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On a certain solution of dual integral equations and its application to contact problems of consolidation

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THE paper presents the method of construction of the solutions of the dual integral equations with trigonometric kernels

$$(1)\qquad \begin{aligned} &\int_0^\infty \Phi(v)G(v)[\cos uv, \sin uv]\,dv = f(u), && u<1,\\ &\int_0^\infty \Phi(v)\,[\cos uv, \sin uv]\,dv = 0, && u>1, \end{aligned}$$

where $\Phi(v)$ is an unknown function, and $G(v)$ is a given function satisfying the condition

$$(2)\qquad G(v) = \frac{a}{v}[1+H(v)]$$

in which a is a constant and $H(v)$ is such a function that the integral

$$(3)\qquad \int_0^\infty H(v)\cos uv\,dv$$

exists. The system of dual integral equations is reduced to a Fredholm equation of the second kind. The results derived are used for solving the problem of a punch acting on a consolidating visco-elastic halfplane. The contact stresses and displacements of the punch are determined effectively.

W pracy podano konstrukcję rozwiązania dualnych równań całkowych z jądrami trygonometrycznymi

$$(1)\qquad \begin{aligned} &\int_0^\infty \Phi(v)G(v)[\cos uv, \sin uv]\,dv = f(u), && u<1,\\ &\int_0^\infty \Phi(v)\,[\cos uv, \sin uv]\,dv = 0, && u>1, \end{aligned}$$

gdzie $\Phi(v)$ jest funkcją niewiadomą, $G(v)$ jest funkcją daną i taką, że

$$(2)\qquad G(v) = \frac{a}{v}[1+H(v)],$$

a — pewna stała; oraz istnieje

$$(3)\qquad \int_0^\infty H(v)\cos uv\,dv.$$

Równania (1) sprowadzono do równania Fredholma drugiego rodzaju. Otrzymane rezultaty wykorzystano do rozwiązania problemu stempla dla konsolidującej półpłaszczyzny lepkosprężystej. Wyznaczono efektywnie naprężenia kontaktowe i przemieszczenia stempla.

В работе дается построение решения дуальных интегральных уравнений с тригонометрическими ядрами:

$$(1)\qquad \begin{aligned} &\int_0^\infty \Phi(v)G(v)[\cos uv, \sin uv]dv = f(u), \qquad u < 1,\\ &\int_0^\infty \Phi(v)[\cos uv, \sin uv]dv = 0, \qquad u < 1, \end{aligned}$$

где $\Phi(v)$ является неизвестной функцией, $G(v)$ — данной функцией и такой, что

$$(2)\qquad G(v) = \frac{a}{v}[1+H(v)]$$

a — некоторая постоянная; а также существует

$$(3)\qquad \int_0^\infty H(v)\cos uv\, dv.$$

Уравнения (1) сведены к уравнению Фредьгольма второго рода. Полученные результаты использованы для решения задачи штампа для консолидующегося вязко-упругого полупространства. Эффективно определены контактные напряжения и перемещения штампа.

1. Introduction

THE paper presents a certain method of solution of integral equations with trigonometric kernels. The following equations are considered:

$$(1.1)\qquad \begin{aligned} &\int_0^\infty \Phi(v)G(v)\cos uv\, dv = f(u), \qquad u < 1,\\ &\int_0^\infty \Phi(v)\cos uv\, dv = 0, \qquad u > 1, \end{aligned}$$

as also the equations

$$(1.2)\qquad \begin{aligned} &\int_0^\infty \Phi(v)G(v)\sin uv\, dv = f(u), \qquad u < 1,\\ &\int_0^\infty \Phi(v)\sin uv\, dv = 0, \qquad u > 1. \end{aligned}$$

Approximate solutions of the dual integral equations with trigonometric kernels were given in [6], the problem being reduced to a system of algebraic equations. G. SZEFER [7] outlines a method of reducing the Eqs. (1.1) and (1.2) to the Fredholm integral equations of the second kind with a weakly singular kernel. The considerations are based on the assumption that $G(v)$ has the form

$$(1.3)\qquad G(v) = v[1+H(v)],$$

where $H(v)$ is a function ensuring the existence and convergence of the integral

$$(1.4) \qquad \int_0^\infty H(v)[\sin uv;\ \cos uv]\,dv.$$

A number of papers dealing with the solution of dual integal equations is encountered in the scientific literature. Most of those papers are devoted to the equations with Hankel kernels and contain numerous effective methods of solution.

The dual integral equations with trigonometric kernels are much less frequently dealt with in the literature.

The application of Fourier transforms to the solution of mixed boundary value problems of mechanics of continua makes it possible to reduce the problems to the dual integral equations (1.1) or (1.2). In a number of problems, like e.g. the contact problems, the function $G(v)$ has the form

$$(1.5) \qquad G(v) = \frac{1}{v}[1+H(v)],$$

function $H(v)$ satisfying the condition (1.4). Such problems cannot be solved by the method presented in [7]. The method of solution to be outlined in this paper deals with the system (1.1), (1.2) with the condition (1.5) and constitutes a generalization of the method by N.N. LEBEDEV – J. S. UFLJAND [9] to the dual integral equations with trigonometric kernels. The method [9] has been generalized previously to the case of dual integral equations with Hankel kernels of arbitrary (even or odd) order [8]. In order to construct the corresponding solutions, the properties of Weber-Schafheitlin integrals will be used. The Eqs. (1.1) and (1.2) will be reduced to a Fredholm integral equation of the second kind what in many cases proves to be advantageous in view of the possibility of constructing the solutions for that type of integral equations.

Due to the applicatory character of our considerations, the conditions of existence of the solutions will not be considered here. All the transformations will be assumed to be possible, and all the integrals — convergent. The results will be used for solving the contact problems of a consolidating viscoelastic halfplane.

2. Solution of the Eqs. (1.1)

First of all let us tackle the problem of solution of the Eqs. (1.1). The function $\Phi(v)$ is sought for in the form

$$(2.1) \qquad \Phi(v) = v\int_0^1 \xi\varphi(\xi)J_1(\xi v)\,d\xi.$$

Inserting (2.1) into $(1.1)_2$ we obtain after transformations

$$(2.2) \qquad \int_0^\infty v\int_0^1 \xi\varphi(\xi)J_1(\xi v)\,d\xi\cos uv\,dv = \int_0^\infty \Big[-\xi\varphi(\xi)J_0(\xi v)|_0^1 + \int_0^1 \big(\xi\varphi(\xi)\big)'J_0(\xi v)\,d\xi\Big]\cos uv\,dv$$

$$= -\varphi(1)\int_0^\infty J_0(v)\cos uv\,dv + \int_0^1 [\xi\varphi(\xi)]'\int_0^\infty J_0(\xi)\cos uv\,dv\,d\xi.$$

On using the properties of Weber-Schafheitlin integrals [4]

$$(2.3)\qquad \int_0^\infty J_0(\xi v)\cos uv\, dv = \begin{cases} \dfrac{1}{\sqrt{\xi^2-u^2}} & \xi > u,\\ 0, & \xi < u,\end{cases}$$

we conclude that the Eq. $(1.1)_2$ is identically satisfied by (2.1).

Let us now substitute the function (2.1) into (1.1)

$$\int_0^\infty v\int_0^1 \xi\varphi(\xi)J_1(\xi v)d\xi G(v)\cos uv\, dv = \int_0^1 \xi\varphi(\xi)\int_0^\infty [1+H(v)]J_1(\xi v)\cos uv\, dv\, d\xi$$

$$= \int_0^1 \xi\varphi(\xi)\int_0^\infty J_1(\xi v)\cos uv\, dv\, d\xi + \int_0^1 \xi\varphi(\xi)\int_0^\infty H(v)J_1(\xi v)\cos uv\, dv\, d\xi = f(u).$$

Using the properties of the Weber-Schafheitlin integral [4]

$$(2.4)\qquad \int_0^\infty J_1(\xi v)\cos uv\, dv = \begin{cases} \dfrac{1}{\xi} - \dfrac{u}{\xi\sqrt{u^2-\xi^2}}, & \xi < u,\\ \dfrac{1}{\xi}\sqrt{\xi^2-u^2}, & \xi > u,\end{cases}$$

we obtain

$$(2.5)\qquad \int_0^u \frac{u\varphi(\xi)}{\sqrt{u^2-\xi^2}}\,\mathrm{d}\xi - \int_0^1 \xi\varphi(\xi)\int_0^\infty H(v)J_1(\xi v)\cos uv\, dv\, d\xi = h(u),$$

where

$$(2.6)\qquad h(u) = -f(u) + \int_0^1 \varphi(\xi)d\xi.$$

In the first integral of (2.5) the substitution is made $\xi = u\sin\theta$, while in the second integral the Bessel function representation is used

$$J_1(\xi v) = \frac{2}{\pi}\int_0^{\pi/2} \sin(\xi v\sin\theta)\sin\theta d\theta.$$

Then we obtain

$$(2.7)\qquad \int_0^{\pi/2} u\varphi(u\sin\theta)d\theta - \frac{2}{\pi}\int_0^{\pi/2}\int_0^1 \xi\varphi(\xi)\int_0^\infty H(v)\sin(\xi v\sin\theta)\sin\theta\cos uv\, dv\, d\xi\, d\theta = h(u).$$

The Eq. (2.7) is now written in the form

$$(2.8)\qquad \int_0^{\pi/2} F(u\sin\theta)d\theta = h(u),$$

where the following definition has been used

$$(2.9)\qquad F(u\sin\theta) = u\varphi(u\sin\theta) - \frac{2}{\pi}\int_0^1 \xi\varphi(\xi)\int_0^\infty H(v)\sin\left(\frac{\xi v}{u}u\sin\theta\right)u\sin\theta\frac{\cos uv}{u}\,\mathrm{d}v\,\mathrm{d}\xi.$$

The Eq. (2.8) is the well-known Schlömilch equation and possesses the solution

$$F(u) = \frac{2}{\pi}\left[h(0)+u\int_0^{\pi/2} h'(u\sin\theta)\,d\theta\right]. \tag{2.10}$$

From the relations (2.9) and (2.10) it follows that

$$u\varphi(u)-\frac{2}{\pi}\int_0^1 \xi\varphi(\xi)\int_0^\infty H(v)\sin\xi v\cos uv\,dv\,d\xi = F(u). \tag{2.11}$$

Certain transformations and substitutions are now made in the Eq. (2.1)

$$\sin\xi v\cos uv = \frac{1}{2}[\sin v(\xi+u)+\sin v(\xi-u)],$$

$$K(u,\ \xi) = \int_0^\infty H(v)\sin v(\xi+u)\,dv+\int_0^\infty H(v)\sin v(\xi-u)\,dv, \tag{2.12}$$

$$\psi(u) = u\varphi(u).$$

Finally, we obtain

$$\psi(u)-\frac{1}{\pi}\int_0^1 K(u,\ \xi)\psi(\xi)\,d\xi = F(u). \tag{2.13}$$

The Eq. (2.13) thus obtained is a Fredholm integral equation of the second kind. Its solution yields the function $\varphi(u)$ and then $\Phi(v)$ what completes the solution of Eqs. (1.1).

3. Solution of the Eqs. (1.2)

The function $\Phi(v)$ is assumed in the form

$$\Phi(v) = v\int_0^1 \varphi(\xi)J_0(\xi v)\,d\xi. \tag{3.1}$$

Its substitution into the Eq. $(1.2)_2$ yields after transformations

$$\int_0^\infty v\int_0^1 \varphi(\xi)J_0(\xi v)\,d\xi\sin uv\,dv = -\frac{d}{du}\int_0^1 \varphi(\xi)\int_0^\infty J_0(\xi v)\cos uv\,dv\,d\xi = 0.$$

This equality is fulfilled due to the Eq. (2.3). Let us now insert (3.1) into $(1.2)_1$,

$$\int_0^1 \varphi(\xi)\int_0^\infty J_0(\xi v)\sin uv\,dv\,d\xi+\int_0^1 \varphi(\xi)\int_0^\infty H(v)J_0(\xi v)\sin uv\,dv\,d\xi = f(u).$$

The Weber-Schafheitlin integral [4] has the properties

$$\int_0^\infty J_0(\xi v)\sin uv\,dv = \begin{cases} 0, & \xi > u, \\ \dfrac{1}{\sqrt{u^2-\xi^2}}, & \xi < u, \end{cases}$$

and hence

$$\int_0^u \frac{\varphi(\xi)}{\sqrt{u^2-\xi^2}}d\xi+\int_0^1 \varphi(\xi)\int_0^\infty H(v)J_0(\xi v)\sin uv\,dv\,d\xi = f(u). \tag{3.2}$$

On substituting in the first integral $\xi = u\sin\theta$, and in the second integral

$$J_0(\xi v) = \int_0^{\pi/2} \cos(\xi v\sin\theta)\,d\theta,$$

we obtain again the Schlömilch equation

$$\int_0^{\pi/2} F(u\sin\theta)\,d\theta = f(u), \tag{3.3}$$

where

$$F(u\sin\theta) = \varphi(u\sin\theta)+\frac{2}{\pi}\int_0^1 \varphi(\xi)\int_0^\infty H(v)\sin uv\cos\left(\frac{\xi v}{u}u\sin\theta\right)dv\,d\xi\Big). \tag{3.4}$$

Using the solution of the Eq. (3.3)

$$F(u) = \frac{2}{\pi}\left[f(0)+u\int_0^{\pi/2} f'(u\sin\theta)\,d\theta\right], \tag{3.5}$$

we obtain the final form of the equation for $\varphi(\xi)$,

$$\varphi(u)+\frac{1}{\pi}\int_0^1 K(u,\xi)\varphi(\xi)\,d\xi = F(u), \tag{3.6}$$

the function $F(u)$ being given by the formula (3.5), and the kernel is determined in the following manner:

$$K(u,\xi) = \int_0^\infty H(v)\sin v(u+\xi)\,dv+\int_0^\infty H(v)\sin v(u-\xi)\,dv. \tag{3.7}$$

We have obtained again the second kind Fredholm integral equation. Its solution may now be used to construct the solution of the system (1.2), and so the principal aim of our consideration is achieved.

4. Pressure of a punch acting on a consolidating viscoelastic halfplane

Let us consider the state of stress and strain in a consolidating viscoelastic halfplane produced by a flat-ended punch of width $2l$ pressed into the halfplane by a force $P(t)$, Fig. 1. The displacement equations of such medium have the form [1, 3, 10]

$$\begin{gathered} N\Delta u(X,t)+(N+M)\varepsilon_{,x}(X,t) = Ap_{,x}(X,t), \\ N\Delta w(X,t)+(N+M)\varepsilon_{,z}(X,t) = Ap_{,z}(X,t), \\ \frac{k}{\gamma}\Delta p(X,t) = \frac{3n}{\alpha w}\dot{p}(X,t)+\dot{\varepsilon}(X,t). \end{gathered} \tag{4.1}$$

Here, the following notations are introduced:

$u(X, t)$, $w(X, t)$ — components of the displacement vector,

$$\varepsilon(X, t) = \frac{\partial u(X, t)}{\partial x} + \frac{\partial w(X, t)}{\partial z},$$

$p(X, t)$ — fluid pressure inside the pores of the medium,

$$X = X(x, z), \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad (\dot{\ }) = \partial/\partial t, \quad t \text{ — time},$$

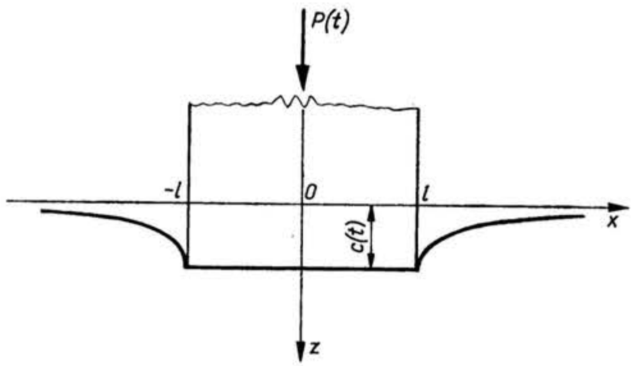

FIG. 1.

k — coefficient of filtration, n — porosity, γ — weight density of the liquid, α_w — modulus of compressibility of the liquid;

$$N^{-1} = \mu^{-1}\Big[1 + \int_0^t K(t-\tau) \ldots d\tau\Big], \quad N = \mu\Big[1 - \int_0^t R(t-\tau) \ldots d\tau\Big],$$

(4.2)

$$A_v^{-1} = \alpha_v^{-1}\Big[1 + \int_0^t K_v(t-\tau) \ldots d\tau\Big], \quad A = \alpha_v\Big[1 - \int_0^t R_v(t-\tau) \ldots d\tau\Big],$$

$$A_p^{-1} = \alpha_p^{-1}\Big[1 + \int_0^t K_p(t-\tau) \ldots d\tau\Big], \quad A_p = \alpha_p\Big[1 - \int_0^t R_p(t-\tau) \ldots d\tau\Big],$$

$$M = \frac{1}{3}(A_v - 2N), \quad A = A_v A_p^{-1},$$

$K(t-\tau)$, $K_v(t-\tau)$, $K_p(t-\tau)$, $R(t-\tau)$, $R_v(t-\tau)$, $R_p(t-\tau)$ — kernels

and resolvents of the kernels of shear and voluminal creep deformation as also the deformation due to pressure of the liquid within the pores; μ, α_v, α_p — moduli of deformation: shear, voluminal and that produced by pressure of the liquid.

Application of the Laplace and Fourier transforms to the system (4.1) yields

$$
\tilde{\bar{u}}(\omega, z, s) = \int_0^\infty \int_0^\infty u(x, z, t)\sin\omega x e^{-st} dx dt,
$$

(4.3)
$$
\begin{bmatrix} \tilde{\bar{w}}(\omega, z, s) \\ \tilde{\bar{p}}(\omega, z, s) \end{bmatrix} = \int_0^\infty \int_0^\infty \begin{bmatrix} w(x, z, t) \\ p(x, z, t) \end{bmatrix} \cos\omega x e^{-st} dx\, dt,
$$

$$
[\bar{A}, \bar{N}, \bar{M}] = \int_0^\infty [A, N, M] e^{-s_1 t} dt,
$$

and the convolution theorem enables us to derive the Laplace transforms of the functions sought for:

(4.4)
$$
\bar{u}(x, z, s) = \frac{2}{\pi} \int_0^\infty \Bigg[C_1(\omega, s) \frac{\bar{N}+\bar{M}}{2\bar{N}} \left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{\bar{N}+\bar{M}} \right) z e^{-\omega z} - C_2(\omega, s) \frac{\bar{A}\omega}{s\bar{B}(2\bar{N}+\bar{M})} e^{-mz} + C_3(\omega, s) e^{-\omega z} \Bigg] \sin\omega x\, d\omega + \bar{u}^*(s),
$$

$$
\bar{w}(x, z, s) = \frac{2}{\pi} \int_0^\infty \Bigg[C_1(\omega, s) \frac{\bar{N}+\bar{M}}{2\bar{N}} \left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{\bar{N}+\bar{M}} \right) z e^{-\omega z} - C_2(\omega, s) \frac{\bar{A}m}{s\bar{B}(2\bar{N}+\bar{M})} e^{-mz} + C_4(\omega, s) e^{-\omega z} \Bigg] \cos\omega x\, d\omega + \bar{w}^*(s),
$$

$$
\bar{p}(x, z, s) = \frac{2}{\pi} \int_0^\infty [C_1(\omega, s) e^{-\omega z} + C_2(\omega, s) e^{-mz}] \cos\omega x\, d\omega.
$$

Here, the notations are used

$$
m^2 = \omega^2 + s\bar{B}, \quad \bar{B} = \frac{\gamma}{k} \left(\frac{3n}{\alpha_w} + \frac{\bar{A}}{2\bar{N}+\bar{M}} \right).
$$

Moreover, the coefficients $C_i(\omega, s)$ should satisfy the condition:

(4.5)
$$
2\bar{N}\omega [C_3(\omega, s) - C_4(\omega, s)] + \left[\frac{3n}{\alpha_w} (3\bar{N}+\bar{M}) + \bar{A} \right] C_1(\omega, s) = 0;
$$

which follows from the method of solution of the system (4.1), [2, 3].

The boundary and initial value problem for the case of a frictionless punch and permeable halfplane boundary takes the form

(4.6)
$$
\begin{aligned}
w(x, 0, t) &= c(t), & -l < x < l, \\
\sigma_z(x, 0, t) &= 0, & x < -l \text{ or } x > l, \\
\sigma_{xz}(x, 0, t) &= 0, & -\infty < x < \infty, \\
p(x, 0, t) &= 0, & -\infty < x < \infty, \\
p(x, z, 0) &= u(x, z, 0) = w(x, z, 0) = 0. &
\end{aligned}
$$

Obviously [3] we have

(4.7)
$$
\begin{aligned}
\sigma_z(X, t) &= 2N\varepsilon_x(X, t) + M\varepsilon(X, t) - Ap(X, t), \\
\sigma_{xz}(X, t) &= 2N\varepsilon_{xz}(X, t).
\end{aligned}
$$

Elimination of the coefficients $C_2(\omega, s)$, $C_3(\omega, s)$, $C_4(\omega, s)$ from $(4.6)_1$–$(4.6)_4$, account being taken of Eqs. (4.5), (4.7), makes it possible to write the mixed boundary condition $(4.6)_{1,2}$ in the form

$$(4.8)\quad \begin{aligned} &\frac{2}{\pi}\int_0^\infty C_1(\omega, s)\left(\frac{3n}{\alpha_w}\,\frac{2\bar{N}+\bar{M}}{2\bar{N}}+\frac{\bar{A}}{2\bar{N}}\right)\frac{\cos\omega x}{\omega}\,d\omega+\bar{w}^*(s) = \bar{c}(s), \qquad x < l,\\ &\frac{2}{\pi}\int_0^\infty C_1(\omega, s)\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\frac{2\bar{A}\bar{N}\omega(m-\omega)}{s\bar{B}(2\bar{N}+\bar{M})}\right]\cos\omega x\,d\omega = 0, \qquad x > l, \end{aligned}$$

where

$$(4.9)\qquad \bar{c}(s) = \int_0^\infty c(t)e^{-st}dt.$$

The following substitution is made in (4.8):

$$(4.10)\quad \begin{aligned} &x = ul, \qquad \omega = vl^{-1},\\ &\bar{\Phi}(v, s) = C_1(v, s)\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\frac{2\bar{A}\bar{N}v(m-v)}{s\bar{B}(2\bar{N}+\bar{M})l^2}\right],\\ &G(v, s) = v^{-1}\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\frac{2\bar{A}\bar{N}v(m-v)}{s\bar{B}(2\bar{N}+\bar{M})l^2}\right]^{-1},\\ &f(s) = \pi\frac{\bar{N}}{2\bar{N}+\bar{M}}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{2\bar{N}+\bar{M}}\right)^{-1}[\bar{c}(s)-\bar{w}^*(s)], \end{aligned}$$

and after simple transformation we obtain

$$(4.11)\quad \begin{aligned} &\int_0^\infty \Phi(v, s)G(v, s)\cos uv\,dv = f(s), \qquad u < 1,\\ &\int_0^\infty \Phi(v, s)\cos uv\,dv = 0, \qquad u > 1. \end{aligned}$$

The function $G(v, s)$ may be transformed as follows:

$$(4.12)\quad \begin{aligned} G(v, s) &= v^{-1}(m+v)\left\{\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}\right](m+v)-\frac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}\right\}^{-1}\\ &= v^{-1}[1-H(v, s)]\,(\bar{N}+\bar{M})^{-1}\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{2\bar{N}+\bar{M}}\right)^{-1}, \end{aligned}$$

with the notation

$$(4.13)\qquad H(v, s) = \frac{sl^2\dfrac{\bar{A}\bar{B}\bar{N}}{2\bar{N}+\bar{M}}}{\left[\dfrac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}\right](m+v)^2-\dfrac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}v(m+v)}.$$

The Eq. (4.11) is now written as

$$(4.14)\qquad \begin{aligned} &\int_0^\infty \frac{1}{v}[1-H(v,s)]\Phi(v,s)\cos uv\,dv = g(s), && u<1,\\ &\int_0^\infty \Phi(v,s)\cos uv\,dv = 0, && u>1,\end{aligned}$$

with the notation

$$(4.15)\qquad g(s) = (\bar{N}+\bar{M})\left(\frac{3n}{\alpha_w}+\frac{\bar{A}}{2\bar{N}+\bar{M}}\right)f(s) = \pi\frac{\bar{N}(\bar{N}+\bar{M})}{2\bar{N}+\bar{M}}[\bar{c}(s)-\bar{w}^*(s)].$$

Thus the mixed boundary value problem is reduced to the dual integral equations (1.1) which were considered in Sect. 1 of this paper. Applying the representation (2.1) to the Eq. (4.14)

$$\Phi(v,s) = v\int_0^1 \xi\varphi(\xi,s)J_1(\xi v)\,d\xi$$

and using the results (2.2) and (2.5), we obtain the equation

$$(4.16)\qquad \int_0^u \frac{u\varphi(\xi,s)}{\sqrt{u^2-\xi^2}}\,d\xi+\int_0^1 \xi\varphi(\xi,s)\int_0^\infty H(v,s)J_1(\xi v)\cos uv\,dv\,d\xi = h(s).$$

Here

$$(4.17)\qquad h(s) = -g(s)+\int_0^1 \varphi(\xi,s)\,d\xi.$$

Further transformation of (4.16) leads to the equation

$$(4.18)\qquad \psi(u,s)+\frac{1}{\pi}\int_0^1 K(u,\xi,s)\psi(\xi,s)\,d\xi = \frac{2}{\pi}h(s),$$

with the following notations:

$$(4.19)\qquad \begin{aligned} &\psi(u,s) = u\varphi(u,s),\\ &K(u,\xi,s) = \int_0^\infty H(v,s)\sin v(\xi+u)\,dv+\int_0^\infty H(v,s)\sin v(\xi-u)\,dv.\end{aligned}$$

The complicated form of the kernel $(4.19)_2$ of the Eq. (4.18) does not allow for determining the accurate solution in a closed form containing elementary functions. The form of the kernel $(4.19)_2$ shows it to be continuous and bounded within the region of its determination, and hence—in view of the continuity of $h(s)$ (4.17) and according to the corresponding theorems of functional analysis—the solution of the Eq. (4.18) is found to be also continuous and bounded. The solution may be determined by approximate methods; thus the difficulty mentioned earlier does not exclude the possibility of analyzing the solution qualitatively.

In what follows we shall determine the contact stresses under the punch, $q(x,t) = \sigma_z(x,t)|_{|x|<l}$, and the corresponding displacements $c(t)$. These results are of primary importance from the points of view of soil and rock mechanics.

The solution of the Eq. (4.18) is written in the form

$$\psi(u,s) = \frac{2}{\pi}h(s) - \frac{1}{\pi}\int_0^1 R(u,\xi,s)\frac{2}{\pi}h(s)d\xi, \tag{4.20}$$

where $R(u,\xi,s)$ is the resolvent of the kernel $K(u,\xi,s)$. Using the notation

$$R^*(u,s) = 1 - \frac{1}{\pi}\int_0^1 R(u,\xi,s)d\xi,$$

the Eq. (4.20) is written as

$$\psi(u,s) = \frac{2}{\pi}h(s)\,R^*(u,s). \tag{4.21}$$

Once the function $\psi(u,s)$ is known, the Laplace transform of the stresses under the punch may easily be determined since, according to the Eqs. $(4.8)_2$, (4.10) and (2.1), we have

$$\bar{q}(u,s) = \frac{2}{\pi}\int_0^\infty C_1(\omega,s)\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\frac{2\bar{A}\bar{N}\omega(m-\omega)}{s\bar{B}(2\bar{N}+\bar{M})}\right]\cos\omega x\,d\omega$$

$$= \int_0^\infty v\int_0^1 \xi\varphi(\xi,s)J_1(\xi v)d\xi\cos uv\,dv.$$

Evaluation of the integral yields

$$\bar{q}(u,s) = -\frac{2}{\pi}\frac{\psi(1,s)}{\sqrt{1-u^2}} + \frac{2}{\pi}\int_u^1 \frac{\psi'_\xi(\xi,s)}{\sqrt{\xi^2-u^2}}d\xi \tag{4.22}$$

and, using the relation (4.21), we obtain

$$\bar{q}(u,s) = -\frac{4}{\pi^2}h(s)\frac{R^*(1,s)}{\sqrt{1-u^2}} + \frac{4}{\pi^2}h(s)\int_u^1 \frac{R^{*\prime}_\xi(\xi,s)}{\sqrt{\xi^2-u^2}}d\xi. \tag{4.23}$$

The punch is acted on by the force $P(t)$, and so

$$2l\int_0^1 \bar{p}(u,s)du = \bar{P}(s), \quad \bar{P}(s) = \int_0^\infty P(t)e^{-st}dt, \tag{4.24}$$

whence, in view of the relation (4.23), we have

$$-\frac{8l}{\pi^2}h(s)\int_0^1\left[\frac{R^*(1,s)}{\sqrt{1-u^2}} - \int_u^1 \frac{R^{*\prime}_\xi(\xi,s)}{\sqrt{\xi^2-u^2}}d\xi\right]du = \bar{P}(s)$$

and

$$h(s) = -\frac{\pi^2}{8l}\frac{\overline{P}(s)}{L(s)} \tag{4.25}$$

with the notation

$$L(s) = \int_0^1 \left[\frac{R^*(1,s)}{\sqrt{1-u^2}} - \int_u^1 \frac{R_\xi^{*\prime}(\xi,s)}{\sqrt{\xi^2-u^2}}d\xi\right]du. \tag{4.26}$$

Substitution of (4.25) into (4.23) yields

$$\overline{q}(u,s) = \frac{\overline{P}(s)}{2lL(s)}\left[\frac{R^*(1,s)}{\sqrt{1-u^2}} - \int_u^1 \frac{R_\xi^{*\prime}(\xi,s)}{\sqrt{\xi^2-u^2}}d\xi\right]. \tag{4.27}$$

The final expression for contact stresses (after the inverse Laplace transform) has the form

$$q(u,t) = \frac{1}{2\pi_i}\int_s \left[\frac{R^*(1,s)}{\sqrt{1-u^2}} - \int_u^1 \frac{R_\xi^{*\prime}(\xi,s)}{\sqrt{\xi^2-u^2}}d\xi\right]\frac{\overline{P}(s)}{2lL(s)}e^{st}ds. \tag{4.28}$$

Let us now pass to the determination of the punch displacement $c(t)$. Like in the two-dimensional problems of single-phase media, we shall determine the relative displacement [5]. Assume the halfplane to be fixed at two points lying on the boundary $z = 0$, symmetric with respect to the $0z$-axis:

$$w(x_0,0,t) = w(-x_0,0,t) = 0 \tag{4.29}$$

and, obviously,

$$\overline{w}(u_0,0,s) = 0, \quad u_0 = x_0 l^{-1}. \tag{4.30}$$

From $(4.4)_2$ it follows that

$$\overline{w}^*(s) = -\frac{2}{\pi}\int_0^\infty C_1(v,s)\left(\frac{3n}{\alpha_w}\frac{2\overline{N}+\overline{M}}{2\overline{N}} + \frac{\overline{A}}{2\overline{N}}\right)\frac{\cos u_0 v}{v}dv. \tag{4.31}$$

On substituting here the relations $(4.10)_1$, $(4.10)_2$ and (4.12) we obtain after transformations

$$\overline{w}^*(s) = -\frac{1}{\pi}\frac{2\overline{N}+\overline{M}}{\overline{N}(\overline{N}+\overline{M})}\left[\int_0^1 \varphi(\xi,s)d\xi - \int_0^{u_0}\frac{u_0\varphi(\xi,s)}{\sqrt{u_0^2-\xi^2}}d\xi - \int_0^1 \xi\varphi(\xi,s)\int_0^\infty H(v,s)J_1(\xi v)\cos u_0 v\, dv\, d\xi\right]. \tag{4.32}$$

From the Eqs. (4.17) and (4.15) we obtain

$$h(s) = -\pi\frac{\overline{N}(\overline{N}+\overline{M})}{2\overline{N}+\overline{M}}\overline{c}(s) + \pi\frac{\overline{N}(\overline{N}+\overline{M})}{2\overline{N}+\overline{M}}\overline{w}^*(s) + \int_0^1 \varphi(\xi,s)d\xi,$$

or, by means of the Eq. (4.32),

$$h(s) = -\pi \frac{\bar{N}(\bar{N}+\bar{M})}{2\bar{N}+\bar{M}} \bar{c}(s) + \int_0^{u_0} \frac{u_0 \varphi(\xi, s)}{\sqrt{u_0^2-\xi^2}} d\xi + \int_0^1 \xi\varphi(\xi, s) \int_0^\infty H(v,s) J_1(\xi v) \cos u_0 v \, dv \, d\xi. \tag{4.33}$$

The Eq. (4.33) is now reduced to the Schlömilch equation (like in Sect. 2), and its solution takes the form

$$\psi(u_0, s) = \frac{1}{\pi} \int_0^1 K(u_0, \xi, s) \psi(\xi, s) d\xi = \frac{2}{\pi} h(s) + \frac{2\bar{N}(\bar{N}+\bar{M})}{2\bar{N}+\bar{M}} c(s), \tag{4.34}$$

where $K(u_0, \xi, s)$ is given by $(4.19)_2$. From the Eqs. (4.21) and (4.25) it follows that

$$\psi(u, s) = -\frac{\pi}{4l} \frac{\bar{P}(s)}{L(s)} R^*(u, s). \tag{4.35}$$

Taking into account the Eqs. (4.25) and (4.35) in the Eq. (4.34) we have, after certain transformations,

$$\bar{c}(s) = \frac{\pi}{8l} \frac{2\bar{N}+\bar{M}}{\bar{N}(\bar{N}+\bar{M})} D(u_0, s) \bar{P}(s), \tag{4.36}$$

where the following notations are introduced:

$$\begin{aligned} D(u_0, s) &= \frac{L(u_0, s)}{L(s)}, \\ L(u_0, s) &= 1 - R^*(u_0, s) - \frac{1}{\pi} \int_0^1 K(u_0, \xi, s) R^*(\xi, s) d\xi. \end{aligned} \tag{4.37}$$

Displacements of the punch are now written in the final form

$$c(t) = \frac{1}{2\pi i} \int_s \frac{\pi}{8l} \frac{2\bar{N}+\bar{M}}{\bar{N}(\bar{N}+\bar{M})} D(u_0, s) \bar{P}(s) e^{st} ds. \tag{4.38}$$

The results (4.28) and (4.38) thus obtained constitute the solution of the contact boundary value problem of the consolidation theory formulated at the beginning of this section.

5. Concluding remarks

1. The results derived in Sections 2 and 3 may be applied to the solutions of boundary value problems of continuum mechanics as it has been shown in Sect. 4 of this paper.

2. The same method of solution may be applied to the case of a punch acting on a consolidating, viscoelastic halfplane with an impermeable boundary. It is easily verified

that the only difference is manifested by different forms of the functions $\Phi(v,s)$, $G(v,s)$, $H(v,s)$ which in the present case are

$$\Phi'(v,s) = C_1(v,s)\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\frac{2\bar{A}\bar{N}\frac{v^2}{m}(m-v)}{s\bar{B}(2\bar{N}+\bar{M})l^2}\right].$$

$$G'(v,s) = v^{-1}\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}-\frac{2\bar{A}\bar{N}\frac{v^2}{m}(m-v)}{s\bar{B}(2\bar{N}+\bar{M})l^2}\right]^{-1}.$$

$$H'(v,s) = \frac{sl^2\frac{\bar{A}\bar{B}\bar{N}}{2\bar{N}+\bar{M}}(m+2v)}{\left[\frac{3n}{\alpha_w}(\bar{N}+\bar{M})+\bar{A}\right]m(m+v)^2-\frac{2\bar{A}\bar{N}}{2\bar{N}+\bar{M}}v^2(m+v)}.$$

The class of regularity of these functions being the same as in the previous case, the method of solution remains unchanged.

3. The integrals occurring in the solutions (4.28) and (4.38) are not elementary; however, they satisfy all the convergence requirements and thus may effectively be evaluated by means of the interpolation polynomials as shown in [2, 3].

4. It follows from the form of Eq. (4.28) that the contact stresses in a two-phase medium possess the same character of singularity as the contact stresses in single-phase media. The same result was obtained in the case of an axi-symmetric punch in [2].

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