

BRIEF NOTES

Thermal effects in plastic-time phenomena

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THE constitutive equations for rheological media in conditions of large deformations and temperature field are derived. The concept of derivation of these equations is based, on the one hand, on the generalized superposition of the reaction of the material and this leads to the non-linear functionals. On the other hand, using the functional differential form, by the assumption of the integrability conditions for this form the result analogous to the previous one is obtained. The integrability conditions couple the effects of a physical reaction of the material expressed in terms of the functions characterizing their rheological properties in the conditions of a temperature field.

IN the presence of temperature field the behaviour of a rheological body is influenced by its thermodynamic state. In order to establish an appropriate constitutive equation describing such a state, it is then necessary to introduce into consideration temperature as an independent state variable and find its derivation on the thermodynamical basis. If the body behaves non-linearly as regards its physical properties and exhibits large deformations, the problem of thermomechanical coupling arises as that of main importance. The coupling of effects has in the case considered a double meaning. One side of the problem lies in the coupling defined by the equation of heat conduction. It indicates the appearance of heat flux due to deformation process. On the other hand, all specified responses of the body should be coupled as being temperature — dependent. The latter coupling expresses the fact that “all phenomena occurring contribute to all effects resulting” in accordance with the principle of equipresence. Therefore, the approach, consisting in postulating or generalizing the corresponding constitutive equation on the basis of premises valid for a linear theory, is not always convincing as regards the physical aspects. In particular, that remark may be referred to the methods using certain generalized rheological models containing additive non-linear devices to derive constitutive relations of thermo-rheological processes.

In some previous papers [1, 2] we proposed a method of generalized superposition to obtain the forms of constitutive functionals and also stated some additional postulates concerned with their non-linearity and its physical significance. It is clear that the procedure of superposing deformational contributions should be performed in full accordance with the general principles of mechanics. The method has been also discussed in connection with the considerations of possible forms of constitutive relations containing time-dependent temperature at the differential level [3, 4].

In the present paper the idea of deriving non-linear constitutive functionals describing thermo-rheological processes by applying the generalized superposition principle is adop-

ted. It is assumed that the body considered undergoes an arbitrary deformation state in the presence of time-dependent temperature field. On the other hand, it is shown which are the necessary conditions to be satisfied in order to be able to obtain equivalent results, if we set up an alternative theory on the differential level. These are the integrability conditions which thus furnish some additional constitutive relations indicating clearly the effect of thermo-mechanical coupling due to non-linearity. Moreover, they point out the significance of the principle of equipresence.

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We restrain our considerations to a certain class of rheological bodies exhibiting instantaneous as well as time-dependent responses, partly of plastic character. They are assumed to be isotropic, homogeneous and mechanically incompressible. Thus, in the isothermal conditions the bodies in question cannot undergo the change of volume due to mechanical loading.

We apply the material description of the process and consider two configurations of the body referred to the two time instants t_0 and t , respectively. By assuming the convected coordinate system θ_k ($k = 1, 2, 3$), we observe a particular element of material through the succession of deformational states within the time interval $\Omega = [t_0, t]$. The straining process of the particle is then given by the contravariant strain tensor e^{ij} being a function of time

$$(1.1) \quad e^{ij} = \int_{\Omega} \omega(t, \tau, t_0) de^{ij}(t, \tau) = \frac{1}{2} \int_{\Omega} dA^{ij},$$

where we put

$$(1.2) \quad e^{ij} = e^{ij}(\theta_k, t, t_0), \quad \bar{\Omega} = (-\infty, \infty),$$

A^{ij} denoting the metric tensor of convected system and

$$(1.3) \quad \omega = \theta(t-\tau)\theta(t-t_0),$$

the multiplier composed of the unit distributions θ .

As the change of time is continuous, the Eq. (1.1) implicates the differential form

$$(1.4) \quad \omega de(t, \tau) = \omega \Phi^{ij}(t, \tau) dt,$$

if Φ^{ij} is assumed as convected time-derivative of a certain physical tensor functions.

As follows from the Eq. (1.4), the constitutive differential is expressed through a "compact" form being responsible for a complex response of the material within the interval defined by ω . On the other hand, the Eq. (1.1) indicates the manner of superposing strain contributions in Ω performed in accordance with the principle of determinism and the assumed rule of summation. Moreover, it imposes the possibility of summing up the responses of the particle at different time instants according to the procedure of convected integration.

The variability of the metric tensor in time is seen from the Eq. (1.1). However, as follows from the definition of convected strain tensor,

$$e^{ij}(\theta_k, t, t_0) = \frac{1}{2} [A^{ij}(\theta_k, t_0) - A^{ij}(\theta_k, t)],$$

this quantity is defined "potentially". It means that its value depends on the difference of the metric tensors referred to the time instants t_0 and t , respectively. Hence, in accordance with such a notion of strain measure, strain tensor does not depend on the intermediate values of metric tensor in Ω . Therefore, as the variability of A^{ij} in Ω does not matter, the final result depending on the difference in the Eq. (1.4), we may conclude that the value of strain is independent of the "time-path".

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In considering the influence of thermal phenomena on the rheological behaviour of the bodies in question, we shall assume stress deviator and temperature as independent physical variables. Thus, all the physical functions appearing in the superposed constitutive form will be dependent on the stress deviator invariants and temperature. In general, it may happen that it will be necessary to take into consideration also the pressure as a dynamical variable. It may appear due to the effect of temperature changes in the presence of geometrical constraints. Introducing the symbol s_k ($k = 1, 2, 3$) to denote the triplet of quantities, we reserve for the sake of generality the notation s_1 for the possible pressure, since the first invariant equals zero.

In observing the deformed particle, let us denote by Δe_n^{ij} strain tensor increment related to the instant t caused by the simultaneous increments of independent variables Δs_n^{ij} and $T_{(n)}$ within the subinterval $\Omega_n = [t_{n-1}, t_n]$, $\Omega_n \in \Omega$. In such a way we are able to define the contribution of partial state in Ω_n to the total deformation observed at t , if the physical weights are prescribed to the latter increments. Thus, by founding the assumptions on the physical premises, we write

$$(2.1) \quad e_n^{ij}(t, \tau) = K \Delta s_n^{ij} + D^{ij} \Delta T_n,$$

where

$$(2.2) \quad \begin{aligned} K &= K\{t, t_n^*, s_k[\tau_n^*, T(t_n^*)], T(\tau_n^*)\}, \\ D^{ij} &= D^{ij}\{\tau_n^*, s_k[\tau_n^*, T(\tau_n^*)], T(\tau_n^*)\}, \end{aligned}$$

denote the generalized memory function and the dilatation tensor dependent on the specified arguments and $\tau_n^* \in \Omega_n$ is an intermediate instant chosen. In general, stress deviator is temperature dependent too, as it is impossible to evaluate and separate stress shares due to external loading and temperature changes in the presence of the coupling effect.

As follows from the physical premises the function K must be temperature dependent, if we consider the thermal state of reference. Since every of such states may be regarded as isothermal, all of them are defined by different physical functions as regards memory. On the other hand, since independently of the state of thermal reference we speak about the same material, its characteristic must be temperature dependent.

The thermal member in the Eq. (2.1) is put apart because of the physical premise. Temperature is the measure of changes of internal energy and the presence of its field creates strain and stress states which may be independent of mechanical causes. In the case

of simultaneous action of mechanical and temperature fields the effects are coupled. It means that one cannot specify them naturally.

The contributions of the Eq. (2.1) we sum within the interval Ω and by passing to the limit we obtain

$$(2.3) \quad e^{ij}(t) = e^{ij}(t, t_0) + \int_{\Omega} K ds^{ij}[\tau, T(\tau)] + \int_{\Omega} D^{ij} dT(\tau).$$

Here we put

$$(2.4) \quad e^{ij}(t, t_0) = K s^{ij}|_{t=t_0} + D^{ij} T|_{t=t_0}.$$

Carrying out integration by parts in the Eq. (2.3) we find

$$(2.5) \quad e^{ij}(t) = \int_{\Omega^*} s^{ij} d_{\tau}[-\theta(t-\tau)K] + \int_{\Omega^*} T d_{\tau}[-\theta(t-\tau)D^{ij}].$$

The above form corresponds to that of a non-linear tensor functional where $\Omega^* = [t_0^-, t^+]$. In the consequence of generalized superposition we obtain an additive expression for strain tensor. Each of the two terms figuring in the Eq. (2.5) may be split into the three ones. The first are related to discontinuity points as given by the instantaneous response contained in K and similarly in D^{ij} . The second terms correspond to the absolutely continuous distributions of effects as regards time-dependent strain state and dilatation changes. The third terms as representing singular distributions are here neglected due to the lack of physical interpretation.

Thus, in accordance with the physical premises, the Eq. (2.5) takes the final form

$$(2.6) \quad e^{ij}(t) = \Phi s^{ij} - \int_{\Omega} s^{ij} \partial_{\tau} K d\tau + D^{ij} T - \int_{\Omega} T \partial_{\tau} D^{ij} d\tau,$$

where we denote

$$(2.7) \quad K = \Phi \{ \tau, s_k[\tau, T(\tau)], T(\tau) \} + H \{ t, \tau, s_k[\tau, T(\tau)], T(\tau) \},$$

Φ being responsible for the instantaneous responses of the body considered and H representing the generalized creep function. In the Eq. (2.6) we make use of the evident condition

$$(2.8) \quad K|_{\tau=t} = \Phi,$$

as for $\tau = t$ we must have

$$(2.9) \quad H|_{\tau=t} = 0.$$

In accordance with the previous assumption that the body in question is isotropic, the dilatation tensor D^{ij} should be expressed through the isotropic tensor δ^{ij} . Thus, we put

$$(2.10) \quad D^{ij} = \{ a + C \{ t, s_k[t, T(t)], T(t) \} \} \delta^{ij},$$

where a is the thermal constant.

In the absence of external loading and constraints the body deforms freely due to temperature changes. Since in this case we have an isotropic expansion defined by

$$(2.11) \quad e^{11} = e^{22} = e^{33} = e^T,$$

the metric tensor referred to the instant t becomes

$$(2.12) \quad A^{ij}(t) = \begin{bmatrix} 12e^{-T} & 0 & 0 \\ 0 & 1-2e^T & 0 \\ 0 & 0 & 1-2e^T \end{bmatrix} = v^{-\frac{2}{3}} \delta^{ij},$$

where v denotes the volumetric deformation. The latter gives the ratio of particle volumes at the time instants t and t_0 , respectively,

$$(2.13) \quad v = (e^T)^{-3},$$

the value of e^T being given by the Eqs. (2.6) and (2.10) as follows

$$(2.14) \quad e^T(t) = (a+C)T - \int_{\Omega} T \partial_{\tau}(a+C) d\tau.$$

It is evident that in the case considered the function C does not depend on the invariants s_k .

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We shall point out that the form of the Eq. (2.6) can also be obtained by setting up the theory on the differential level. Thus we assume the differential form of constitutive equation for a non-linear material deforming in the temperature field as follows

$$(3.1) \quad de^{ij} = K ds^{ij} + D^{ij} dT.$$

Here the notations used are the same as previously.

The Eq. (3.1) can be put into the alternative differential representation taking into account the distinction of independent physical variables

$$(3.2) \quad de^{ij} = \partial_{s^{kl}} e^{ij} ds^{kl} + \partial_T e^{ij} dT,$$

where by identification we find

$$(3.3) \quad \partial_{s^{kl}} e^{ij} = K \delta^{ik} \delta^{jl}, \quad \partial_T e^{ij} = D^{ij}.$$

As we postulate the Eq. (3.1) to be a total differential, we should have

$$(3.4) \quad \partial_T K = \partial_{s^{kl}} D^{ij} \delta^{ik} \delta^{jl}.$$

On the other hand, by assuming differentiability of the independent variables with respect to the parameter τ , we can write instead of the Eq. (3.1)

$$(3.5) \quad de^{ij}(t, \tau) = [\partial_{s^{kl}} e^{ij} \partial_{\tau} s^{kl} + \partial_T e^{ij} \partial_{\tau} T] d\tau.$$

Since, according to our assumption the Eq. (3.4) holds, by integrating the Eq. (3.2), or, alternatively, the Eq. (3.5), we obtain the final result as being analogous to the Eq. (2.6)

$$(3.6) \quad e^{ij}(t) = \Phi s^{ij} - \int_{\Omega} s^{ij} \partial_{\tau} K d\tau + \int_{\Omega} D^{ij} d_{\tau} T d\tau.$$

It follows from the form of the Eqs. (3.4) and (3.6) that "all phenomena occurring contribute to all effects resulting" as emphasized especially by the integrability conditions. On the other hand, the latter gives an additional constitutive relation between physical functions characterizing non-linear thermo-rheological behaviour.

References

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