\quad \eta_{,t} = r_{,t}\sin\theta_{,t},\quad \xi_{,0} = r_{,0}\sin\theta_{,0},\quad \eta_{,0} = r_{,0}\sin\theta_{,0},$$

we have

$$\frac{\delta S}{\delta \xi_t} = \cos\theta_t \frac{\delta S}{\delta r_t} - \frac{\sin\theta_t}{r_t}\frac{\delta S}{\delta\theta_t},\quad \frac{\delta S}{\delta\eta_t} = \sin\theta_t\frac{\delta S}{\delta r_t} + \frac{\cos\theta_t}{r_t}\frac{\delta S}{\delta\theta_t};$$

and therefore

$$\left(\frac{\delta S}{\delta\xi_t}\right)^2 + \left(\frac{\delta S}{\delta\eta_t}\right)^2 = \left(\frac{\delta S}{\delta r_t}\right)^2 + \frac{1}{r_t^2}\left(\frac{\delta S}{\delta\theta_t}\right)^2,\quad \left(\frac{\delta S}{\delta\xi_{,t}}\right)^2 + \left(\frac{\delta S}{\delta\eta_{,t}}\right)^2 = \left(\frac{\delta S}{\delta r_{,t}}\right)^2 + \frac{1}{r_{,t}^2}\left(\frac{\delta S}{\delta\theta_{,t}}\right)^2,$$

$$\frac{\delta S}{\delta\xi_t}\frac{\delta S}{\delta\xi_{,t}} + \frac{\delta S}{\delta\eta_t}\frac{\delta S}{\delta\eta_{,t}} = \left(\frac{\delta S}{\delta r_t}\frac{\delta S}{\delta r_{,t}} + \frac{1}{r_t r_{,t}}\frac{\delta S}{\delta\theta_t}\frac{\delta S}{\delta\theta_{,t}}\right)\cos(\theta_{,t}-\theta_t) + \left(\frac{1}{r_t}\frac{\delta S}{\delta\theta_t}\frac{\delta S}{\delta r_{,t}} - \frac{1}{r_{,t}}\frac{\delta S}{\delta\theta_{,t}}\frac{\delta S}{\delta r_t}\right)\sin(\theta_{,t}-\theta_t).$$

The first of the above two partial differential equations gives

$$\boxed{\begin{aligned}0 = {} & \frac{\delta S}{\delta t} + \frac{M+m}{2Mm}\left\{\left(\frac{\delta S}{\delta r_t}\right)^2 + \left(\frac{\delta S}{r_t\delta\theta_t}\right)^2 + \left(\frac{\delta S}{\delta\zeta_t}\right)^2\right\} + \frac{M+m}{2Mm}\left\{\left(\frac{\delta S}{\delta r_{,t}}\right)^2 + \left(\frac{\delta S}{r_{,t}\delta\theta_{,t}}\right)^2 + \left(\frac{\delta S}{\delta\zeta_{,t}}\right)^2\right\} - \frac{Mm}{\sqrt{r_t^2+\zeta_t^2}} \\ & - \frac{Mm_,}{\sqrt{r_{,t}^2+\zeta_{,t}^2}} + \frac{\cos(\theta_{,t}-\theta_t)}{M}\left(\frac{\delta S}{\delta r_t}\frac{\delta S}{\delta r_{,t}} + \frac{\delta S}{r_t\delta\theta_t}\frac{\delta S}{r_{,t}\delta\theta_{,t}}\right) + \frac{\sin(\theta_{,t}-\theta_t)}{M}\left(\frac{\delta S}{\delta r_{,t}}\frac{\delta S}{r_t\delta\theta_t} - \frac{\delta S}{\delta r_t}\frac{\delta S}{r_{,t}\delta\theta_{,t}}\right) \\ & + \frac{1}{M}\frac{\delta S}{\delta\zeta_t}\frac{\delta S}{\delta\zeta_{,t}} - \frac{mm_,}{\sqrt{r_t^2+r_{,t}^2-2r_t r_{,t}\cos(\theta_{,t}-\theta_t)+(\zeta_{,t}-\zeta_t)^2}},\end{aligned}}$$

and the second may be similarly transformed. The first five terms may be made to vanish by employing elliptic values; the remaining terms give the perturbations.

We shall assume that the inclinations are neglected, and shall put $\zeta_t = \zeta_0 = 0$, $\zeta_{,t} = \zeta_{,0} = 0$. It is easy to perceive that in a system attracting according to Newton's law all the linear coordinates may be multiplied by any one common factor l (besides altering all the positions by any common motion of rotation), and all the masses may be multiplied by any other common factor $l_,$, provided that the time t is multiplied by the factor $l^{\frac{3}{2}}l_,^{-\frac{1}{2}}$. And then, in the general expression for the principal function of relative motion,

$$S = \int_0^t T_, dt + \int_0^t U\,dt,$$

the coefficient U will be multiplied by $l_,^2/l$ and $T_,$ will be multiplied by $l_, l^2/l^3 l_,^{-1} = l_,^2/l$, and therefore $U dt$ and $T_, dt$ and finally S itself will each be multiplied by $l_,^{\frac{3}{2}} l^{\frac{1}{2}}$. Hence, in the present system, in which the orbits are both in one plane,

$$S = M^{\frac{3}{2}} r_0^{\frac{1}{2}} \times \text{funct.}\left(\frac{m}{M}, \frac{m_,}{M}, \frac{r_t}{r_0}, \frac{r_{,0}}{r_0}, \frac{r_{,t}}{r_0}, \theta_t - \theta_0, \theta_{,0}-\theta_0, \theta_{,t}-\theta_0, nt\right)$$

rigorously, or, if we choose to express it so, then

$$S = Mm\sqrt{\frac{r_t+r_0}{2(M+m)}}\,\kappa + Mm_,\sqrt{\frac{r_{,t}+r_{,0}}{2(M+m_,)}}\,\kappa_, + \frac{mm_,}{\sqrt{M}}W,$$

$$W = \sqrt{r_0+r_{,0}}\,\text{funct.}\left(\lambda, \lambda_,, \iota, \iota_,, \frac{r_t-r_0}{r_t+r_0}, \frac{r_{,t}-r_{,0}}{r_{,t}+r_{,0}}, \theta_{,0}-\theta_0\right),$$

κ, λ, ι having the same significance as before for the first orbit and $\kappa_,$, $\lambda_,$, $\iota_,$ similar quantities for the second orbit. In rigour, W involves m/M, $m_,/M$, but these small ratios may be treated as equal to zero in W, if we wish only to deduce those perturbations which are of the first order with respect to the disturbing masses. For greater symmetry we might put

$$W = \sqrt{\frac{r_{,t}+r_t+r_{,0}+r_0}{2}}\,\text{funct.}\left(\frac{\theta_{,t}-\theta_t+\theta_{,0}-\theta_0}{2},\ \lambda,\ \lambda_,,\ \iota,\ \iota_,,\ \frac{r_t-r_0}{r_t+r_0},\ \frac{r_{,t}-r_{,0}}{r_{,t}+r_{,0}}\right).$$

Thus, if we put for abridgment, as before, $\alpha=\frac{1}{2}(r_t+r_0)$, $\mu=M+m$, and similarly

$$\alpha_,=\tfrac{1}{2}(r_{,t}+r_{,0}),\quad \mu_,=M+m_,;$$

and if we put also

$$\beta=\frac{r_t-r_0}{r_t+r_0},\quad \beta_,=\frac{r_{,t}-r_{,0}}{r_{,t}+r_{,0}},\quad \vartheta=\frac{\theta_{,t}+\theta_{,0}}{2}-\frac{\theta_t+\theta_0}{2},$$

we may write

$$\boxed{S=Mm\sqrt{\frac{\alpha}{\mu}}\,\kappa+Mm_,\sqrt{\frac{\alpha_,}{\mu_,}}\,\kappa_,+mm_,\sqrt{\frac{\alpha+\alpha_,}{M}}\,\psi(\vartheta,\ \lambda,\ \lambda_,,\ \iota,\ \iota_,,\ \beta,\ \beta_,);}$$

and may form two partial differential equations relatively to the function ψ as follows.

S is explicitly a function of α, $\alpha_,$, λ, $\lambda_,$, β, $\beta_,$, ι, $\iota_,$ and ϑ, involving also the masses; its variation may therefore be thus expressed

$$\delta S=\frac{\delta S}{\delta\alpha}\delta\alpha+\frac{\delta S}{\delta\alpha_,}\delta\alpha_,+\frac{\delta S}{\delta\lambda}\delta\lambda+\frac{\delta S}{\delta\lambda_,}\delta\lambda_,+\frac{\delta S}{\delta\beta}\delta\beta+\frac{\delta S}{\delta\beta_,}\delta\beta_,+\frac{\delta S}{\delta\iota}\delta\iota+\frac{\delta S}{\delta\iota_,}\delta\iota_,+\frac{\delta S}{\delta\vartheta}\delta\vartheta.$$

Now

$$\sin\lambda^2=\left(\sin\frac{\theta_t-\theta_0}{2}\right)^2+\beta^2\left(\cos\frac{\theta_t-\theta_0}{2}\right)^2,$$

and so

$$\delta\lambda=\frac{1-\beta^2}{2}\frac{\sin(\theta_t-\theta_0)}{\sin 2\lambda}(\delta\theta_t-\delta\theta_0)+\frac{1+\cos(\theta_t-\theta_0)}{\sin 2\lambda}\beta\delta\beta.$$

Also

$$\left(\cos\frac{\theta_t-\theta_0}{2}\right)^2=\frac{\cos\lambda^2}{1-\beta^2},\quad \left(\sin\frac{\theta_t-\theta_0}{2}\right)^2=\frac{\sin\lambda^2-\beta^2}{1-\beta^2},$$

that is,

$$\frac{1+\cos(\theta_t-\theta_0)}{\sin 2\lambda}=\frac{\operatorname{cotan}\lambda}{1-\beta^2},\quad \frac{(1-\beta^2)\sin(\theta_t-\theta_0)}{\sin 2\lambda}=+\sqrt{1-\beta^2\operatorname{cosec}\lambda^2},$$

therefore

$$\delta\lambda=\tfrac{1}{2}\sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\,(\delta\theta_t-\delta\theta_0)+\operatorname{cotan}\lambda\frac{\beta\delta\beta}{1-\beta^2}.$$

Now

$$\beta=\frac{r_t-r_0}{r_t+r_0},\quad \alpha=\tfrac{1}{2}(r_t+r_0),$$

and hence

$$\delta\beta=\frac{\delta r_t-\delta r_0}{2\alpha}-\frac{\beta}{2\alpha}(\delta r_t+\delta r_0)=\frac{1-\beta}{2\alpha}\delta r_t-\frac{1+\beta}{2\alpha}\delta r_0.$$

Also

$$\iota=-\lambda+\tfrac{1}{2}t\sqrt{\frac{\mu}{\alpha^3}},$$

therefore

$$\delta\iota = -\delta\lambda + \frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\,\delta t - \frac{3t}{4}\sqrt{\frac{\mu}{\alpha^5}}\,\delta\alpha = -\delta\lambda + \frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\,\delta t - \frac{3(\lambda+\iota)}{2\alpha}\,\delta\alpha.$$

Hence

$$\frac{\delta S}{\delta\alpha}\delta\alpha + \frac{\delta S}{\delta\beta}\delta\beta + \frac{\delta S}{\delta\lambda}\delta\lambda + \frac{\delta S}{\delta\iota}\delta\iota = \frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\frac{\delta S}{\delta\iota}\delta t + \left\{\frac{\delta S}{\delta\alpha} - \frac{3}{2}\frac{\lambda+\iota}{\alpha}\frac{\delta S}{\delta\iota}\right\}\frac{\delta r_t + \delta r_0}{2}$$

$$+\left\{\frac{\delta S}{\delta\beta} + \operatorname{cotan}\lambda\frac{\beta}{1-\beta^2}\left(\frac{\delta S}{\delta\lambda} - \frac{\delta S}{\delta\iota}\right)\right\}\frac{(1-\beta)\,\delta r_t - (1+\beta)\,\delta r_0}{2\alpha} + \sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\left(\frac{\delta S}{\delta\lambda} - \frac{\delta S}{\delta\iota}\right)\frac{\delta\theta_t - \delta\theta_0}{2},$$

that is,

$$\delta S = \frac{1}{2}\left(\sqrt{\frac{\mu}{\alpha^3}}\frac{\delta S}{\delta\iota} + \sqrt{\frac{\mu_{,}}{\alpha_{,}^3}}\frac{\delta S}{\delta\iota_{,}}\right)\delta t + \left(\frac{\delta S}{\delta\alpha} - \frac{3}{2}\frac{\lambda+\iota}{\alpha}\frac{\delta S}{\delta\iota}\right)\frac{\delta r_t + \delta r_0}{2} + \left(\frac{\delta S}{\delta\alpha_{,}} - \frac{3}{2}\frac{\lambda_{,}+\iota_{,}}{\alpha_{,}}\frac{\delta S}{\delta\iota_{,}}\right)\frac{\delta r_{,t} + \delta r_{,0}}{2}$$

$$+\left\{\frac{\delta S}{\delta\beta} + \frac{\beta\operatorname{cotan}\lambda}{1-\beta^2}\left(\frac{\delta S}{\delta\lambda} - \frac{\delta S}{\delta\iota}\right)\right\}\frac{(1-\beta)\,\delta r_t - (1+\beta)\,\delta r_0}{2\alpha} + \left\{\frac{\delta S}{\delta\beta_{,}} + \frac{\beta_{,}\operatorname{cotan}\lambda_{,}}{1-\beta_{,}^2}\left(\frac{\delta S}{\delta\lambda_{,}} - \frac{\delta S}{\delta\iota_{,}}\right)\right\}$$

$$\times\frac{(1-\beta_{,})\,\delta r_{,t} - (1+\beta_{,})\,\delta r_{,0}}{2\alpha_{,}} + \sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\left(\frac{\delta S}{\delta\lambda} - \frac{\delta S}{\delta\iota}\right)\frac{\delta\theta_t - \delta\theta_0}{2} + \sqrt{1-\beta_{,}^2\operatorname{cosec}\lambda_{,}^2}$$

$$\times\left(\frac{\delta S}{\delta\lambda_{,}} - \frac{\delta S}{\delta\iota_{,}}\right)\frac{\delta\theta_{,t} - \delta\theta_{,0}}{2} + \frac{\delta S}{\delta\vartheta}\frac{\delta\theta_{,t} - \delta\theta_t + \delta\theta_{,0} - \delta\theta_0}{2}.$$

From this equation we immediately deduce the partial derivatives of S with respect to the variables $t, r_t, r_{,t}, \theta_t, \theta_{,t}, r_0, r_{,0}, \theta_0$, and $\theta_{,0}$, and we have then to substitute these values in the two partial differential equations (for the case of null inclinations)

$$0 = \frac{\delta S}{\delta t} + \frac{M+m}{2Mm}\left\{\left(\frac{\delta S}{\delta r_t}\right)^2 + \left(\frac{\delta S}{r_t\delta\theta_t}\right)^2\right\} + \frac{M+m_{,}}{2Mm_{,}}\left\{\left(\frac{\delta S}{\delta r_{,t}}\right)^2 + \left(\frac{\delta S}{r_{,t}\delta\theta_{,t}}\right)^2\right\} - \frac{Mm}{r_t} - \frac{Mm_{,}}{r_{,t}}$$

$$+\frac{\cos(\theta_{,t}-\theta_t)}{M}\left(\frac{\delta S}{\delta r_t}\frac{\delta S}{\delta r_{,t}} + \frac{\delta S}{r_t\delta\theta_t}\frac{\delta S}{r_{,t}\delta\theta_{,t}}\right) + \frac{\sin(\theta_{,t}-\theta_t)}{M}\left(\frac{\delta S}{\delta r_{,t}}\frac{\delta S}{r_t\delta\theta_t} - \frac{\delta S}{\delta r_t}\frac{\delta S}{r_{,t}\delta\theta_{,t}}\right)$$

$$-\frac{mm_{,}}{\sqrt{r_{,t}^2 + r_t^2 - 2r_{,t}r_t\cos(\theta_{,t}-\theta_t)}},$$

and

$$0 = \frac{\delta S}{\delta t} + \frac{M+m}{2Mm}\left\{\left(\frac{\delta S}{\delta r_0}\right)^2 + \left(\frac{\delta S}{r_0\delta\theta_0}\right)^2\right\} + \frac{M+m_{,}}{2Mm_{,}}\left\{\left(\frac{\delta S}{\delta r_{,0}}\right)^2 + \left(\frac{\delta S}{r_{,0}\delta\theta_{,0}}\right)^2\right\} - \frac{Mm}{r_0} - \frac{Mm_{,}}{r_{,0}}$$

$$+\frac{\cos(\theta_{,0}-\theta_0)}{M}\left(\frac{\delta S}{\delta r_0}\frac{\delta S}{\delta r_{,0}} + \frac{\delta S}{r_0\delta\theta_0}\frac{\delta S}{r_{,0}\delta\theta_{,0}}\right) + \frac{\sin(\theta_{,0}-\theta_0)}{M}\left(\frac{\delta S}{\delta r_{,0}}\frac{\delta S}{r_0\delta\theta_0} - \frac{\delta S}{\delta r_0}\frac{\delta S}{r_{,0}\delta\theta_{,0}}\right)$$

$$-\frac{mm_{,}}{\sqrt{r_{,0}^2 + r_0^2 - 2r_{,0}r_0\cos(\theta_{,0}-\theta_0)}}.$$

We have

$$\theta_{,t} - \theta_t = \vartheta + \frac{\theta_{,t}-\theta_{,0}}{2} - \frac{\theta_t - \theta_0}{2}, \quad \theta_{,0} - \theta_0 = \vartheta - \frac{\theta_{,t}-\theta_{,0}}{2} + \frac{\theta_t-\theta_0}{2},$$

and

$$\cos\frac{\theta_t-\theta_0}{2} = \frac{\cos\lambda}{\sqrt{1-\beta^2}}, \quad \sin\frac{\theta_t-\theta_0}{2} = \frac{\sin\lambda}{\sqrt{1-\beta^2}}\sqrt{1-\beta^2\operatorname{cosec}\lambda^2},$$

$$\cos\frac{\theta_{,t}-\theta_{,0}}{2} = \frac{\cos\lambda_{,}}{\sqrt{1-\beta_{,}^2}}, \quad \sin\frac{\theta_{,t}-\theta_{,0}}{2} = \frac{\sin\lambda_{,}}{\sqrt{1-\beta_{,}^2}}\sqrt{1-\beta_{,}^2\operatorname{cosec}\lambda_{,}^2};$$

therefore

$$\cos(\theta_{,\iota}-\theta_{\iota})=K\cos\vartheta-L\sin\vartheta,\quad \sin(\theta_{,\iota}-\theta_{\iota})=K\cos\vartheta+L\sin\vartheta,$$

where we have put for abridgment

$$K=\cos\left(\frac{\theta_{,\iota}-\theta_{,0}}{2}-\frac{\theta_{\iota}-\theta_0}{2}\right)=\frac{\cos\lambda\cos\lambda_,+\sin\lambda\sin\lambda_,\sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\sqrt{1-\beta_,^2\operatorname{cosec}\lambda_,^2}}{\sqrt{1-\beta^2}\sqrt{1-\beta_,^2}},$$

$$L=\sin\left(\frac{\theta_{,\iota}-\theta_{,0}}{2}-\frac{\theta_{\iota}-\theta_0}{2}\right)=\frac{\cos\lambda\sin\lambda_,\sqrt{1-\beta_,^2\operatorname{cosec}\lambda_,^2}-\sin\lambda\cos\lambda_,\sqrt{1-\beta^2\operatorname{cosec}\lambda^2}}{\sqrt{1-\beta^2}\sqrt{1-\beta_,^2}}.$$

Thus, making the above substitutions, the first partial differential equation becomes *rigorously* (for the case of null inclinations)

$$\boxed{\begin{aligned}
0=&\frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\frac{\delta S}{\delta\iota}+\frac{1}{2}\sqrt{\frac{\mu_,}{\alpha_,^3}}\frac{\delta S}{\delta\iota_,}-\frac{Mm}{\alpha(1+\beta)}-\frac{Mm_,}{\alpha_,(1+\beta_,)}\\
&-\frac{mm_,}{\sqrt{\alpha^2(1+\beta)^2+\alpha_,^2(1+\beta_,)^2-2\alpha\alpha_,(1+\beta)(1+\beta_,)(K\cos\vartheta-L\sin\vartheta)}}\\
&+\frac{M+m}{8Mm\alpha^2}\left\{\alpha\frac{\delta S}{\delta\alpha}+(1-\beta)\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}+\frac{\beta\cot\lambda}{1+\beta}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\right\}^2\\
&+\frac{M+m_,}{8Mm_,\alpha_,^2}\left\{\alpha_,\frac{\delta S}{\delta\alpha_,}+(1-\beta_,)\frac{\delta S}{\delta\beta_,}-\tfrac{3}{2}(\lambda_,+\iota_,)\frac{\delta S}{\delta\iota_,}+\frac{\beta_,\cot\lambda_,}{1+\beta_,}\left(\frac{\delta S}{\delta\lambda_,}-\frac{\delta S}{\delta\iota_,}\right)\right\}^2\\
&+\frac{M+m}{8Mm\alpha^2(1+\beta)^2}\left\{\sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)-\frac{\delta S}{\delta\vartheta}\right\}^2\\
&+\frac{M+m_,}{8Mm_,\alpha_,^2(1+\beta_,)^2}\left\{\sqrt{1-\beta_,^2\operatorname{cosec}\lambda_,^2}\left(\frac{\delta S}{\delta\lambda_,}-\frac{\delta S}{\delta\iota_,}\right)+\frac{\delta S}{\delta\vartheta}\right\}^2\\
&+\frac{K\cos\vartheta-L\sin\vartheta}{4M\alpha\alpha_,}\left\{\alpha\frac{\delta S}{\delta\alpha}+(1-\beta)\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}+\frac{\beta\cot\lambda}{1+\beta}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\right\}\\
&\times\left\{\alpha_,\frac{\delta S}{\delta\alpha_,}+(1-\beta_,)\frac{\delta S}{\delta\beta_,}-\tfrac{3}{2}(\lambda_,+\iota_,)\frac{\delta S}{\delta\iota_,}+\frac{\beta_,\cot\lambda_,}{1+\beta_,}\left(\frac{\delta S}{\delta\lambda_,}-\frac{\delta S}{\delta\iota_,}\right)\right\}\\
&+\frac{K\cos\vartheta-L\sin\vartheta}{4M\alpha\alpha_,(1+\beta)(1+\beta_,)}\left\{\sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)-\frac{\delta S}{\delta\vartheta}\right\}\\
&\times\left\{\sqrt{1-\beta_,^2\operatorname{cosec}\lambda_,^2}\left(\frac{\delta S}{\delta\lambda_,}-\frac{\delta S}{\delta\iota_,}\right)+\frac{\delta S}{\delta\vartheta}\right\}+\frac{K\sin\vartheta+L\cos\vartheta}{4M\alpha\alpha_,(1+\beta)}\left\{\sqrt{1-\beta^2\operatorname{cosec}\lambda^2}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)-\frac{\delta S}{\delta\vartheta}\right\}\\
&\times\left\{\alpha_,\frac{\delta S}{\delta\alpha_,}+(1-\beta_,)\frac{\delta S}{\delta\beta_,}-\tfrac{3}{2}(\lambda_,+\iota_,)\frac{\delta S}{\delta\iota_,}+\frac{\beta_,\cot\lambda_,}{1+\beta_,}\left(\frac{\delta S}{\delta\lambda_,}-\frac{\delta S}{\delta\iota_,}\right)\right\}\\
&-\frac{K\sin\vartheta+L\cos\vartheta}{4M\alpha\alpha_,(1+\beta_,)}\left\{\sqrt{1-\beta_,^2\operatorname{cosec}\lambda_,^2}\left(\frac{\delta S}{\delta\lambda_,}-\frac{\delta S}{\delta\iota_,}\right)+\frac{\delta S}{\delta\vartheta}\right\}\\
&\qquad\times\left\{\alpha\frac{\delta S}{\delta\alpha}+(1-\beta)\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}+\frac{\beta\cot\lambda}{1+\beta}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}\right)\right\};
\end{aligned}}$$

and the second partial differential equation may be *rigorously* formed from this by merely changing the signs of β, $\beta_,$, ϑ.

[**6.** *Approximations in the case of nearly circular orbits.*]

If we entirely neglect β and $\beta_,$, or rather suppose them rigorously to vanish, then the partial differential equations become

$$0=\frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\frac{\delta S}{\delta\iota}+\frac{1}{2}\sqrt{\frac{\mu_,}{\alpha_,^3}}\frac{\delta S}{\delta\iota_,}-\frac{Mm}{\alpha}-\frac{Mm_,}{\alpha_,}-\frac{mm_,}{\sqrt{\alpha^2+\alpha_,^2-2\alpha\alpha_,\cos(\vartheta+\lambda_,-\lambda)}}$$
$$+\frac{M+m}{8Mm\alpha^2}\left\{\alpha\frac{\delta S}{\delta\alpha}+\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}\right\}^2+\frac{M+m_,}{8Mm_,\alpha_,^2}\left\{\alpha_,\frac{\delta S}{\delta\alpha_,}+\frac{\delta S}{\delta\beta_,}-\tfrac{3}{2}(\lambda_,+\iota_,)\frac{\delta S}{\delta\iota_,}\right\}^2$$
$$+\frac{M+m}{8Mm\alpha^2}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}-\frac{\delta S}{\delta\vartheta}\right)^2+\frac{M+m_,}{8Mm_,\alpha_,^2}\left(\frac{\delta S}{\delta\lambda_,}-\frac{\delta S}{\delta\iota_,}+\frac{\delta S}{\delta\vartheta}\right)^2$$
$$+\frac{\cos(\vartheta+\lambda_,-\lambda)}{4M\alpha\alpha_,}\left\{\alpha\frac{\delta S}{\delta\alpha}+\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}\right\}\left\{\alpha_,\frac{\delta S}{\delta\alpha_,}+\frac{\delta S}{\delta\beta_,}-\tfrac{3}{2}(\lambda_,+\iota_,)\frac{\delta S}{\delta\iota_,}\right\}$$
$$+\frac{\cos(\vartheta+\lambda_,-\lambda)}{4M\alpha\alpha_,}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}-\frac{\delta S}{\delta\vartheta}\right)\left(\frac{\delta S}{\delta\lambda_,}-\frac{\delta S}{\delta\iota_,}+\frac{\delta S}{\delta\vartheta}\right)$$
$$+\frac{\sin(\vartheta+\lambda_,-\lambda)}{4M\alpha\alpha_,}\left(\frac{\delta S}{\delta\lambda}-\frac{\delta S}{\delta\iota}-\frac{\delta S}{\delta\vartheta}\right)\left\{\alpha_,\frac{\delta S}{\delta\alpha_,}+\frac{\delta S}{\delta\beta_,}-\tfrac{3}{2}(\lambda_,+\iota_,)\frac{\delta S}{\delta\iota_,}\right\}$$
$$-\frac{\sin(\vartheta+\lambda_,-\lambda)}{4M\alpha\alpha_,}\left(\frac{\delta S}{\delta\lambda_,}-\frac{\delta S}{\delta\iota_,}+\frac{\delta S}{\delta\vartheta}\right)\left\{\alpha\frac{\delta S}{\delta\alpha}+\frac{\delta S}{\delta\beta}-\tfrac{3}{2}(\lambda+\iota)\frac{\delta S}{\delta\iota}\right\},$$

together with the other similar partial differential equation derived from this by changing the sign of ϑ and those of $\frac{\delta S}{\delta\vartheta}$, $\frac{\delta S}{\delta\beta}$, $\frac{\delta S}{\delta\beta_,}$. In these equations no power of any mass is neglected nor any of ι, $\iota_,$; but if we put

$$S=Mm\kappa\sqrt{\frac{\alpha}{\mu}}+Mm_,\kappa_,\sqrt{\frac{\alpha_,}{\mu_,}}+mm_,\psi\sqrt{\frac{\alpha+\alpha_,}{M}}$$

and neglect $m^2m_,$ and $mm_,^2$, the equations become, after being reduced by the partial differential equations of elliptic motion and divided by $mm_,/16$,

$$0=\sqrt{\frac{\mu}{\alpha^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\kappa-3\lambda\frac{\delta\kappa}{\delta\iota}-3\iota\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\alpha\psi}{\alpha+\alpha_,}\pm 2\frac{\delta\psi}{\delta\beta}-3\lambda\frac{\delta\psi}{\delta\iota}-3\iota\frac{\delta\psi}{\delta\iota}\right)$$
$$+\sqrt{\frac{\mu_,}{\alpha_,^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\kappa_,-3\lambda_,\frac{\delta\kappa_,}{\delta\iota_,}-3\iota_,\frac{\delta\kappa_,}{\delta\iota_,}\right)\left(\frac{\alpha_,\psi}{\alpha+\alpha_,}\pm 2\frac{\delta\psi}{\delta\beta_,}-3\lambda_,\frac{\delta\psi}{\delta\iota_,}-3\iota_,\frac{\delta\psi}{\delta\iota_,}\right)$$
$$+4\sqrt{\frac{\mu}{\alpha^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\frac{\delta\kappa}{\delta\lambda}-\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\delta\psi}{\delta\lambda}\mp\frac{\delta\psi}{\delta\vartheta}\right)+4\sqrt{\frac{\mu_,}{\alpha_,^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\frac{\delta\kappa_,}{\delta\lambda_,}-\frac{\delta\kappa_,}{\delta\iota_,}\right)\left(\frac{\delta\psi}{\delta\lambda_,}\pm\frac{\delta\psi}{\delta\vartheta}\right)$$
$$-4\sqrt{\frac{\mu}{\alpha^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\frac{\delta\kappa}{\delta\lambda}-\frac{\delta\kappa}{\delta\iota}-2\right)\frac{\delta\psi}{\delta\iota}-4\sqrt{\frac{\mu_,}{\alpha_,^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\frac{\delta\kappa_,}{\delta\lambda_,}-\frac{\delta\kappa_,}{\delta\iota_,}-2\right)\frac{\delta\psi}{\delta\iota_,}$$
$$+\frac{M}{\sqrt{\mu\alpha}\sqrt{\mu_,\alpha_,}}\cos(\pm\vartheta+\lambda_,-\lambda)\left\{\left(\kappa-3\lambda\frac{\delta\kappa}{\delta\iota}-3\iota\frac{\delta\kappa}{\delta\iota}\right)\left(\kappa_,-3\lambda_,\frac{\delta\kappa_,}{\delta\iota_,}-3\iota_,\frac{\delta\kappa_,}{\delta\iota_,}\right)\right.$$
$$\left.+4\left(\frac{\delta\kappa}{\delta\lambda}-\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\delta\kappa_,}{\delta\lambda_,}-\frac{\delta\kappa_,}{\delta\iota_,}\right)\right\}$$
$$+\frac{2M}{\sqrt{\mu\alpha}\sqrt{\mu_,\alpha_,}}\sin(\pm\vartheta+\lambda_,-\lambda)\left\{\left(\kappa_,-3\lambda_,\frac{\delta\kappa_,}{\delta\iota_,}-3\iota_,\frac{\delta\kappa_,}{\delta\iota_,}\right)\left(\frac{\delta\kappa}{\delta\lambda}-\frac{\delta\kappa}{\delta\iota}\right)-\left(\kappa-3\lambda\frac{\delta\kappa}{\delta\iota}-3\iota\frac{\delta\kappa}{\delta\iota}\right)\left(\frac{\delta\kappa_,}{\delta\lambda_,}-\frac{\delta\kappa_,}{\delta\iota_,}\right)\right\}$$
$$-\frac{16}{\sqrt{\alpha^2+\alpha_,^2-2\alpha\alpha_,\cos(\pm\vartheta+\lambda_,-\lambda)}}.$$

This double equation is rigorous so far as powers of ι are concerned, though β, $\beta_,$ and $mm_,^2$, $m^2m_,$ have been neglected. But if we change κ and $\kappa_,$ to their approximate values $3\lambda+\iota$ and $3\lambda_,+\iota_,$, and then neglect ι, $\iota_,$, the double equation becomes

$$0=\frac{1}{2}\sqrt{\frac{\mu}{\alpha^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\frac{\delta\psi}{\delta\lambda}\mp\frac{\delta\psi}{\delta\vartheta}\right)+\frac{1}{2}\sqrt{\frac{\mu_,}{\alpha_,^3}}\sqrt{\frac{\alpha+\alpha_,}{M}}\left(\frac{\delta\psi}{\delta\lambda_,}\pm\frac{\delta\psi}{\delta\vartheta}\right)+\frac{M\cos(\pm\vartheta+\lambda_,-\lambda)}{\sqrt{\mu\alpha}\sqrt{\mu_,\alpha_,}}$$
$$-\frac{1}{\sqrt{\alpha^2+\alpha_,^2-2\alpha\alpha_,\cos(\pm\vartheta+\lambda_,-\lambda)}}.$$

If we now put

$$n=\sqrt{\frac{\mu}{\alpha^3}},\quad n_,=\sqrt{\frac{\mu_,}{\alpha_,^3}},$$

and therefore

$$\sqrt{\mu\alpha}=n\alpha^2,\quad \sqrt{\mu_,\alpha_,}=n_,\alpha_,^2,\quad \frac{\lambda}{n}=\frac{\lambda_,}{n_,},\quad \frac{\lambda\alpha^{\frac{3}{2}}}{\sqrt{\mu}}=\frac{\lambda_,\alpha_,^{\frac{3}{2}}}{\sqrt{\mu_,}},$$

and make as a sufficient approximation $\mu/M=\mu_,/M=1$, we shall change the double equation to the following:

$$0=\sqrt{\lambda^{-\frac{2}{3}}+\lambda_,^{-\frac{2}{3}}}\left\{\lambda\left(\frac{\delta\psi}{\delta\lambda}\mp\frac{\delta\psi}{\delta\vartheta}\right)+\lambda_,\left(\frac{\delta\psi}{\delta\lambda_,}\pm\frac{\delta\psi}{\delta\vartheta}\right)\right\}+2\lambda^{\frac{1}{3}}\lambda_,^{\frac{1}{3}}\cos(\pm\vartheta+\lambda_,-\lambda)$$
$$-\frac{2}{\sqrt{\lambda^{-\frac{4}{3}}+\lambda_,^{-\frac{4}{3}}-2\lambda^{-\frac{2}{3}}\lambda_,^{-\frac{2}{3}}\cos(\pm\vartheta+\lambda_,-\lambda)}},$$

that is,

$$0=\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}}\left(\frac{\delta\psi}{\delta\lambda_,}+\frac{\lambda}{\lambda_,}\frac{\delta\psi}{\delta\lambda}\right)+2\left(\frac{\lambda}{\lambda_,}\right)^{\frac{1}{3}}\cos(\lambda_,-\lambda)\cos\vartheta-\frac{1}{\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{4}{3}}-2\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}\cos(\vartheta+\lambda_,-\lambda)}}$$
$$-\frac{1}{\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{4}{3}}-2\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}\cos(-\vartheta+\lambda_,-\lambda)}},$$

and

$$0=\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}}\left(1-\frac{\lambda}{\lambda_,}\right)\frac{\delta\psi}{\delta\vartheta}-2\left(\frac{\lambda}{\lambda_,}\right)^{\frac{1}{3}}\sin(\lambda_,-\lambda)\sin\vartheta-\frac{1}{\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{4}{3}}-2\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}\cos(\vartheta+\lambda_,-\lambda)}}$$
$$+\frac{1}{\sqrt{1+\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{4}{3}}-2\left(\frac{\lambda}{\lambda_,}\right)^{-\frac{2}{3}}\cos(-\vartheta+\lambda_,-\lambda)}}.$$

We have seen that $\frac{\lambda_,}{\lambda}=\frac{n_,}{n}=\left(\frac{\alpha}{\alpha_,}\right)^{\frac{3}{2}}$; if then we put for a moment $\alpha/\alpha_,=\alpha'$ and $\lambda_,-\lambda=\lambda'$, we shall have

$$\frac{\lambda_,}{\lambda}=\alpha'^{\frac{3}{2}},\quad \lambda=\frac{\lambda'}{\alpha'^{\frac{3}{2}}-1},\quad \lambda_,=\frac{\alpha'^{\frac{3}{2}}\lambda'}{\alpha'^{\frac{3}{2}}-1},$$

$$\frac{\delta\psi}{\delta\lambda_,}=\frac{\delta\psi}{\delta\lambda'}+\frac{2}{3}\frac{\delta\psi}{\delta\alpha'}\frac{\lambda_,^{-\frac{1}{3}}}{\lambda^{\frac{2}{3}}},\quad \frac{\delta\psi}{\delta\lambda}=-\frac{\delta\psi}{\delta\lambda'}-\frac{2}{3}\frac{\delta\psi}{\delta\alpha'}\frac{\lambda_,^{\frac{2}{3}}}{\lambda^{\frac{5}{3}}};$$

and hence

$$\frac{\delta\psi}{\delta\lambda_{,}}+\frac{\lambda}{\lambda_{,}}\frac{\delta\psi}{\delta\lambda}=\left(1-\frac{\lambda}{\lambda_{,}}\right)\frac{\delta\psi}{\delta\lambda^{\backprime}}.$$

Therefore

$$0=\sqrt{1+\alpha^{\backprime}}\,(1-\alpha^{\backprime-\frac{3}{2}})\frac{\delta\psi}{\delta\lambda^{\backprime}}+\frac{2}{\sqrt{\alpha^{\backprime}}}\cos\lambda^{\backprime}\cos\vartheta-\frac{1}{\sqrt{1+\alpha^{\backprime 2}-2\alpha^{\backprime}\cos(\vartheta+\lambda^{\backprime})}}-\frac{1}{\sqrt{1+\alpha^{\backprime 2}-2\alpha^{\backprime}\cos(-\vartheta+\lambda^{\backprime})}},$$

and

$$0=\sqrt{1+\alpha^{\backprime}}\,(1-\alpha^{\backprime-\frac{3}{2}})\frac{\delta\psi}{\delta\vartheta}-\frac{2}{\sqrt{\alpha^{\backprime}}}\sin\lambda^{\backprime}\sin\vartheta-\frac{1}{\sqrt{1+\alpha^{\backprime 2}-2\alpha^{\backprime}\cos(\vartheta+\lambda^{\backprime})}}+\frac{1}{\sqrt{1+\alpha^{\backprime 2}-2\alpha^{\backprime}\cos(-\vartheta+\lambda^{\backprime})}}.$$

Treating $\alpha^{\backprime}$ as constant, this gives

$$\sqrt{1+\alpha^{\backprime}}\,(1-\alpha^{\backprime-\frac{3}{2}})\,\delta\psi=\frac{2}{\sqrt{\alpha^{\backprime}}}(\sin\lambda^{\backprime}\sin\vartheta\,\delta\vartheta-\cos\lambda^{\backprime}\cos\vartheta\,\delta\lambda^{\backprime})+\frac{\delta\vartheta+\delta\lambda^{\backprime}}{\sqrt{1+\alpha^{\backprime 2}-2\alpha^{\backprime}\cos(\vartheta+\lambda^{\backprime})}}$$

$$+\frac{\delta\lambda^{\backprime}-\delta\vartheta}{\sqrt{1+\alpha^{\backprime 2}-2\alpha^{\backprime}\cos(-\vartheta+\lambda^{\backprime})}},$$

that is,

$$\sqrt{1+\alpha^{\backprime}}\,(1-\alpha^{\backprime-\frac{3}{2}})\,\psi=\text{funct.}\,(\alpha^{\backprime})-\frac{2}{\sqrt{\alpha^{\backprime}}}\sin\lambda^{\backprime}\cos\vartheta+\int_{\vartheta-\lambda^{\backprime}}^{\vartheta+\lambda^{\backprime}}\frac{d\theta^{\backprime}}{\sqrt{1+\alpha^{\backprime 2}-2\alpha^{\backprime}\cos\theta^{\backprime}}}.$$

But the arbitrary function of $\alpha^{\backprime}$, introduced by this integration, must be identically zero because otherwise it would not vanish with $\lambda^{\backprime}$, as it must do, since ψ does so, independently of $\alpha^{\backprime}$ and ϑ. Thus, finally

$$\sqrt{\alpha+\alpha_{,}}\,\psi=\sqrt{\alpha_{,}}\sqrt{1+\alpha^{\backprime}}\,\psi=-\frac{2\alpha\alpha_{,}}{\alpha^{\frac{3}{2}}-\alpha_{,}^{\frac{3}{2}}}\cos\vartheta\sin(\lambda_{,}-\lambda)+\frac{\alpha^{\frac{3}{2}}\alpha_{,}^{\frac{3}{2}}}{\alpha^{\frac{3}{2}}-\alpha_{,}^{\frac{3}{2}}}\int_{\vartheta-\lambda_{,}+\lambda}^{\vartheta+\lambda_{,}-\lambda}\frac{d\theta^{\backprime}}{\sqrt{\alpha^2+\alpha_{,}^2-2\alpha\alpha_{,}\cos\theta^{\backprime}}},$$

which may also be written

$$\psi=\frac{(\lambda^{-\frac{2}{3}}+\lambda_{,}^{-\frac{2}{3}})^{-\frac{1}{2}}}{\lambda_{,}-\lambda}\left\{-2\lambda^{\frac{1}{3}}\lambda_{,}^{\frac{1}{3}}\cos\vartheta\sin(\lambda_{,}-\lambda)+\int_{\vartheta-\lambda_{,}+\lambda}^{\vartheta+\lambda_{,}-\lambda}\frac{d\theta^{\backprime}}{\sqrt{(\lambda^{-\frac{4}{3}}+\lambda_{,}^{-\frac{4}{3}}-2\lambda^{-\frac{2}{3}}\lambda_{,}^{-\frac{2}{3}}\cos\theta^{\backprime})}}\right\},$$

or in the alternate form

$$\boxed{\psi=\left\{\left(\frac{\lambda_{,}}{\lambda}\right)^{\frac{1}{3}}+\left(\frac{\lambda}{\lambda_{,}}\right)^{\frac{1}{3}}\right\}^{-\frac{1}{2}}\frac{\sqrt{\lambda\lambda_{,}}}{\lambda_{,}-\lambda}\left\{-2\cos\vartheta\sin(\lambda_{,}-\lambda)+\int_{\vartheta-\lambda_{,}+\lambda}^{\vartheta+\lambda_{,}-\lambda}\left\{\left(\frac{\lambda_{,}}{\lambda}\right)^{\frac{2}{3}}+\left(\frac{\lambda}{\lambda_{,}}\right)^{\frac{2}{3}}-2\cos\theta^{\backprime}\right\}^{-\frac{1}{2}}d\theta^{\backprime}\right\}.}$$