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Basic static problems of elastic micropolar-media

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By the method of singular integral equations and singular potentials some static boundary value problems of the moment theory of elasticity are studied. The case under consideration is that in which the normal displacement and rotation components are given as well as tangential force and couple stress components, or the case in which normal force and couple stress components and tangential displacement and rotation components are prescribed. Such problems arise as a result of contact between elastic micropolar media and may be of interest from the point of view of mechanics. They are interesting also from the mathematical point of view because they can be reduced to singular integral equations with complicated kernels.

Niektóre statyczne zagadnienia brzegowe momentowej teorii sprężystości były badane metodą osobliwych równań całkowych i osobliwych potencjałów. Rozważany jest przypadek, w którym dane są zarówno składowe przemieszczenia w kierunku normalnym i obrót, jak i siła styczna i składowe naprężeń momentowych bądź przypadek, w którym dane są siła normalna i składowe naprężeń momentowych oraz przemieszczenie w kierunku stycznym i obrót. Problemy tego typu występują w trakcie kontaktu sprężystych ośrodków mikropolarnych i budzą stałe zainteresowanie mechaników. Są one również atrakcyjne z matematycznego punktu widzenia, gdyż prowadzą do osobliwych równań całkowych ze złożonymi jądrami.

Методом сингулярных интегральных уравнений и сингулярных потенциалов исследуются граничные задачи статики моментной теории упругости, когда на границе среды заданы нормальные составляющие смещения и вращения и касательные составляющие силового и моментного напряжения или когда заданы нормальные составляющие силового и моментного напряжения и касательные составляющие смещения и вращения. Задачи такого рода возникают при соприкосновении упругих микрополярных сред и могут иметь интерес с точки зрения механики. Эти задачи интересны и с точки зрения математики, так как непосредственное сведение их к интегральным уравнениям, приводит к сингулярным уравнениям весьма сложной структуры.

1. Static state of a micropolar medium

In the study of the moment theory of elasticity sometimes referred to as the micropolar or asymmetric theory the point of departure are the following axioms([1]):

A homogeneous isotropic elastic medium having a centre of elastic symmetry is a region $\mathscr{D}$ in a three-dimensional Euclidean space E^3 and an ordered set of seven real numbers $\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta$ satisfying the conditions

$$(1.1) \quad \varrho > 0, \quad \mu > 0, \quad 3\lambda+2\mu > 0, \quad \alpha > 0, \quad v > 0, \quad 3\varepsilon+2v > 0, \quad \beta > 0.$$

This medium will be denoted by $\mathscr{D}(\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta)$ or simply, if a mistake is exclu-

([1]) The foundations of the moment theory of elasticity are discussed in many works, of which let us mention Refs. [1 to 6] and the monograph [7]. Detailed historical information can also be found in those references ded, by $\mathscr{D}$. The quantity ϱ will be referred to as the density of the medium $\mathscr{D}$, the remaining quantities being elastic constants.

In this work we shall consider a homogeneous and isotropic elastic body having a centre of elastic symmetry, therefore the usual term, which is somewhat lengthy, will be replaced by that of "elastic medium".

Let $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)$ and $\mathscr{G} = (\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3)$ be real vectors defined in the region $\mathscr{D}$.

The static state of the elastic medium $\mathscr{D}(\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta)$ corresponding to a mass force $\mathscr{F}$ and a mass moment $\mathscr{G}$, is an ordered set of four numbers $[u, \omega, \tau, \mu]$, where

I) $u = (u_1, u_2, u_3)$ and $\omega = (\omega_1, \omega_2, \omega_3)$ are real vectors and $\tau = ||\tau_{ij}||_{3\times 3}$ and $\mu = ||\mu_{ij}||_{3\times 3}$ are real matrices, defined in the region $\mathscr{D}$.

(1.2) $$\text{II) } u, \omega \in C^2(\mathscr{D}) \cap C^1(\bar{\mathscr{D}}) \quad \text{and} \quad \tau, \mu \in C^1(\mathscr{D});$$

(1.2′) $$\text{III) } \frac{\partial \tau_{ij}}{\partial x_i} + \varrho \mathscr{F}_j = 0,$$

$$\frac{\partial \mu_{ij}}{\partial x_i} + \varepsilon_{jik}\tau_{ik} + \varrho \mathscr{G}_j = 0, \quad j = 1, 2, 3;$$

(1.3) $$\text{IV) } \tau_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + (\mu+\alpha)\frac{\partial u_j}{\partial x_i} + (\mu-\alpha)\frac{\partial u_i}{\partial x_j} - 2\alpha\varepsilon_{ijk}\omega_k;$$

(1.3′) $$\mu_{ij} = \varepsilon \delta_{ij} \frac{\partial \omega_k}{\partial x_k} + (v+\beta)\frac{\partial \omega_j}{\partial x_i} + (v-\beta)\frac{\partial \omega_i}{\partial x_j}, \quad i, j = 1, 2, 3,$$

where δ_{ij} is Kronecker delta, ε_{ijk}—the Levi-Civita symbol and $x = (x_1, x_2, x_3)$—a point in the space E^3.

The Eqs. (1.2) and (1.2′) are the fundamental equations of static state and (1.3) and (1.3′) express Hooke law of the moment theory of elasticity(²). The quantities u, ω, τ and μ are the vector of displacement and rotation and the tensor of force stress and moment stress, respectively.

It should be observed that the relation $2\omega = \mathrm{rot}\, u$ between the vectors of displacement and rotation known from the classical theory and sometimes assumed in the moment theory of elasticity is not assumed in the present paper.

2. Basic equations in displacement and rotation components. The stress tensor

On substituting (1.3) into (1.3′) and (1.2) into (1.2′) we obtain the general equations of static state of the elastic medium $\mathscr{D}(\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta)$ in components of displacement and rotation due to the mass force $\mathscr{F}$ and the mass moment $\mathscr{G}$:

(2.1) $$(\mu+\alpha)\Delta u + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\omega + 2\alpha\,\mathrm{rot}\,u + \varrho\mathscr{F} = 0,$$

(2.1′) $$(v+\beta)\Delta\omega + (\varepsilon+v-\beta)\,\mathrm{grad\,div}\,\omega + 2\alpha\,\mathrm{rot}\,u - 4\alpha\omega + \varrho\mathscr{G} = 0,$$

where Δ is the Laplacian operator.

(²) Repeated Latin index in a term means summation with respect to that index from 1 to 3. Repeated Greek index does not mean summation.

If $[u, \omega, \tau, \mu]$ is the static state of the elastic body $\mathscr{D}(\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta)$, u and ω belong to the class $C^2(\mathscr{D}) \cap C^1(\bar{\mathscr{D}})$ and satisfy the relations (1.3) and (1.3'). The inverse statement is also valid if u and ω are determined from the relations (2.1) and (2.1') and belong to the class $C^2(\mathscr{D}) \cap C^1(\bar{\mathscr{D}})$, then $[u, \omega, \tau, \mu]$, where the matrices τ and μ have been determined from (1.3) and (1.3') represent the static state of the medium $\mathscr{D}(\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta)$.

Thus, the problem of determining the static state, which is the fundamental problem of the present paper, reduces to that of determining a pair of vectors, μ and ω, of the class $C^2(\mathscr{D}) \cap C^1(\bar{\mathscr{D}})$, from the relations (2.1) and (2.1').

The Eqs. (2.1) and (2.1') can be written in a matricial form. Let us introduce the following differential matrix operator

$$\mathscr{M}(\partial_x) = \|\mathscr{M}_{ij}(\partial_x)\|_{6\times 6}$$

and represent it in the form

$$\mathscr{M}(\partial_x) = \left\| \begin{matrix} \mathscr{M}^{(1)}(\partial_x), & \mathscr{M}^{(2)}(\partial_x) \\ \mathscr{M}^{(3)}(\partial_x), & \mathscr{M}^{(4)}(\partial_x) \end{matrix} \right\|,$$

$$\mathscr{M}^{(k)}(\partial_x) = \|\mathscr{M}^{(k)}_{ij}(\partial_x)\|_{3\times 3}, \quad k = 1, 2, 3, 4,$$

where

$$\mathscr{M}^{(1)}_{ij}(\partial_x) = (\mu+\alpha)\delta_{ij}\Delta + (\lambda+\mu-\alpha)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$\mathscr{M}^{(2)}_{ij}(\partial_x) = \mathscr{M}^{(3)}_{ij}(\partial_x) = -2\alpha\varepsilon_{ijk}\frac{\partial}{\partial x_k},$$

$$\mathscr{M}^{(4)}_{ij}(\partial_x) = \delta_{ij}[(v+\beta)\Delta - 4\alpha] + (\varepsilon+v-\beta)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Now (2.1), (2.1') can be written in the form

$$\mathscr{M}(\partial_x)\mathscr{U} + \varrho\mathscr{H} = 0,$$

where $\mathscr{U} = (\mathscr{U}_1, \ldots, \mathscr{U}_6), \mathscr{H} = (\mathscr{H}_1, \ldots, \mathscr{H}_6)$, $\mathscr{U}_i = u_i$, $i = 1, 2, 3$ and $U_i = \omega_{i-3}$, $i = 4, 5, 6$; $\mathscr{H}_i = \mathscr{F}_i$, $i = 1, 2, 3$ and $\mathscr{H}_i = \mathscr{G}_{i-3}$, $i = 4, 5, 6$. $\mathscr{U}$ and $\mathscr{H}$ will sometimes be written in the form $\mathscr{U} = (u, \omega)$, $\mathscr{H} = (\mathscr{F}, \mathscr{G})$.

Let $n = (n_1, n_2, n_3)$ denote any unit vector. The force stress at a point x in the direction n is a vector $\tau^{(n)} = (\tau^{(n)}_1, \tau^{(n)}_2, \tau^{(n)}_3)$ where $\tau^{(n)}_j(x) = \tau_{ij}(x)n_i$ and the moment stress at a point x in the direction n is a vector $\mu^{(n)} = (\mu^{(n)}_1, \mu^{(n)}_2, \mu^{(n)}_3)$ where $\mu^{(n)}_j(x) = \mu_{ij}(x)n_i$.

Let $[u, \omega, \tau, \mu]$ be the static state of the elastic body $\mathscr{D}(\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta)$. Then, from (1.3) and (1.3') we find

$$\tau^{(n)}_i = \lambda n_i \operatorname{div} u + (\mu-\alpha)\frac{\partial u_j}{\partial x_i}n_j + (\mu+\alpha)\frac{\partial u_i}{\partial x_j}n_j + 2\alpha\varepsilon_{ijk}n_j\omega_k, \tag{2.3}$$

$$\mu^{(n)}_i = \varepsilon n_i \operatorname{div}\omega + (v-\beta)\frac{\partial \omega_j}{\partial x_i}n_j + (v+\beta)\frac{\partial \omega_i}{\partial x_j}n_j. \tag{2.3'}$$

Let us introduce the differential matrix operator

(2.4) $$T(\partial_x, n) = ||T_{ij}(\partial_x, n)||_{6\times 6} \equiv \begin{Vmatrix} T^{(1)}(\partial_x, n) & T^{(2)}(\partial_x, n) \\ T^{(3)}(\partial_x, n) & T^{(4)}(\partial_x, n) \end{Vmatrix},$$

where

(2.5) $$T^{(k)}(\partial_x, n) = ||T^{(k)}(\partial_x, n)||_{3\times 3}, \quad k = 1, 2, 3, 4;$$

$$T_{ij}^{(1)}(\partial_x, n) = \lambda n_i \frac{\partial}{\partial x_j} + (\mu-\alpha) n_j \frac{\partial}{\partial x_i} + (\mu+\alpha)\delta_{ij}\frac{\partial}{\partial n},$$

(2.6) $$T_{ij}^{(2)}(\partial_x, n) = -2\alpha\varepsilon_{ijk}n_k, \quad T_{ij}^{(3)}(\partial_x, n) = 0,$$

(2.7) $$T_{ij}^{(4)}(\partial_x, n) = \varepsilon n_i \frac{\partial}{\partial x_j} + (v-\beta) n_j \frac{\partial}{\partial x_i} + (v+\beta)\delta_{ij}\frac{\partial}{\partial n}$$

and

(2.8) $$\tau^{(n)}(x) = T^{(1)}(\partial_x, n)u(x) + T^{(2)}(\partial_x, n)\omega(x),$$
$$\tau_i^{(n)}(x) = [T(\partial_x, n)\mathscr{U}(x)]_i, \quad \text{for} \quad i = 1, 2, 3,$$

(2.9) $$\mu^{(n)}(x) = T^{(4)}(\partial_x, n)\omega(x),$$
$$\mu_i^{(n)}(x) = [T(\partial_x, n)\mathscr{U}(x)]_{i+3}, \quad \text{for} \quad i = 1, 2, 3,$$

T and $T^{(k)}$ will be termed stress operators.

3. Basic problems

There are in the moment theory sixteen problems corresponding to the four problems of the classical theory of elasticity. These problems are formulated thus:

Find the static state $[u, \omega, \tau, \mu]$ of an elastic body $\mathscr{D}(\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta)$ if a displacement vector and a rotation vector are prescribed at the boundary of that body [problem (I.I)] or the vector of moment stress [problem (I.II)], or the normal component of the rotation vector and the tangential components of the vector of moment stress [problem (I.III)], or the normal component of the vector of moment strain and tangential components of the rotation vector [problem (I.IV)].

The problems (II.I) to (II.IV) are formulated in an analogous manner, the stress vector being prescribed instead of the displacement vector. In the problems (III.I) to (III.IV) the normal component of the displacement vector and the tangential components of the vector of force stress are prescribed instead of the displacement vector. In the problems (IV.I) to (IV.IV) the displacement vector is replaced by the tangential components of the displacement vector and the normal component of the force stress vector.

A method for investigating the fundamental problems of the classical theory of elasticity and thermoelasticity with the use of the theory of singular integral equations is described in [8]. This method can be used to study all the problems above. The construction of fundamental solutions and certain elastic potentials is explained in the monograph [7]. The problems (I.I), (II.II), (I.II), (II.I) are studied in Refs. [9, 10, 11].

In the present paper we shall study the problems (III.III) and (IV.IV), to which correspond the third and the fourth problem of the classical theory of elasticity (see [12, 13]). Similarly to the classical case their study comes up against certain difficulties.

In what follows a finite region in E^3 will be denoted by $\mathscr{D}^+$, its boundary by S and the complement of the set $\mathscr{D}^+ \cup S$ by $\mathscr{D}^-$.

The above problems will be studied for the elastic body $\mathscr{D}^+(\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta)$ and $\mathscr{D}^-(\varrho, \lambda, \mu, \alpha, \varepsilon, v, \beta)$ as well. In the first case the problem is termed internal and will be denoted by the symbol $(p, q)^+$ $(p, q = \mathrm{I, II, III, IV})$ and in the second case—external and will be denoted by the symbol $(p, q)^-$ $(p, q = \mathrm{I, II, III, IV})$.

In addition to being of interest for themselves, external problems occur, with the method used in the present paper, as auxiliary problems for the solution of internal problems. The same may be said for the internal problems, which occur as auxiliary problems for the solution of external problems.

Let us observe that, for the statement of the problems $(p, q)^\pm$ $(p, q = \mathrm{I, II, III, IV})$, the vectors of displacement and rotation $u(z)$ and $\omega(z)$ and the vectors of force and moment stress $\tau^{(n)}(z)$ and $\mu^{(n)}(z)$, when $x \in S$, are considered to constitute the following limits(³)

$$[u(x)]^\pm \equiv \lim_{\mathscr{D}^\pm \ni z \to x} u(z), \quad [\omega(x)]^\pm \equiv \lim_{\mathscr{D}^\pm \ni z \to x} \omega(z),$$

$$[\tau^{(n)}(x)]^\pm \equiv \lim_{\mathscr{D}^\pm \ni z \to x} \tau^{(n)}(z), \quad [\mu^{(n)}(x)]^\pm \equiv \lim_{\mathscr{D}^\pm \ni z \to x} \mu^{(n)}(z),$$

where $\tau^{(n)}(z)$ and $\mu^{(n)}(z)$ are to be found from (2.8) and (2.9) and n is a unit vector normal to the surface S at a point x external with reference to $\mathscr{D}^+$.

The problems $(\mathrm{III.\,III})^\pm$ and $(\mathrm{IV.\,IV})^\pm$ to be studied in the present paper are equivalent to the following problems.

The problem $(\mathrm{III.\,III})^\pm$. In a region $\mathscr{D}^\pm$ find vectors $u = (u_1, u_2, u_3)$ and $\omega(\omega_1, \omega_2, \omega_3)$ or a vector $\mathscr{U} = (u, \omega)$ of the class $C^2(\mathscr{D}^\pm) \cap C^1(\overline{\mathscr{D}}^\pm)$ satisfying the relation (2.2) in the region $\mathscr{D}^\pm$ and the boundary conditions $\forall_y \in S$:

$$(3.1)\quad [T^{(1)}(\partial_y, n)u(y) + T^{(2)}(\partial_y, n)\omega(y) - n\{T^{(1)}(\partial_y, n)u(y) + T^{(2)}(\partial_y, n)\omega(y)\}_n]^\pm = \mathring{f}^{(1)}(y),$$

$$(3.1')\quad [u(y)]_n^\pm = \mathring{f}_4^{(1)}(y),$$

$$(3.2)\quad [T^{(4)}(\partial_y, n)\omega(y) - n\{T^{(4)}(\partial_y, n)\omega(y)\}_n]^\pm = \mathring{f}^{(2)}(y),$$

$$(3.2')\quad [\omega(y)]_n^\pm = \mathring{f}_4^{(2)}(y),$$

where $\mathring{f}_i^{(1)}$ and $\mathring{f}_i^{(2)}$ $(i = 1, 2, 3, 4)$ are real functions prescribed on S.

The problem $(\mathrm{IV.\,IV})^\pm$. In a region $\mathscr{D}^\pm$ find vectors $u = (u_1, u_2, u_3)$ and $\omega = (\omega_1, \omega_2, \omega_3)$ or a vector $\mathscr{U} = (u, \omega)$ of the class $C^2(\mathscr{D}^\pm) \cap C^1(\overline{\mathscr{D}}^\pm)$, satisfying the relation (2.2) in the region $\mathscr{D}^\pm$ and the boundary conditions $\forall y \in S$

$$(3.3)\quad [u(y) - n\{u(y)\}_n]^\pm = \mathring{g}^{(1)}(y),$$

$$(3.3')\quad [T^{(1)}(\partial_y, n)u(y) + T^{(2)}(\partial_y, n)\omega(y)]_n^\pm = \mathring{g}_4^{(1)}(y),$$

$$(3.4)\quad [\omega(y) - n\{\omega(y)\}_n]^\pm = \mathring{g}^{(2)}(y),$$

$$(3.4')\quad [T^{(4)}(\partial_y, n)\omega(y)]_n^\pm = \mathring{g}_4^{(2)}(y),$$

where $\mathring{g}_i^{(1)}$ and $\mathring{g}_i^{(2)}$ $(i = 1, 2, 3, 4)$ are real functions prescribed on S.

(³) When there is a double symbol $\pm$ or $\mp$ in a statement it should be understood as an abbreviated expression of two statements, for the upper and the lower symbols.

Using the notations of the foregoing section the problems $(III.III)^{\pm}$ and $(IV.IV)^{\pm}$ can be formulated thus. Find in the region $\mathscr{D}^{\pm}$ a six-component vector $\mathscr{U}$ of the class $C^2(\mathscr{D}^{\pm}) \cap C^1(\overline{\mathscr{D}}^{\pm})$ satisfying the equation (2.2) in the region $\mathscr{D}^{\pm}$ and the boundary condition (3.1)–(3.2′) in the case (III. III) and (3.3)–(3.4′) in the case $(IV. IV)^{\pm}$ in which $u_i = \mathscr{U}_i$ and $\omega_i = \mathscr{U}_{i+3}$ $(i = 1, 2, 3)$.

4. Uniqueness theorems

Let us assume that $\mathscr{U} = (u, \omega)$, $u = (u_1, u_2, u_3)$, $\omega = (\omega_1, \omega_2, \omega_3)$, $S \in \text{Л}_1(\alpha)$, $\alpha \geqslant 0$, $\mathscr{U} \in C^2(\mathscr{D}^{\pm}) \cap C^1(\overline{\mathscr{D}}^{\pm})$ and $\mathscr{M}(\partial_x)\mathscr{U}$ is absolutely integrable over the region $\mathscr{D}^{\pm}$. Then, the following formula is valid

$$(4.1)\quad \int\limits_{\mathscr{D}^+} \{\mathscr{U}\mathscr{M}(\partial_x)\mathscr{U} + E(\mathscr{U}, \mathscr{U})\}\,dx = \int\limits_{S} \{[u - n(n\cdot u)]^+ \, [T^{(1)}u + T^{(2)}\omega - n(nT^{(1)}u + nT^{(2)}\omega)]^+ + [\omega - n(n\omega)]^+ [T^{(4)}\omega - n(nT^{(4)}\omega)]^+ + [nu]^+ [nT^{(1)}u + nT^{(2)}\omega]^+ + [n\omega]^+ [nT^{(4)}\omega]^+\}\,dS,$$

where

$$(4.2)\quad E(\mathscr{U}, \mathscr{U}) = \frac{3\lambda + 2\mu}{3}\left(\frac{\partial u_i}{\partial x_i}\right)^2 + \frac{\mu}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)^2 + \frac{\alpha}{2}\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j} + 2\varepsilon_{kji}\omega_k\right)^2 + \frac{3\varepsilon + 2v}{3}\left(\frac{\partial \omega_i}{\partial x_i}\right)^2 + \frac{v}{2}\sum_{i,j=1}^{3}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial \omega_k}{\partial x_k}\right)^2 + \frac{\beta}{2}\left(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j}\right)^2.$$

The formula (4.1) will be referred to as Green formula. It can easily be proved by means of the Gauss-Ostrogrodzki theorem. Let us observe that, under the condition (1.1), we have $E(\mathscr{U}, \mathscr{U}) \geqslant 0$.

Let now $\mathscr{U} = (u, \omega)$ be a solution of the equation $\mathscr{M}(\partial_x)\mathscr{U} = 0$ in the region $\mathscr{D}^-$, of class $C^2(\mathscr{D}^-) \cap C^1(\overline{\mathscr{D}}^-)$ satisfying, in the neighbourhood of an infinitely remote point, the conditions

$$(4.3)\quad u_j(x) = O(|x|^{-1}), \qquad \omega_j(x) = o(|x|^{-1}), \qquad j = 1, 2, 3;$$

$$(4.3')\quad \frac{\partial u_j(x)}{\partial x_i} = o(|x|^{-1}), \qquad \frac{\partial \omega_j(x)}{\partial x_i} = O(|x|^{-1}), \qquad i, j = 1, 2, 3,$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Under these conditions, by applying the Eq. (4.1) in the region $\mathscr{D}^- \cap \{x \in E^3 \,\big|\, |x| < R\}$, where R is a sufficiently large number, and passing to the limit for $R \to \infty$, we obtain

$$(4.4)\quad \int\limits_{\mathscr{D}^-} E(\mathscr{U}, \mathscr{U})\,dx = -\int\limits_{S} \{[u - n(n\,u)]^- \, [T^{(1)}u + T^{(2)}\omega - n(n\,T^{(1)}u + nT^{(2)}\omega)]^- + [\omega - n(n\,\omega)]^- \, [T^{(4)}\omega - n(n\,T^{(4)}\omega)]^- + [n\,u]^- [n\,T^{(1)}u + n\,T^{(2)}\omega]^- + [n\,\omega]^- [n\,T^{(4)}\omega]^-\}\,dS,$$

which will also be termed Green formula.

From (4.2) it follows that any solution of the equation $E(\mathscr{U}, \mathscr{U}) = 0$ of class $C^1(\mathscr{D}^{\pm})$ is given by the formula

$$u = [a \times x] + b, \quad \omega = a,$$

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ are arbitrary constant vectors.

Hence we can easily prove the following theorems:

THEOREM 4.1. *The problems* (III. III)$^+$ *and* (IV. IV)$^+$ *have no more than one solution.*

THEOREM 4.2. *The problems* (III. III)$^-$ *and* (IV. IV)$^-$ *have no more than one solution satisfying the conditions* (4.3) *and* (4.3′).

5. The volume potential

The term of volume potential will be used to denote the integral

$$W(\mathscr{X})(x) = \frac{\varrho}{2} \int_{\mathscr{D}^+} \Psi(x-y)\mathscr{X}(y)\,dy, \tag{5.1}$$

where $\Psi(x)$ is the fundamental solution of the equation $\mathscr{M}(\partial_x)\mathscr{U} = 0$ (see [7])

$$\Psi = \|\Psi_{pq}\|_{6\times 6} = \left\| \begin{matrix} \Psi^{(1)}, & \Psi^{(2)} \\ \Psi^{(3)}, & \Psi^{(4)} \end{matrix} \right\|, \quad \Psi^{(l)} = \|\Psi^{(l)}_{kj}\|_{3\times 3}, \quad l = 1, 2, 3, 4,$$

(5.2)

$$\Psi^{(1)}_{kj}(x) = \frac{\delta_{kj}}{2\pi}\left[\frac{1}{\mu}\frac{1}{|x|} - \frac{\alpha}{\mu(\mu+\alpha)}\frac{\exp(-\sigma_2|x|)}{|x|}\right] + \frac{1}{2\pi\mu}\frac{\partial^2}{\partial x_k \partial x_j}\left[-\frac{(\lambda+\mu)|x|}{2(\lambda+2\mu)} + \frac{\beta+v}{4\mu}\frac{\exp(-\sigma_2|x|)}{|x|}\right],$$

$$\Psi^{(2)}_{kj}(x) = \Psi^{(3)}_{kj}(x) = \frac{1}{4\pi\mu}\varepsilon_{jkp}\frac{\partial}{\partial x_p}\frac{1-\exp(-\sigma_2|x|)}{|x|},$$

$$\Psi^{(4)}_{kj}(x) = \frac{\delta_{kj}}{2\pi(\beta+v)}\frac{\exp(-\sigma_2|x|)}{|x|} + \frac{1}{8\pi}\frac{\partial^2}{\partial x_k \partial x_j} \times \left[\frac{\exp(-\sigma_1|x|)-\exp(-\sigma_2|x|)}{\alpha|x|} - \frac{\exp(-\sigma_2|x|)-1}{\mu|x|}\right],$$

$$\sigma_1 = \left(\frac{4\alpha}{\varepsilon+2v}\right)^{\frac{1}{2}}, \quad \sigma_2 = \left(\frac{4\alpha}{(\mu+\alpha)(\beta+v)}\right)^{\frac{1}{2}}.$$

We shall prove the following theorem:

THEOREM. 5.1. *If $\mathscr{H} \in C^{0,\alpha}(\mathscr{D}^+)$, then $W \in C^2(\mathscr{D}^+) \cap C^1(\bar{\mathscr{D}}^+)$ and*

$$\mathscr{M}(\partial_x) W(\mathscr{H}) + \varrho \mathscr{H} = 0 \tag{5.3}$$

in the region $\mathscr{D}^+$. If $\mathscr{H} \in C^{0,\alpha}(\mathscr{D}^-)$ and $\mathscr{H}$ is a finite-value function, the integral

$$W(\mathscr{H})(x) = \frac{\varrho}{2} \int_{\mathscr{D}^-} \Psi(x-y) \mathscr{H}(y) dy \tag{5.4}$$

is a solution of the Eq. (5.3) in the region $\mathscr{D}^-$ and, belongs to the class $C^2(\mathscr{D}^-) \cap C^1(\bar{\mathscr{D}}^-)$ and satisfies the conditions (4.3) and (4.3′).

If follows that if V is a solution of (2.2) in the region $\mathscr{D}^\pm$, $\mathscr{U} = V - W(\mathscr{H})$ is a solution of the homogeneous equation $\mathscr{M}(\partial_x)\mathscr{U} = 0$ in the same region. In addition, if V satisfies the conditions (4.3) and (4.3′), the same conditions are satisfied by $\mathscr{U}$.

The boundary conditions for $\mathscr{U}$ will involve the volume potential and its derivatives on S. Such a variation of the boundary data does not change their regularity character assumed in the present work (see Sect. 6).

Thus, the problems $(\text{III. III})^\pm$ and $(\text{IV. IV})^\pm$ reduce for $\mathscr{H} = 0$ to $(\text{III.III})^\pm$ and $(\text{IV.IV})^\pm$, respectively. Let us observe that the problems $(\text{III, III})^-$ and $(\text{IV. IV})^-$, with the additional conditions (4.3), (4.3′), reduce for $\mathscr{H} = 0$ to $(\text{III.III})^-$ and $(\text{IV.IV})^-$ with the same additional conditions.

In what follows it will be assumed, without limiting the generality of the considerations, that $\mathscr{H} = 0$. Let us observe that the volume potential (5.4) satisfies the conditions (4.3) and (4.3′) for weaker limitations than that of finite-value of $\mathscr{H}$.

6. Transformation of the boundary conditions

In what follows it will be assumed (without repeating that assumption each time verbally) that

$$\begin{gathered} S \in \text{Л}_2(h'), \quad 0 < h' \leqslant 1, \quad \overset{\bullet}{f}{}^{(1)}, \overset{\bullet}{f}{}^{(2)}, \overset{\bullet}{g}{}_4^{(1)}, \overset{\bullet}{g}{}_4^{(2)} \in C^{1,h}(S); \\ \overset{\bullet}{f}{}_4^{(1)}, \overset{\bullet}{f}{}_4^{(2)}, \overset{\bullet}{g}{}^{(1)}, \overset{\bullet}{g}{}^{(2)} \in C^{2,h}(S), \quad 0 < h < h'. \end{gathered} \tag{6.1}$$

We shall express the boundary conditions of the problems $(\text{III.III})^\pm$ and $(\text{IV.IV})^\pm$ in a form more convenient for further considerations. It is easy to see that the following relationships hold

$$n\, T^{(2)}(\partial_y, n)\, \omega = -2\alpha \varepsilon_{ijk} n_i \omega_j = 0, \tag{6.2}$$

$$n_j \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial n} = [n \times \operatorname{rot} u]_i, \tag{6.3}$$

$$T^{(1)}(\partial_y, n) u = 2\mu \frac{\partial u}{\partial n} + \lambda n \operatorname{div} u + (\mu - \alpha)[n \times \operatorname{rot} u], \tag{6.4}$$

$$n\, T^{(1)}(\partial_y, n) u = 2\mu n \frac{\partial u}{\partial n} + \lambda \operatorname{div} u, \tag{6.4'}$$

$$T^{(1)}(\partial_y, n) u - n(n\, T^{(1)}(\partial_y, n) u) = 2\mu \left[\frac{\partial u}{\partial n} - n\left(n \frac{\partial u}{\partial n}\right)\right] + (\mu - \alpha)\, [n \times \operatorname{rot} u]. \tag{6.5}$$

By virtue of (6.2) and (6.5) the boundary condition (3.1) takes the form

$$\left[(\mu-\alpha)\,[n\times \operatorname{rot} u(y)]+2\mu\left\{\frac{\partial u(y)}{\partial n}-n\left(n\,\frac{\partial u(y)}{\partial n}\right)\right\}+2\alpha[n\times\omega(y)]\right]^{\pm}=\overset{*}{f}{}^{(1)}(y). \tag{6.6}$$

Let us consider the sets

$$\text{Ш}^{\pm}(y;d)\equiv \text{Ш}(y;d)\cap\overline{\mathscr{D}}{}^{\pm},\qquad S(y;d)\equiv \text{Ш}(y;d)\cap S,$$

where $\text{Ш}(y;d)$ is a spherical region, with its centre at the point y, its radius being d, and d denoting the Lapunov radius.

Let x be any point in $\text{Ш}\,(y;d)$. Let us pass a straight line through the point x, parallel to the normal $n(y)$. This straight line will intersect $S(y;d)$ at a single point which will be denoted by x'. Let us determine in $\text{Ш}(y;d)$ a function ν_i $(i=1,2,3)$ such that $\nu_i(x)=$ $=n_i(x')$. Let $\nu=(\nu_1,\nu_2,\nu_3)$. We have, of course $\nu\in C^1(\text{Ш}(y;d)$ and $\forall x\in \text{Ш}(y;d)$: $\nu_1^2(x)+\nu_2^2(x)+\nu_3^2(x)=1$.

The condition (6.6) can now be written by means of the limit

$$\lim_{\text{Ш}^{\pm}(y;d)\ni x\to y}\left\{(\mu-\alpha)[\nu(x)\times\operatorname{rot}u(x)]+2\mu\left[\frac{\partial u(x)}{\partial\nu(x)}-\nu(x)\left(\nu(x)\,\frac{\partial u(x)}{\partial \nu(x)}\right)\right]\right.$$
$$\left.+2\alpha[\nu(x)\times\omega(x)]\right\}=\overset{*}{f}{}^{(1)}(y). \tag{6.6'}$$

By virtue of the identities

$$\nu(x)\cdot\frac{\partial u(x)}{\partial\nu(x)}=\frac{\partial}{\partial\nu(x)}\,[u(x)]_{\nu(x)}-u_j(x)\frac{\partial \nu_j(x)}{\partial\nu(x)},$$

$$\frac{\partial u(x)}{\partial\nu(x)}=-[\nu(x)\times\operatorname{rot}u(x)]+\nu_j(x)\operatorname{grad}u_j(x), \tag{6.7}$$

$$\nu_j(x)\operatorname{grad}u_j(x)=\operatorname{grad}[u(x)]_{\nu x}-u_j(x)\operatorname{grad}\nu_j(x), \tag{6.8}$$

the expression (6.6′) can be rewritten thus

$$\lim_{\text{Ш}^{\pm}(y;d)\ni x\to y}\left\{-(\mu+\alpha)[\nu(x)\times\operatorname{rot}u(x)]+2\mu\left(\operatorname{grad}-\nu(x)\frac{\partial}{\partial\nu}\right)[u(x)]_{\nu(x)}\right.$$
$$\left.-2\mu u_j(x)\left(\operatorname{grad}-\nu(x)\frac{\partial}{\partial\nu(x)}\right)\nu_j(x)+2\alpha[\nu(x)\times\omega(x)]\right\}=\overset{*}{f}{}^{(1)}(y).$$

If we observe that

$$\left(\operatorname{grad}-\nu(x)\frac{\partial}{\partial\nu(x)}\right)_k\equiv\frac{\partial}{\partial x_k}-\nu_k(x)\frac{\partial}{\partial\nu(x)}=\mathscr{D}_k,$$

where $\mathscr{D}_k$ is the Günther operator (see [8]), we have from (3.1′) the equation

$$\lim_{\text{Ш}^{\pm}(y;d)\ni x\to y}\left(\operatorname{grad}-\nu(x)\frac{\partial}{\partial\nu(x)}\right)[u(x)]_{\nu(x)}=\left(\operatorname{grad}-n\frac{\partial}{\partial n}\right)\overset{*}{f}{}_4^{(1)}(y)$$

by virtue of which the boundary condition (3.1) takes the form

$$\left[(\mu+\alpha)[n\times\operatorname{rot}u(y)]+2\mu u_j(y)\left(\operatorname{grad}-n\frac{\partial}{\partial n}\right)n_j-2\alpha[n\times\omega(y)]\right]^{\pm}$$
$$=-\overset{*}{f}{}^{(1)}(y)+2\mu\left(\operatorname{grad}-n\frac{\partial}{\partial n}\right)\overset{*}{f}{}_4^{(1)}(y).$$

The condition (3.2) can be transformed in the same manner.

Thus, the boundary conditions of the problem (III.III)$^{\pm}$ take the form

$$(6.9)\quad \left[(\mu+\alpha)\left(n_j\frac{\partial u_j}{\partial y_k}-n_j\frac{\partial u_k}{\partial y_j}\right)+2\mu u_j\mathscr{D}_k n_j-2\alpha\varepsilon_{kij}n_i\omega_j\right]^{\pm}=f_k^{(1)},\quad [n_j u_j]^{\pm}=f_4^{(1)},$$

$$(6.10)\quad \left[(v+\beta)\left(n_j\frac{\partial \omega_j}{\partial y_k}-n_j\frac{\partial \omega_k}{\partial y_j}\right)+2v\omega_j\mathscr{D}_k n_j\right]^{\pm}=f_k^{(2)},\ (k=1,2,3),\ [n_j\omega_j]^{\pm}=f_4^{(2)}.$$

where

$$f^{(1)}=-\overset{*}{f}{}^{(1)}+2\mu\left(\operatorname{grad}-n\frac{\partial}{\partial n}\right)\overset{*}{f}{}_4^{(1)},\quad f_4^{(1)}=\overset{*}{f}{}_4^{(1)};$$

$$f^{(2)}=-\overset{*}{f}{}^{(2)}+2v\left(\operatorname{grad}-n\frac{\partial}{\partial n}\right)\overset{*}{f}{}_4^{(2)},\quad f_4^{(2)}=\overset{*}{f}{}_4^{(2)}.$$

The boundary condition of the problem (IV.IV)$^{\pm}$ can be transformed in an analogous manner. We have

$$[u_k-n_k n_j u_j]^{\pm}=g_k^{(1)},\quad k=1,2,3;\quad \left[(\lambda+2\mu)\frac{\partial u_j}{\partial y_j}-2\mu n_j u_j(\mathscr{D}_k n_k)\right]^{\pm}=g_4^{(1)},$$

$$[\omega_k-n_k n_j\omega_j]^{\pm}=g_k^{(2)},\quad k=1,2,3;\quad \left[(\varepsilon+2v)\frac{\partial \omega_j}{\partial y_j}-2vn_j\omega_j(\mathscr{D}_k n_k)\right]^{\pm}=g_4^{(2)},$$

where

$$g^{(1)}=\overset{*}{g}{}^{(1)},\quad g_4^{(1)}=\overset{*}{g}{}_4^{(1)}+2\mu\mathscr{D}_j\overset{*}{g}{}_j^{(1)},\quad g^{(2)}=\overset{*}{g}{}^{(2)},\quad g_4^{(2)}=\overset{*}{g}{}_4^{(2(}+2v\mathscr{D}_j\overset{*}{g}{}_j^{(2)}.$$

Let us introduce the following differential matrix operators

$$H(\partial_y,n)=[H_{pq}(\partial_y,n)]_{8\times6}=\begin{bmatrix}H^{(1)}(\partial_y,n) & H^{(2)}(\partial_y,n)\\ H^{(3)}(\partial_y,n) & H^{(4)}(\partial_y,n)\end{bmatrix},$$

$$R(\partial_y,n)=[R_{pq}(\partial_y,n)]_{8\times6}=\begin{bmatrix}R^{(1)}(\partial_y,n) & R^{(2)}(\partial_y,n)\\ R^{(3)}(\partial_y,n) & R^{(4)}(\partial_y,n)\end{bmatrix},$$

where

$$H^{(l)}=||H_{\gamma j}^{(l)}||_{4\times3},\quad R^{(l)}=||R_{\gamma j}^{(l)}||_{4\times3},\quad l=1,2,3,4;$$

$$H_{\gamma j}^{(1)}=\left[(\mu+\alpha)n_j\frac{\partial}{\partial y_\gamma}-(\mu+\alpha)\delta_{\gamma j}\frac{\partial}{\partial n}+2\mu\mathscr{D}_\gamma n_j\right](1-\delta_{\gamma4})+n_j\delta_{\gamma4},$$

$$H_{\gamma j}^{(2)}=2\alpha\varepsilon_{\gamma ji}n_i(1-\delta_{\gamma4}),\quad H_{\gamma j}^{(3)}=0,$$

$$H_{\gamma j}^{(4)}=\left[(v+\beta)n_j\frac{\partial}{\partial y_\gamma}-(v+\beta)\delta_{\gamma j}\frac{\partial}{\partial n}+2v\mathscr{D}_\gamma n_j\right](1-\delta_{\gamma4})+n_j\delta_{\gamma4},$$

$$R_{\gamma j}^{(1)}=(\delta_{\gamma j}-n_\gamma n_j)(1-\delta_{\gamma4})+\left[(\lambda+2\mu)\frac{\partial}{\partial y_j}-2\mu n_j(\mathscr{D}_i n_i)\right]\delta_{\gamma4},\quad R_{\gamma j}^{(2)}=0,$$

$$R_{\gamma j}^{(3)}=0,\quad R_{\gamma j}^{(4)}=(\delta_{\gamma j}-n_\gamma n_j)(1-\delta_{\gamma4})+\left[(\varepsilon+2v)\frac{\partial}{\partial y_j}-2vn_j(\mathscr{D}_i n_i)\right]\delta_{\gamma4}\ (^4).$$

(4) In these formulae and everywhere in what follows the values of $\mathscr{D}_k$, n_k, ε_{kij} and $\partial/\partial y_k$ will be considered for $k=4$, to be the same as for $k=1$.

With these notations the problems $(\mathrm{III.III})^{\pm}$ and $(\mathrm{IV.IV})^{\pm}$ can be formulated as follows.

Find in the region $\mathscr{D}^{\pm}$ a vector $\mathscr{U} = (u, \omega)$ of class $C^2(\mathscr{D}^{\pm}) \cap C^1(\overline{\mathscr{D}}^{\pm})$ which satisfies the equation $\mathscr{M}(\partial_x)\mathscr{U} = 0$ and, in the case of the problem $(\mathrm{III.III})^{\mp}$ the boundary condition

$$[H(\partial_y, n)\mathscr{U}]^{\pm} = f, \quad [f = (f_1^{(1)}, \ldots, f_4^{(1)}, f_1^{(2)}, \ldots, f_4^{(2)})],$$

or, in the case of the problem $(\mathrm{IV.IV})^{\pm}$,— the boundary condition

$$[R(\partial_y, n)\mathscr{U}]^{\pm} = g, \quad [g = (g_1^{(1)}, \ldots, g_4^{(1)}, g_1^{(2)}, \ldots, g_4^{(2)})],$$

where $f = (f_1, \ldots, f_8)$, $g = (g_1, \ldots, g_8)$ are real vectors prescribed on S, of class $C^{1,h}(S)$, satisfying the conditions

$$f_k n_k = 0, \quad f_{4+k} n_k = 0, \quad g_k n_k = 0, \quad g_{4+k} n_k = 0.$$

These problems will be denoted in what follows by the symbols $(\mathrm{III.III})_f^{\pm}$, $(\mathrm{IV.IV})_g^{\pm}$, respectively.

7. The potentials and their properties

Let us consider the following vectors of the potential type

$$V(\varphi)(x) = \int_S [R(\partial_y, n)\Psi(y-x)]'\varphi(y)\,d_y S, \tag{7.1}$$

$$W(\varphi)(x) = \int_S [H(\partial_y, n)\Psi(y-x)]'\varphi(y)\,d_y S, \tag{7.2}$$

where $\varphi = \varphi_1, \ldots, \varphi_8$. $[\ ]'$ denotes the transposed matrix in square brackets.

Bearing in mind the identity $[\Psi(y-x)]' = \Psi(x-y)$ it can be shown that if $\varphi \in L(S)$, then $V(\varphi)$ and $W(\varphi)$ belong to the class $C^{\infty}(\mathscr{D}^{\pm})$, satisfy the conditions (4.3) and (4.3′) and

$$\forall x \in E^3 \backslash S : \mathscr{M}(\partial_x) V(\varphi)(x) = 0, \quad \mathscr{M}(\partial_x) W(\varphi)(x) = 0.$$

Let us observe that the value of the potential $V(\varphi)[W(\varphi)]$ does not vary if the density φ is replaced by χ, where $\chi = (\chi_1, \ldots, \chi_8)$ and $\chi_k = \varphi_k - n_k n_j \varphi_j$, $\chi_{4+k} = \varphi_{4+k} - n_k n_j \varphi_{4+j}$, $k = 1, 2, 3$; $\chi_4 = \varphi_4$, $\chi_8 = \varphi_8$. This can easily be found from the equations

$$n_l R_{lj}^{(m)}(\partial_y, n) = 0, \quad [n_l H_{lj}^{(m)}(\partial_y, n) = 0], \quad m = 1, 2, 3, 4; \quad j = 1, 2, 3. \tag{7.3}$$

It is obvious that $n_k \chi_k = 0$, $n_k \chi_{4+k} = 0$. It is concluded that, without changing the value of the potential $V(\varphi)[W(\varphi)]$, we can assume that the conditions

$$\forall y \in S: \; n_k \varphi_k(y) = 0, \quad n_k \varphi_{4+k}(y) = 0, \tag{7.4}$$

are satisfied.

We shall now demonstrate the following theorems

THEOREM 7.1. *If* $\varphi \in C^{0,h}(S)$ *and satisfies the conditions* (7.4) *then, for any* $z \in S$, *there exist* $[H(\partial_z, \nu)V(\varphi)(z)]^{\pm}$ *and* $[R(\partial_z, \nu)W(\varphi)(z)]^{\pm}$ *belonging to the class* $C^{0,h}(S)$ *and*

$$[H(\partial_z, \nu)V(\varphi)(z)]^{\pm} = \mp\varphi(z) + \int_S H(\partial_z, \nu)\,[R(\partial_y, n)\Psi(y-z)]'\varphi(y)d_yS,$$

$$[R(\partial_z, \nu)W(\varphi)(z)]^{\pm} = \pm\varphi(z) + \int_S R(\partial_z, \nu)[\hat{H}(\partial_y, n)\Psi(y-z)]'\varphi(y)d_yS,$$

where $\nu = n(z)$, $n = n(y)$ *and the integrals in the right-hand members are understood in the sense of their principal value.*

If $\varphi \in C^{1,h}(S)$ *then* $V(\varphi)$ *and* $W(\varphi)$ *belong to the class* $C^1(\overline{\mathscr{D}}^+)[C^1(\overline{\mathscr{D}}^-)]$.

THEOREM 7.2. *If* $\varphi \in C^{0,h}(S)$ *and satisfies the conditions* (7.4), *there exist* $[R(\partial_z, \nu)V(\varphi)(z)]$ *and* $[R(\partial_z, \nu)\ V(\varphi)(z)]$ *belonging to the class* $C^{0,h}(S)$ *and equal to each other.*

8. Integral equations

Solution of the boundary value problems $(\mathrm{III.III})^{\pm}$ and $(\mathrm{IV.IV})^{\pm}$ will be sought for in the form (7.1) and (7.2), respectively, with the sought for density φ of class $C^{0,h}(S)$ and satisfying the conditions (7.4). Then, bearing in mind the Theorem 7.1 we obtain the integral equations

$$\mp\varphi(z) + \int_S H(\partial_z, \nu)[R(\partial_y, n)\Psi(y-z)]'\varphi(y)d_yS = f(z), \qquad (\mathrm{III.\ III})_f^{\pm}$$

$$\pm\varphi(z) + \int_S R(\partial_z, \nu)[H(\partial_y, n)\Psi(y-z)]'\varphi(y)d_yS = g(z), \qquad (\mathrm{IV.\ IV})_g^{\pm}.$$

By $(\mathrm{III.III})^+$ we denote the operator, generated by the left-hand member of the equation $(\mathrm{III.III})_f^+$. The notations $(\mathrm{III.III})^-$ and $(\mathrm{IV.IV})^{\pm}$ will have an analogous sense.

THEOREM 8.1. *The operators* $(\mathrm{III.III})^+$, $(\mathrm{IV.IV})^-$, $(\mathrm{III.III})^-$ *and* $(\mathrm{IV.IV})^+$ *in the spaces* $L_p^{(8)}(S)$ *and* $L_{p'}^{(8)}(S)$ *where* $\frac{1}{p} + \frac{1}{p'} = 1$ *are adjoint.*

This statement follows from the identity

$$[R(\partial_z, \nu)[H(\partial_y, n)\Psi(z-y)]']' = H(\partial_y, n)[R(\partial_z, \nu)\Psi(y-z)]', \tag{8.1}$$

which can easily be verified. The symbol $L_p^{(8)}(S)$ denotes the space of vectors having the form $v = (v_1, \ldots, v_8)$ summed over S in the p-th power, the norm being

$$||v|| = \Big\{\int_S \Big(\sum_{i=1}^{8} (|v_i|^2)\Big)^{P/2} d_yS\Big\}^{1/P}.$$

Let us investigate the singular integral equations obtained. For this purpose it is necessary to write expressions for the operators $(\mathrm{III.III})^{\pm}$ and $(\mathrm{IV.IV})^{\pm}$.

Let us represent the fundamental solution (5.2) in the form

$$\Psi_{kj}^{(1)}(x) = \frac{1}{4\pi}\left[\delta_{kj}\frac{\lambda+2\mu+\alpha}{(\mu+\alpha)(\lambda+2\mu)}\frac{1}{|x|} + \frac{\lambda+\mu-\alpha}{(\mu+\alpha)(\lambda+2\mu)}\frac{x_k x_j}{|x|^3}\right] + \chi_{kj}^{(1)}(x),$$

$$(8.2)\qquad \Psi_{kj}^{(2)}(x) = \frac{\alpha}{2\pi(\mu+\alpha)(v+\beta)}\varepsilon_{kjp}\frac{x_p}{|x|} + \chi_{kj}^{(2)}(x),$$

$$\Psi_{kj}^{(4)}(x) = \frac{1}{4\pi}\left[\delta_{kj}\frac{\varepsilon+2v+\beta}{(v+\beta)(\varepsilon+2v)}\frac{1}{|x|} + \frac{\varepsilon+v-\beta}{(v+\beta)(\varepsilon+2v)}\frac{x_k x_j}{|x|^3}\right] + \chi_{kj}^{(3)}(x),$$

where $\chi_{kj}^{(1)}$, $\chi_{kj}^{(2)}$ and $\chi_{kj}^{(3)}$ are functions continuous at every point of the space E^3. These functions have their first derivatives bounded in E^3. Their second derivatives have a singularity at the origin of coordinates only, of order not higher than $|x|^{-1}$.

By means of the expression of the fundamental solution (8.2) just obtained it is easy to write the singular part of the kernels of the operators $(\mathrm{III.III})^{\pm}$ and $(\mathrm{IV.IV})^{\pm}$. We have

$$(8.3)\qquad H(\partial_z, v)\,[R(\partial_y, n)\Psi(y-z)]' = \mathscr{P}(y,z) + \mathscr{Q}(y,z),$$

where

$$\mathscr{P}(y,z) = [\mathscr{P}_{rq}(y,z)]_{8\times 8} = \begin{bmatrix}\mathscr{P}^{(1)}(y,z) & \mathscr{P}^{(2)}(y,z)\\ \mathscr{P}^{(3)}(y,z) & \mathscr{P}^{(4)}(y,z)\end{bmatrix}, \qquad \mathscr{P}^{(m)}(y,z) = [\mathscr{P}_{kj}^{(m)}(y,z)]_{4\times 4}$$

$$m = 1,2,3,4,$$

$$\mathscr{P}_{j\gamma}^{(1)}(y,z) = \frac{1}{2\pi}(1-\delta_{j4})(1-\delta_{\gamma4})\left[(v_\gamma - n_\gamma v_l n_l)\frac{\partial}{\partial z_j}\frac{1}{|y-z|} + (n_\gamma n_j - \delta_{j\gamma})\frac{\partial}{\partial v}\frac{1}{|y-z|}\right]$$

$$+\frac{\mu}{\pi}(1-\delta_{j4})\,\delta_{\gamma4}\left[(\mathscr{D}_i n_i)\left(n_j\frac{\partial}{\partial v}\frac{1}{|y-z|} - v_l n_l\frac{\partial}{\partial z_j}\frac{1}{|y-z|}\right) - \frac{\partial}{\partial z_i}\frac{1}{|y-z|}(\mathscr{D}_\gamma v_i)\right]$$

$$-\frac{\delta_{\gamma4}\,\delta_{j4}}{2\pi}\frac{\partial}{\partial v}\frac{1}{|y-z|},$$

$$\mathscr{P}_{j\gamma}^{(2)}(y,z) = \frac{\alpha}{\pi}(1-\delta_{j4})\,\delta_{\gamma4}\,\varepsilon_{jil}n_l\frac{\partial}{\partial y_i}\frac{.1}{|y-z|}, \qquad \mathscr{P}_{j\gamma}^{(3)}(y,z) = 0,$$

$\mathscr{P}_{j\gamma}^{(4)}(y,z)$ is expressed in exactly the same manner as $\mathscr{P}_{j\gamma}^{(1)}(y,z)$ except that μ is replaced by v. $\mathscr{Q}(y,z)$ is a matrix having the form $[\mathscr{Q}_{ik}(y,z)]_{8\times 8}$, where $\mathscr{Q}_{ik}(y,z)$ is a kernel of weak singularity. More exactly (see [8]) $\mathscr{Q}_{ik} \in G(1,h,h)$ is $S\times S$.

From the identity (8.1) it follows that an analogous expression is valid for the kernel

$$R(\partial_z, v)[H(\partial_y, n)\Psi(y-z)]'.$$

By means of the expression (8.3) it is easy to evaluate the determinant of the symbolic matrix Θ_λ of the operator

$$(\mathrm{III.\,III})_\lambda^{+}(\varphi)(z) \equiv -\varphi(z) + \lambda\int_S H(\partial_z, v)[R(\partial_y, n)\Psi(y-z)]'\varphi(y)\,d_yS,$$

where λ is a complex parameter. We find $\det\Theta_\lambda = 1$.

It follows (see [8]) that the Fredholm theorems are valid for the operator (III.III)$^+$ in the space $L_p^{(8)}(S)$. For the operator (III.III)$^-$ in the space $L_p^{(8)}(S)$ the same can be shown in an analogous manner. Now from the theorem (see [8], p. 166) we have the following

THEOREM 8.2. *Iff belongs to the class $C^{0,h}(S)[C^{1,h}(S)]$ any solution of the equations* (III.III)$_f^{\pm}$ *and* (IV.IV)$_f^{\pm}$ *of the class $L_p^{(8)}(S)$ belongs to the class $C^{0,h}(S)[C^{1,h}(S)]$.*

Thus, we have the following

THEOREM. 8.3. *The equations* (III.III)$_0^{\pm}$ *and* (IV.IV)$_0^{\pm}$ *have a finite number of linearly independent solutions in the space $C^{0,h}(S)[C^{1,h}(S)]$. The equations* (III.III)$_0^+$ *and* (IV.IV)$_0^-$ *and also* (III.III)$_0^-$ *and* (IV.IV)$_0^+$ *have the same number of linearly independent solutions in the space $C^{0,h}(S)[C^{1,h}(S)$.* If $f \in C^{0,h}(S)$ $[g \in C^{1,h}(S)]$ then, *for solvabilty of the Eq.* (III.III)$_f^{\pm}$ [(IV.IV)$_g^{\pm}$] *in the space $C^{0,h}(S)$ it is necessary and sufficient that the conditions*

$$\int_S (f\cdot \varphi^{(k)})dS = 0, \quad \left[\int_S (g\cdot \psi^{(k)})dS = 0\right],$$

are satisfied, where $\varphi^{(k)}[\psi^{(k)}]$ — is the complete set of linearly independent solutions of the equation (IV.IV)$_0^{\pm}$, [(III.III)$_0^{\mp}$].

9. The homogeneous integral equations

Let us investigate the homogeneous singular integral equations (III.III)$_0^{\pm}$ and (IV.IV)$_0^{\pm}$. From the Eqs. (7.3) we easily obtain the following

THEOREM 9.1. *Every solution φ of the homogeneous equation* (III.III)$_0^{\pm}$ [(IV.IV)$_0^{\pm}$)] *of class $C^{0,h}(S)$ satisfies the conditions* (7.4).

THEOREM 9.2. *The equations* (III.III)$_0^{\pm}$ *and* (IV.IV)$_0^{\pm}$ *have only a trivial solution in the space $C^{0,h}(S)$.*

P r o o f. Let us assume that the equation (III.III)$_0^{\pm}$ has a non-trivial solution φ of class $C^{0,h}(S)$. Then φ satisfies the conditions (7.4) [see (9.1)]. Let us consider the potential $V(\varphi)$ [see (7.1)]. From the Theorem 7.1 it follows that

$$[H(\partial_z, \nu)V(\varphi)(z)]^+ = -\varphi(z) + \int_S H(\partial_z, \nu)[R(\partial_y, n)\Psi(y-z)]'\varphi(y)d_yS,$$

and, since φ is a solution of the equation (III.III)$_0^+$,

$$\forall z \in S\colon [H(\partial_z, \nu)V(\varphi)(z)]^+ = 0.$$

It follows that $V(\varphi)$ is a solution of the problem (III.III)$_0^+$. But the problem (III.III)$_0^+$ has only the trivial solution $\forall x \in \mathcal{D}^+ : V(\varphi)(x) = 0$. From the equation it follows that

$$\forall z \in S : [R(\partial_z, \nu)V(\varphi)(z)]^+ = 0.$$

But, by virtue of the Theorem 7.2,

$$\forall z \in S : [R(\partial_z, \nu) V(\varphi)(z)]^- = 0.$$

Let us consider the potential $V(\varphi)$ in the region $\mathscr{D}^-$. By virtue of (9.1) $V(\varphi)$, is a solution of the problem (IV.IV)$^-$. This problem has only the trivial solution $\forall x \in \mathscr{D}^- : V(\varphi)(x) = 0$.

From the Theorem 7.1 we find

$$\forall z \in S : 2\varphi(z) = [H(\partial_z, \nu) V(\varphi)(z)]^- - [H(\partial_z, \nu) V(\varphi)(z)]^+ = 0.$$

The contradiction obtained shows that the equation (III.III)$_0^+$ has only a trivial solution in the space $C^{0,h}(S)$. Similarly it can be shown that the equation (III.III)$_0^-$ has only a trivial solution in the space $C^{0,h}(S)$. It follows (see Theorem 8.2) the equations (IV.IV)$_0^-$ and (IV.IV)$_0^\pm$ have only a trivial solution.

10. The existence of solutions of boundary value problems

From what was shown above it follows that:

THEOREM 10.1. *The problem* (III.III)$_f^+$ [(III.III)$_f^-$] *has a solution for any vectro* $f \in C^{1,h}(S)$ *and it is unique. This solution can be represented in the form* (7.1) *where* φ *is determined from the integral equation* (III.III)$_f^+$ [(IV.IV)$_f^-$].

THEOREM 10.2. *The problem* (IV.IV)$_g^+$ [(IV.IV)$_g^-$] *has a solution for any vector* $g \in C^{i,h}(S)$ *and it is unique. This solution can be represented in the form* (7.2) *where* φ *is to be determined from the integral equation* (IV.IV)$_g^+$ [(IV. IV)$_g^-$].

All the theorems above could have been demonstrated with much weaker limitations on the limiting functions, namely for $\overset{*}{f}{}^{(l)}, \overset{*}{g}{}_4^{(l)} \in C^{0,h}(S)$ (see Sect. 6) and $\overset{*}{f}{}_4^{(l)}, \overset{*}{g}{}^{(l)} \in C^{1,h}(S)$ $(l = 1, 2)$, but this has not be done, for the sake of simplicity.

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