

## Microstress approach in thermoplasticity(\*)

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THE presented paper, leaning on earlier author's papers [1, 2], shows their generalization to non-isothermal slow plastic deformation processes. A technical polycrystalline material is described as a special kind of a composite of two material constituents, where the second constituent simulates the influence of obstacles to plastic deformation. Elastic modulus of the composite and plastic limits of the constituents depend on temperature. The consequences of the presented theoretical approach are compared with two kinds of experimental observations and a good qualitative agreement is found.

Niniejsza praca, wykorzystując wcześniejsze koncepcje autora [1, 2], stanowi ich uogólnienie na nieizotermiczne wolne procesy deformacji plastycznej. Techniczny materiał polikrystaliczny opisano jako specjalny rodzaj kompozytu dwuskładnikowego, w którym drugi składnik gra rolę barier przy odkształceniu plastycznym. Moduł sprężysty kompozytu oraz granice plastyczności poszczególnych składników zależą od temperatury. Otrzymane wyniki teoretyczne porównano z dwoma rodzajami obserwacji doświadczalnych i stwierdzono dobrą zgodność jakościową.

Настоящая работа, базируя на более ранних работах автора [1, 2], составляет их обобщение на неизотермические свободные процессы пластической деформации. Технический поликристаллический материал описан как специальный род двухкомпонентного композита, в котором второй компонент играет роль барьеров при пластической деформации. Модуль упругости композита и пределы пластичности отдельных компонентов зависят от температуры. Полученные теоретические результаты сравнены с двумя родами экспериментальных наблюдений и констатировано хорошее качественное совпадение.

### Notations

- $\bar{\sigma}_{ij}$  macroscopic stress tensor,
- $\bar{s}_{ij}$  macroscopic stress deviator,
- $s_{ijn}$  mean value of the stress deviator in the  $n$ -th material constituent,
- $s_{ijn}^r$  residual value of  $s_{ijn}$ ,
- $s'_{ijn}$  deviator expressing the heterogeneity of distribution of  $s_{ijn}$ ,
- $s_{ijn}^{r'}$  residual value of  $s'_{ijn}$ ,
- $\bar{\epsilon}_{ij}$  macroscopic strain tensor,
- $\bar{e}_{ij}$  macroscopic strain deviator,
- $e_{ijn}$  mean value of the strain deviator in the  $n$ -th material constituent,
- $E$  Young's modulus,
- $\nu$  Poisson's ratio,
- $\mu = (1 + \nu)/E$ ,
- $G = 1/2\mu$ ,
- $d\lambda_n$  scalar measure of the incremental plastic deformation in the  $n$ -th material constituent,
- $\vartheta_n$  volume fraction of the  $n$ -th material constituent,

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- $\eta_n$  positive material constant relating to the  $n$ -th material constituent, depending on the structure of the material,  
 $k_n$  plastic limit under pure shear loading of the  $n$ -th material constituent,  
 $c = 2k_n/\sqrt{3}$ ,  
 $T$  absolute temperature,  
 $T^*$  room temperature,  
 $c^*, \mu^*$  values of  $c, \mu$  at room temperature,  
 $A, B, C$  material constants — functions of  $\vartheta_a, \vartheta_b, \eta_a, \eta_b$ ,  
 $X, X^*, Y, Y^*, v, v^*$  quantities defined by the Eqs. (3.2) and (3.11),  
 $f_n$  yield function of the  $n$ -th material constituent.

In earlier papers of the author (cf. [1, 2]) a continuum theory of isothermal time-independent plasticity was formulated, the basic idea of which was that a technical polycrystalline material was considered as a conglomerate of two kinds of microparticles: the particles of the basic material that deform easily and the fixed obstacles.

In this paper we consider the possibility of moderate temperature change which exerts influence upon the elastic constants and the criteria of yielding. Furthermore, the obstacles are supposed here to be capable of undergoing also plastic deformation if the temperature is elevated.

## 1. Basic relations

Principally the analysed material is arrived at as a special kind of a quasi-homogeneous quasi-isotropic composite of two material constituents which do not differ in their elastic properties, but differ in their resistance to plastic deformation. The  $a$ -material is characterized by a low resistance and the  $b$ -material by a high resistance. The last named material constituent simulates the influence of fixed obstacles which are assumed to be concentrated mostly at grain boundaries owing to impurities, precipitates the change of orientation, an "obstacle" in itself (cf. [1] and experimental works quoted in the paper).

The constitutive equations of both kinds of microparticles are supposed to be of the Prandtl-Reuss type as follows:

$$(1.1) \quad de_{ij} = \mu ds_{ij} + s_{ij} d\lambda, \quad d\lambda \geq 0,$$

where  $e_{ij}, s_{ij}$  are the deviatoric strain and stress respectively,  $\mu = (1+\nu)/E$  is the elastic compliance and  $d\lambda$  is the scalar measure of plastic deformation.

With low stresses the response of both material constituents is elastic ( $d\lambda = 0$ ), the elastic compliance  $\mu$  is constant and the material behaves thus as homogeneous (stage I).

With higher stresses plastic deformation in the  $a$ -material steps in with the  $b$ -material remaining elastic (stage II).

Only under special conditions — relatively high temperature — is the  $b$ -material also supposed to be capable of undergoing plastic deformation (stage III).

As to the first invariants of strain due to stressing and thermal expansion, they are supposed to have the same course as in a homogeneous material and may thus be analysed separately from the deviatoric parts.

The basic relations for a quasi-homogeneous composite consisting of two kinds of material constituents may be formulated as follows (cf. e.g. [5, 6, 7, 1]):

$$(1.2) \quad \vartheta_a \sigma_{ija} + \vartheta_b \sigma_{ijb} = \bar{\sigma}_{ij},$$

$$(1.3) \quad \vartheta_a \varepsilon_{ija} + \vartheta_b \varepsilon_{ijb} = \bar{\varepsilon}_{ij},$$

$$(1.4) \quad \bar{\sigma}_{ij} \bar{\varepsilon}_{ij} = \bar{\sigma}_{ij} \bar{\varepsilon}_{ij},$$

where  $\sigma_{ij}$ ,  $\varepsilon_{ij}$  are the stress- and strain-tensors, respectively,  $\vartheta_a (= 1 - \vartheta_b)$  is the volume fraction of the  $a$ -material constituent,  $\sigma_{ija}$  is the arithmetic mean of  $\sigma_{ij}$  in the  $a$ -material and  $\bar{\sigma}_{ij}$  is the arithmetic mean of  $\sigma_{ij}$  in the composite.

The first of the three equations expresses the static equilibrium condition, the second one the kinematic composition of deformation and the third one expresses the equality between the work of macroscopic stress and strain and the mean work of the respective micro-quantities.

By linear combinations it can easily be shown that the first two equations hold equally good for the deviatoric parts and the first invariants separately.

From the Eq. (1.4) it follows

$$\bar{s}_{ij} \bar{e}_{ij} + 3\bar{\sigma} \bar{\varepsilon} = \bar{s}_{ij} \bar{e}_{ij} + 3\bar{\sigma} \bar{\varepsilon},$$

where  $\sigma = \sigma_{ii}/3$ ,  $\varepsilon = \varepsilon_{ii}/3$  and according to our supposition  $\varepsilon$  is distributed as in a homogeneous material, so that  $\bar{\sigma} \bar{\varepsilon} = \bar{\sigma} \bar{\varepsilon}$ , and thus

$$(1.5) \quad \bar{s}_{ij} \bar{e}_{ij} = \bar{s}_{ij} \bar{e}_{ij}.$$

With some simplifying assumptions concerning the law of distribution of stress and strain in the composite, it was shown — primarily in [1], but more simply and precisely in [3] — that

$$(1.6) \quad \bar{s}_{ij} d\bar{e}_{ij} = \sum_{n=a,b} \vartheta_n (s_{ijn} de_{ijn} + \frac{1}{\eta_n} s'_{ijn} de'_{ijn})$$

with

$$(1.7) \quad de_{ijn} = \mu ds_{ijn} + s_{ijn} d\lambda_n,$$

$$(1.8) \quad de'_{ijn} = de_{ijn} - d\bar{e}_{ij} = \mu ds'_{ijn} + s'_{ijn} d\lambda_n,$$

( $n = a, b$ , no summation over  $n$ )

$\eta_n \geq 0$  being positive material constants.

The methods of determination of  $\eta_n$ ,  $\vartheta_n$  from macroscopic tests are shown in [4]: Combining (1.5) with (1.6), (1.2) and (1.3) we arrive at:

$$0 = \delta(\bar{s}_{ij} \bar{e}_{ij} - \bar{s}_{ij} \bar{e}_{ij}) = \vartheta_a \vartheta_b \left( s_{ija} - s_{ijb} + \frac{s'_{ija}}{\eta_a} - \frac{s'_{ijb}}{\eta_b} \right) (\delta e_{ija} - \delta e_{ijb}).$$

The last equation must be fulfilled for any  $\delta e_{ija}$ ,  $\delta e_{ijb}$  and any stage of the process and thus

$$(1.9) \quad s_{ija} - s_{ijb} + \frac{s'_{ija}}{\eta_a} - \frac{s'_{ijb}}{\eta_b} = 0,$$

throughout the process.

Combining further the Eqs. (1.2), (1.3), (1.7), (1.8), (1.9) and their differentiated forms it can easily be deduced that

$$(1.10) \quad d\bar{e}_{ij} = \mu d\bar{s}_{ij} + \vartheta_a s_{ija} d\lambda_a + \vartheta_b s_{ijb} d\lambda_b,$$

$$(1.11) \quad ds_{ija} = d\bar{s}_{ij} - \frac{\vartheta_b}{\mu(\vartheta_a \eta_a + \vartheta_b \eta_b + \eta_a \eta_b)} \{[(\vartheta_a \eta_a + \vartheta_b \eta_b) s_{ija} - \eta_b s'_{ija}] d\lambda_a - [(\vartheta_a \eta_a + \vartheta_b \eta_b) s_{ijb} - \eta_a s'_{ijb}] d\lambda_b\}.$$

$$(1.12) \quad ds'_{ija} = \frac{\eta_a}{\mu(\vartheta_a \eta_a + \vartheta_b \eta_b + \eta_a \eta_b)} \{[\vartheta_b \eta_b s_{ija} - (\vartheta_a + \eta_b) s'_{ija}] d\lambda_a - [\vartheta_b \eta_b s_{ijb} + \vartheta_b s'_{ijb}] d\lambda_b\},$$

where  $\bar{s}_{ij}$ ,  $\bar{e}_{ij}$  are macroscopic deviatoric stress and strain,  $s_{ija}$ ,  $s_{ijb}$  are the arithmetic means of the deviatoric stress in the respective material constituents and  $s'_{ija}$ ,  $s'_{ijb}$  are deviators that describe the heterogeneity of distribution of  $s_{ija}$ ,  $s_{ijb}$  and are defined by the Eq. (1.8) (for details see [3]). The values of  $s_{ijb}$ ,  $s'_{ijb}$  in any stage may be obtained from the Eqs. (1.2) and (1.9).

To complete the constitutive equation it is necessary to express  $d\lambda_a$ ,  $d\lambda_b$  on the basis of the criteria of yielding in the two material constituents. In our concept the mechanical state of the material constituents is described by the deviators  $s_{ijn}$ ,  $s'_{ijn}$  and the yield criterion is formulated therefore in terms of these variables:

$$(1.13) \quad f_n(s_{ijn}, s'_{ijn}, k_n) = 0, \quad n = a, b,$$

where the quantities  $k_a$ ,  $k_b$  representing the plastic limits depend on temperature.

By differentiation of the Eq. (1.13) we have

$$(1.14) \quad \frac{\partial f_n}{\partial s_{ijn}} ds_{ijn} + \frac{\partial f_n}{\partial s'_{ijn}} ds'_{ijn} + \frac{\partial f_n}{\partial T} dT = 0, \quad (n = a, b, \text{ no summation over } n)$$

and inserting  $ds_{ija}$ ,  $ds'_{ija}$  from the Eqs. (1.11) and (1.12) and  $ds_{ijb}$ ,  $ds'_{ijb}$  from the differentiated Eqs. (1.2) — rewritten for deviatoric components — and (1.9) we arrive at a system of two linear equations for  $d\lambda_a$ ,  $d\lambda_b$ :

$$(1.15) \quad d\lambda_a \left\{ -\vartheta_b (\vartheta_a \eta_a + \vartheta_b \eta_b) s_{ija} - \eta_b s'_{ija} \right\} \frac{\partial f_a}{\partial s_{ija}} + \eta_a [\vartheta_b \eta_b s_{ija} - (\vartheta_a + \eta_b) s'_{ija}] \frac{\partial f_a}{\partial s'_{ija}} \left. \right\} + d\lambda_b \left\{ \vartheta_b [(\vartheta_a \eta_a + \vartheta_b \eta_b) s_{ijb} - \eta_a s'_{ijb}] \frac{\partial f_a}{\partial s_{ija}} - \eta_a [\vartheta_b \eta_b s_{ijb} + \vartheta_b s'_{ijb}] \frac{\partial f_a}{\partial s'_{ija}} \right\} + \mu (\vartheta_a \eta_a + \vartheta_b \eta_b + \eta_a \eta_b) \left( \frac{\partial f_a}{\partial s_{ija}} d\bar{s}_{ij} + \frac{\partial f_a}{\partial T} dT \right) = 0.$$

The second equation is obtainable by the interchange of the indexes  $a, b$ .

All the above equations were written for the case of an active plastic process in both the material constituents, i.e. "stage III". The respective equations for "stage II" may be obtained by setting  $d\lambda_b = 0$  and for "stage I" by setting  $d\lambda_a = d\lambda_b = 0$ .

Once the yield criteria are formulated the decision, if the process in the respective material constituent is active, is simple:

by definition

$$d\lambda_n = 0 \quad \text{for } f_n < 0,$$

$$\text{or } f_n = 0, \quad \frac{\partial f_n}{\partial s_{ijn}} d\bar{s}_{ij} + \frac{\partial f_n}{\partial T} dT \leq 0.$$

For  $d\lambda_n = 0$ , one of the two equations (1.15) in which there appear the derivatives of  $f_n$ , must be omitted.

In our forthcoming qualitative demonstrations we shall consider the simplest variant of the criterion:

$$(1.16) \quad f_n \equiv s_{ijn} s_{ijn} - 2k_n^2 = 0, \quad (\text{summation over } i, j, \text{ no summation over } n)$$

or

$$(1.17) \quad f_n = (\bar{s}_{ij} + s_{ijn}^r) (\bar{s}_{ij} + s_{ijn}^r) - 2k_n^2 = 0,$$

with

$$(1.18) \quad s_{ijn}^r = s_{ijn} - \bar{s}_{ij}$$

meaning residual microstresses due to the preceding plastic deformation and elastic unloading.

It is easy to see that in this case

$$(1.19) \quad \frac{\partial f_n}{\partial \bar{s}_{ij}} = 2s_{ijn},$$

i.e. the directions of the increments of plastic deformation in the Eq. (1.10) coincide with the normal directions to the respective yield surfaces (1.19).

## 2. Tempering of specimens prestrained plastically by tension

To illustrate the concept outlined in Sect. 1 let us consider plastic deformation of a metallic polycrystalline specimen at room temperature due to tension  $\bar{\sigma}_{11} > 0$ . In such a process the obstacles do not undergo plastic deformation ( $d\lambda_b = 0$ ) and temperature is constant ( $dT = 0$ ).

For the stress components during the process it holds

$$\frac{2}{3} \bar{\sigma}_{11} = \bar{s}_{11} = -2\bar{s}_{22} = -2\bar{s}_{33} > 0, \quad s_{11a} = -2s_{22a} = -2s_{33a} > 0,$$

and the residual stresses after unloading are

$$s_{11a}^r = -2s_{22a}^r = -2s_{33a}^r < 0, \quad s_{11a}^r = -2s_{22a}^r = -2s_{33a}^r > 0.$$

The last two inequalities may be deduced from the Eqs. (1.11) and (1.12) as follows.

For  $s_{11a} > 0$ ,  $s_{11a}^r \geq 0$  it holds:

$$(2.1) \quad [(\vartheta_a \eta_a + \vartheta_b \eta_b) s_{11a} - \eta_b s_{11a}^r] > [\vartheta_b \eta_b s_{11a} - (\vartheta_a + \eta_b) s_{11a}^r]$$

as  $\vartheta_a$ ,  $\vartheta_b$ ,  $\eta_a$ ,  $\eta_b$  are positive. On the plastic limit  $s_{11a}^r = 0$ ,  $s_{11a} = \bar{s}_{11} > 0$ ;  $s_{11a}^r$  increases further with plastic deformation, but it cannot overpass the value of  $[s_{11a} \vartheta_b \eta_b / (\vartheta_a + \eta_b)]$

as the increase is proportional to  $[\vartheta_b \eta_b s_{11a} - (\vartheta_a + \eta_b) s'_{11a}]$ . It holds therefore throughout the drawing:

$$ds'_{11a} = ds''_{11a} > 0$$

and by the Eqs. (1.11) and (2.1):

$$ds'_{11a} = ds_{11a} - d\bar{s}_{11} < 0, \quad \text{q.e.d.}$$

For residual stresses it holds furthermore according to (1.2):

$$(2.2) \quad s'_{jba} = -\frac{\vartheta_a}{\vartheta_b} s'_{jba},$$

and as  $\vartheta_b \ll \vartheta_a$  it follows:

$$(2.3) \quad |s'_{jba}| \gg |s'_{jba}|.$$

Now if the temperature of the prestrained unloaded specimen is elevated, two mechanisms of plastic deformation may occur:

a) Temperature lowers  $k_a$  as much as the residual stresses cause plastic deformation in the basic material;

b) Temperature lowers  $k_b$  — i.e. weakens the obstacles — as much as the residual stresses cause plastic deformation of the obstacles.

(The decrease of flow stress with increasing temperature is a generally observed phenomenon — see e.g. [8, 9, 10, 11]. In some cases — cf. [8] — the flow stress divided by the respective shear modulus was observed to be constant in some interval of temperature, but as the shear modulus decreases with temperature (cf. [12]) it means that even in this case the flow stress decreases).

The basic material is weaker, but residual stresses are much higher in the obstacles as stated by (2.3).

In both cases ( $d\lambda_a > 0$ ,  $d\lambda_b = 0$  or  $d\lambda_b > 0$ ,  $d\lambda_a = 0$ ) residual microstresses relax according to (1.11), (1.12), (2.2) (1.2), (1.1) in both the material constituents, but in the first case the respective plastic deformation is, according to (1.10):

$$(2.4) \quad \begin{aligned} d\bar{\epsilon}_{11} &\doteq d\bar{\epsilon}_{11} = \vartheta_a s'_{11a} d\lambda_a < 0, \\ d\bar{\epsilon}_{22} &\doteq d\bar{\epsilon}_{22} = \vartheta_a s'_{22a} d\lambda_a > 0, \end{aligned}$$

i.e. contraction in the direction of the previous drawing and expansion in the transverse direction.

In the second case:

$$(2.5) \quad \begin{aligned} d\bar{\epsilon}_{11} &\doteq d\bar{\epsilon}_{11} = \vartheta_b s'_{11b} d\lambda_b > 0, \\ d\bar{\epsilon}_{22} &\doteq d\bar{\epsilon}_{22} = \vartheta_b s'_{22b} d\lambda_b > 0, \end{aligned}$$

i.e. expansion in the direction of the previous drawing and contraction in the transverse direction.

Both these phenomena were observed by the authors of the experimental works [13 and 14].

For the case of carbon steels, which were investigated in the paper [13], the observed course of deformation is schematically illustrated in Fig. 1. (full lines  $\bar{e}_{11}$ ,  $\bar{e}_{22}$ ). Such a course of deformation may be explained by the analysis outlined in the foregoing section if the law of change of  $k_a$  and  $k_b$  with temperature is supposed to have such a character that at first the basic material deforms plastically ( $k_a = \sqrt{3}s_{11a}^r/2$ ), which results in contraction of the specimen in the direction of previous tension and expansion in the transverse direction [cf. the Eq. (2.4)]. With the temperature further increasing, the obstacles are substantially weakened, their plastic deformation steps in ( $k_b = \sqrt{3}s_{11b}^r/2$ ) and predo-

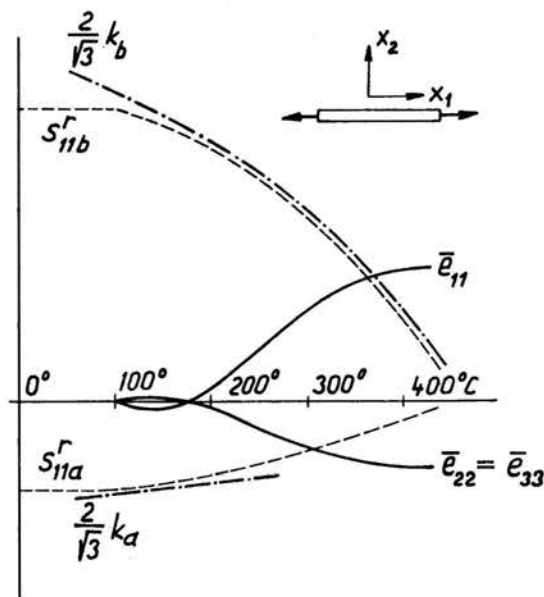


FIG. 1.

minates [cf. the Eq. (2.5)]. However, residual microstresses  $s_{ija}$ ,  $s_{ijb}$  relax in the course of this process [cf. the Eqs. (1.11) and (1.2)] and the deformation rate tends to zero.

Alternatively the process may be illustrated by the change of the yield surfaces as is done in Fig. 2. Accepting the yield criterion (1.17) the yield surfaces are — in the space of the macrostresses  $\bar{s}_{ij}$  — hyperspheres, their shift from the central position being given by the residual microstresses  $s_{ijn}^r$ . The central sections of the hyperspheres are schematically illustrated in Fig. 2. The microstresses  $s_{ijn}^r$  have opposite signs in the *a*- and *b*-material constituents [cf. the Eq. (2.2)] and therefore the centres of the hyperspheres  $f_a, f_b$  are shifted in opposite directions. At room temperature the surface  $f_a^1$  is far smaller than  $f_b^1$  and is enclosed in it, i.e. the elastic domain is limited by  $f_a^1$  only and the obstacles cannot deform plastically. With increasing temperature the diameters of the surfaces decrease (i.e.  $k_a, k_b$  decrease). At the moment when the zero-point of the macroscopic stress gets onto the yield surface ( $f_a^2$ ), plastic deformation steps in the *a*-material which, according to (2.4), represents contraction in the  $x_1$ -direction and expansion in the transverse direction. With the plastic deformation increasing, residual stresses are being reduced, which means that the centres of the yield surfaces are shifted towards the zero-point. The decrease of

the diameter of  $f_b$  (i.e. the weakening of the obstacles) with the increase of temperature may be so rapid that the zero-point gets onto this second yield surface (position  $f_b^3$  in Fig. 2) and this new plastic deformation, which takes an opposite direction, preponderates over the first one. Then the shift of the centre of  $f_a$  due to the decrease of the residual stress  $s'_{ija}$  may be larger than the decrease of the diameter of  $f_a$  and the zero-point falls into the area limited by  $f_a$ , i.e. plastic deformation proceeds only in the  $b$ -material

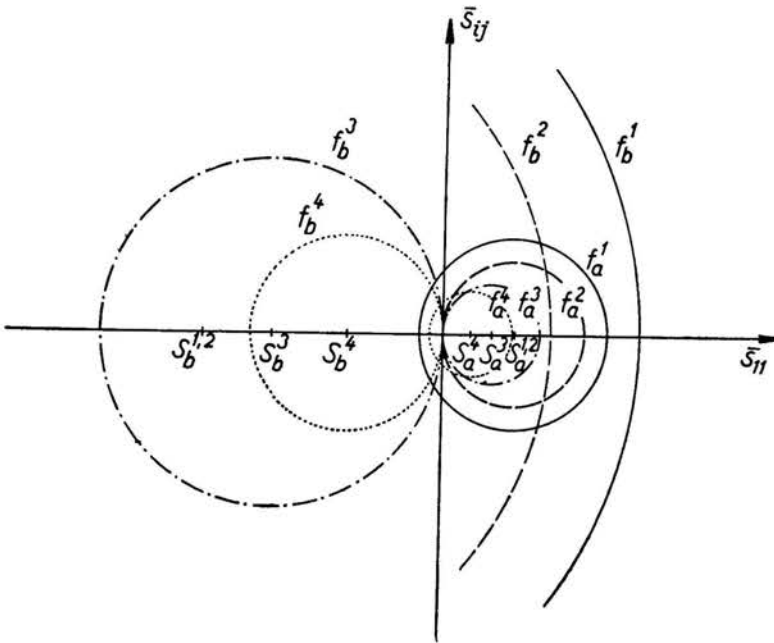


FIG. 2.

and in the direction of the outward normal to  $f_b$  (positions  $f_a^4$  and  $f_b^4$  in Fig. 2). This means expansion in the  $x_1$ -direction and contraction in the transverse direction [cf. the Eq. (2.5)].

In both the quoted experimental works their authors connect the observed deformations with the change of microstresses.

The consequences of the presented theoretical approach agree also with other experimental observations recorded in [13]:

a) The rate of deformation decreased in the range of temperature over  $300^\circ\text{C}$  — i.e. microstresses causing the deformation were already small in this range;

b) The time of tempering did not affect the observed deformations;

c) The density changes were very small, expansion in the  $x_1$ -direction was accompanied by contraction in the transverse directions;

d) Deformations during cooling corresponded to normal isotropic thermal contraction only;

e) A new tempering after cooling did not lead to new changes in the material up to the temperature of the first tempering and then it proceeded in the same way as if the interruption had not existed;



f) Presence of carbide in the material increased the observed effect compared with other steels with the same content of carbon, i.e. the presence of hard carbide particles led to a more substantial effect of the obstacles.

### 3. Influence of the temperature of prestraining upon a new deformation process at room temperature

Let us now consider two successive deformation process-drawings in the same direction  $x_1$ . The second process proceeds at room temperature, the first process at different temperatures. The extent of plastic deformation in the first process is supposed fixed and we are going to investigate the influence of the temperature of the first process upon the second one. The temperature changes are considered to be relatively small in this case so that  $d\lambda_b = 0$  in both the processes, plastic deformation steps in only in the  $a$ -material and only due to loading and the change of microstresses is connected only with this plastic deformation.

By integration of the Eqs. (1.10) to (1.12) with the yield criterion (1.16) for  $n = a$  and with  $d\lambda_b = 0$  the following description may be deduced for the first process that starts from the virgin state (for details see [4]):

$$(3.1) \quad Y = Av - B \ln(1 - v), \quad X = -C \ln(1 - v)$$

with

$$(3.2) \quad Y = \frac{\bar{s}_{11} - c}{c}, \quad X = \frac{\bar{e}_{11}^p}{\mu c}, \quad v = \frac{\vartheta_a + \eta_b}{\vartheta_b \eta_b} \frac{s'_{11a}}{c},$$

$$\bar{s}_{11} = -2\bar{s}_{22} = -2\bar{s}_{33}, \quad s_{11a} = -2s_{22a} = -2s_{33a},$$

$$s'_{11a} = -2s'_{22a} = -2s'_{33a}, \quad \bar{e}_{11}^p = -2\bar{e}_{22}^p = -2\bar{e}_{33}^p,$$

$$(3.3) \quad A = \frac{\vartheta_b^2 \eta_b^2}{\eta_a (\vartheta_a + \eta_b)^2}, \quad B = \frac{\vartheta_a \vartheta_b (\vartheta_a \eta_a + \vartheta_b \eta_b + \eta_a \eta_b)}{\eta_a (\vartheta_a + \eta_b)^2},$$

$$C = \frac{\vartheta_a (\vartheta_a \eta_a + \vartheta_b \eta_b + \eta_a \eta_b)}{\eta_a (\vartheta_a + \eta_b)}, \quad c = 2k_a / \sqrt{3}.$$

After unloading there remain in the material residual microstresses (see Sect. 2):

$$s'_{11a} = -2s'_{22a} = -2s'_{33a} < 0, \quad s'_{11a} = -2s'_{22a} = -2s'_{33a} > 0.$$

The respective components in the  $b$ -material would follow from the Eqs. (1.2) and (1.9).

According to the yield criterion (1.16) and the Eqs. (3.2)<sub>1</sub> and (3.2)<sub>5</sub> the following relations hold true:

$$(3.4) \quad s'_{11a} = s_{11a} - \bar{s}_{11} = c - cY - c = -cY.$$

Now we suppose that after unloading the temperature of the specimen was changed to room temperature which we denote by  $T^*$ . This means that  $c$  and  $\mu$  changed to new values  $c^*$ ,  $\mu^*$ . The specimen is drawn anew in the same direction as before. New plastic

deformation will begin when the new plastic limit is reached in the  $a$ -material, which — according to (1.16) and (3.3)<sub>4</sub> — is:

$$(3.5) \quad s_{11a}^L = c^*.$$

Let us denote the respective new macroscopic plastic limit by  $\bar{s}_{11}^L$ . It holds:

$$(3.6) \quad s_{11a}^L = s_{11a}^r + \bar{s}_{11}^L$$

and inserting from (3.4) and (3.5) into (3.6) we obtain:

$$(3.7) \quad \bar{s}_{11}^L = c^* + cY^m,$$

where  $Y^m$  is the value of  $Y$  reached in the first process before unloading.

If the maximum fixed deformation  $[\bar{e}_{11}^{pm}]$  that had been reached in the first process was large enough, the first addend on the right-hand side of (3.1)<sub>1</sub> may be neglected and it holds approximately:

$$(3.8) \quad Y^m \doteq \frac{B}{C} X^m = \frac{B}{\mu c C} [\bar{e}_{11}^{pm}],$$

and inserting (3.8) into (3.7):

$$(3.9) \quad \bar{s}_{11}^L = c^* + \frac{B}{C} [\bar{e}_{11}^{pm}] \frac{1}{\mu}.$$

Let us further express the deviatoric stress  $\bar{s}_{11}^M$  that corresponds in the second process to some fixed plastic deformation  $[\bar{e}_{11}^{PM}]$ . Integrating again the Eqs. (1.2) to (1.4) with new initial values  $\bar{s}_{11}^L, s_{11a}^L = s_{11a}^r$  we arrive at the following description of the second process:

$$(3.10) \quad X^* = -C \ln \frac{1-v^*}{1-v^{*L}}, \quad Y^* = A(v^* - v^{*L}) - B \ln \frac{1-v^*}{1-v^{*L}},$$

with

$$(3.11) \quad X^* = \frac{\bar{e}_{11}^p}{\mu^* c^*}, \quad Y^* = \frac{\bar{s}_{11} - \bar{s}_{11}^L}{c^*}, \quad v^* = \frac{\vartheta_a + \eta_b}{\vartheta_b \eta_b} \cdot \frac{s_{11a}^r}{c^*}.$$

Again we suppose that the fixed deformation  $[\bar{e}_{11}^{PM}]$  is large enough so that the first addend on the right-hand side of (3.10)<sub>2</sub> may be neglected.

Then it holds approximately:

$$(3.12) \quad Y^{*M} \doteq \frac{B}{C} X^{*M} = \frac{B}{\mu^* c^* C} [\bar{e}_{11}^{PM}],$$

and inserting (3.12) and (3.9) into (3.11)<sub>2</sub> gives:

$$(3.13) \quad \bar{s}_{11}^M = c^* + \frac{B}{C} \frac{[\bar{e}_{11}^{PM}]}{\mu^*} + \frac{B}{C} \frac{[\bar{e}_{11}^{pm}]}{\mu}.$$

If the temperature of the first process is changed, nothing in both the expressions (3.9) and (3.13) will change but  $\mu$ .

Let us suppose the temperature of the first process to be in the range from  $-200^{\circ}\text{C}$  to  $+200^{\circ}\text{C}$ . In this range the change of the Young modulus  $E$  is approximately linear in most metals (cf. [12]) and the change of

$$\frac{1}{\mu} = \frac{E}{1+\nu} = 2G$$

is approximately linear as well (cf. [8, 9]).

Introducing the change of temperature from the first process to the second process

$$(3.14) \quad \Delta T = T^* - T,$$

and the shear modulus  $G$  instead of  $\mu$ , the relations (3.9) and (3.13) may be rewritten in another form:

$$(3.9)' \quad \bar{s}_{11}^I = c^* + 2\frac{B}{C} [e_{11}^{pm}] \left( G^* - \frac{\partial G}{\partial T} \Delta T \right),$$

$$(3.13)' \quad \bar{s}_{11}^M = c^* + 2\frac{B}{C} G^* [e_{11}^{pm}] + 2\frac{B}{C} [e_{11}^{pm}] \left( G^* - \frac{\partial G}{\partial T} \Delta T \right),$$

where  $\partial G/\partial T$  is negative and in the considered range of temperature constant.

What the above equations say is that in the second process both the deviatoric plastic limit and the deviatoric stress after a fixed plastic deformation display a linear decrease with the temperature increasing in the first process ( $T$  increasing means  $\Delta T$  decreasing). The same holds of course for the normal components  $\bar{\sigma}_{11}^I$  and  $\bar{\sigma}_{11}^M$  in regard to the independence of the first invariant  $\bar{\sigma}_{ii}$  on the temperature of the first process.

In the paper [15] (p. 218) the results of experiments are referred to where the dependence of  $\bar{\sigma}_{11}^M$  on the temperature of the first process was investigated and really a decrease with the temperature was found which was very closely linear.

It is important to note that, according to our analysis, the decrease is connected with the change of the elastic modulus  $G$  and independent of  $c$  in spite of the fact that the studied process is plastic. This theoretical conclusion seems to be corroborated by the quoted experiments. In the experiments, performed on copper specimens, the observed decrease was linear with the same slope in the whole range from  $-200^{\circ}\text{C}$  to  $+200^{\circ}\text{C}$ , the change of the elastic modulus is — according to [12] — in this range also linear to a good degree of approximation, but the change of the plastic limit of copper is — according to [8] — strongly non-linear, substantially differing in the interval  $-200^{\circ}\text{C}$  to  $0^{\circ}\text{C}$  from that in the interval  $0^{\circ}\text{C}$  to  $+200^{\circ}\text{C}$ .

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