Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 1, pp. 13–30, Warszawa 1976

On phenomenological description of rock-like materials with account for kinetics of brittle fracture

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A MODEL is proposed for a class of solids (rocks, concretes, ceramics) exhibiting physical non-linearity and stress path dependence as a consequence of structural rearrangement due to brittle cracking. The incremental stress-strain relations being presented are related to the change of internal parameter, which characterizes the intensity and geometry of crack field. The kinetics of cracking process is formulated analytically. The final failure criteria, depending on stress-history, may be derived using the introduced concepts. Some correlations between computed and experimental results are discussed.

W pracy przedstawiono model materiałów wykazujących m.in. efekty fizycznej nieliniowości i wrażliwości na historię obciążenia wskutek inicjacji i rozwoju kruchych spękań w trakcie procesu. Celem rozważań jest opisanie zachowania się materiałów skałopodobnych (skały, betony, ceramiki). W celu uchwycenia zależności między makroskopowymi efektami mechanicznymi a procesem pękania wprowadzono do związków fizycznych parametr wewnętrzny, charakteryzujący intensywność i geometrię pola spękań i szczelin. Zaproponowano analityczny opis kinetyki kruchych spękań; odpowiedni związek ma postać przyrostową podobnie jak równania konstytutywne. Opierając się na przedstawionej koncepcji można wyprowadzić warunki końcowego zniszczenia, będące funkcją drogi w przestrzeni naprężenia. Otrzymano wyniki wykazujące zgodność z danymi doświadczalnymi.

В работе представлена модель материалов, обладающих, между прочим, эффектами физической нелинейности и чувствительностью на историю нагрузки вследствие инициирования и развития хрупких трещин по ходу процесса. Целью рассуждений является описание поведения скалообразных материалов (скалы, бетоны, керамика). С целью выражения зависимости между макроскопическими механическими эффектами и процессом разрушения введен в физические соотношения внутренний параметр, характеризующий интенсивность и геометрию поля трещин и щелей. Предложено аналитическое описание кинетики хрупких трещин; соответствующее соотношение имеет вид в приростах, аналогично как определяющие уравнения. Опираясь на представленную концепцию, можно вывести условия остаточного разрушения, будучие функцией пути в пространстве напряжений. Получены результаты, показывающие совпадание с экспериментальными данными.

1. Introduction

THE purpose of this paper is to propose a simplified material model for predicting the time-independent stress-strain relations and failure of rock-like materials (i.e. rocks, concretes, mortars, ceramics) under general stress conditions.

The fundamental property of this class of materials is progressive fracturing which determines the type of behaviour observed in the process. Coupling between material response and fracturing takes place in a wide range of loads, also in the range of engineering applications. The study of failure is also essential for a more efficient description of crushing processes, which are widely employed in technology.

In spite of a growing interest in the study of mechanical behaviour of rocks and concretes, their behaviour and properties are still not fully understood, especially for complex stress states. In experiments, the physical non-linearity and volume changes are observed. Moreover, shear and volumetric deformations are coupled, similarly as in a familiar dilatancy effect, typical for soil behaviour. Generally, the material under consideration exhibits the stress- and strain path-dependent behaviour. Typical stress-strain curves are discussed in the next section together with brittle cracking mechanism and final fracture patterns found by some investigators. The relationship between the intensity and geometry of crack-field, from one side, and stress state from the other is subsequently discussed. This relation, determining the kinetics of the brittle cracking, is an important step for establishing the mechanical model, developed in Sect. 3.

In what follows, the macroscopic mechanical behaviour in any process is related to internal (structural) rearrangement of a material which is simulated by a change of internal parameter. It is the tensor parameter which characterizes the intensity and geometry of crack field in a body under consideration. Mathematically, the incremental stress-strain relations are connected with the change of parameter describing the fracturing process. The idea of such approach for brittle-plastic materials, like rocks and concretes, has been suggested by MRÓZ [1].

It is shown that, following this concept, one may obtain the failure criteria in the stress-space, which depend on the stress path. Some examples are given in Sect. 4. There is demonstrated qualitatively good correlation between predicted and experimental results. The model is admissible, when progressive brittle cracking and also brittle failure mechanism (both defined in Sect. 2) are taken into account. The latter takes place in rock-like materials, when moderate hydrostatic pressures exist.

Since the actual concern of this study is phenomenological description of fracturing of rocks and concrete, no mechanisms of damage on the microscale level are considered. For instance, we do not introduce the notion of microstresses, but use the continuous mechanics-concept of Cauchy-stress also when describing cracking mechanisms.

2. Material response and progressive crack-field growth

2.1. Mechanical behaviour of rock-like materials

Typical stress-strain curves for uniaxial and multiaxial compression are of the type shown in Fig. 1. They are decomposed into spherical and deviatoric components, plotted separately. A detailed consideration of the curves shows interesting features of the material response. The first is that material displays non-linear behaviour from the beginning of the structural rearrangement, due to crack-field growth, see also Fig. 2 [9]. The extent of domain of non-linearity depends on the type of loading applied. Since large hydrostatic pressure inhibits crack growth, the linear response is observed in the wide range of stresses in processes with high initial confining pressures applied (see, for example, the segment O–B of the curve e, Fig. 1a, b).

The character of dilatational response is quite complex and exhibits some important features. The volume decrease takes place in the first phase of loading when compressive loads are considered. This is due to the closure of initial voids in material. Afterwards, the relevant volume increasing is observed as a consequence of fracturing accompanied by an increase in porosity. In Fig. 1a is evident the deviation of volumetric curves for processes in which shearing occurs and those processes corresponding to hydrostatic loading only. It is the result of coupling of shear and volumetric deformations, called sometimes

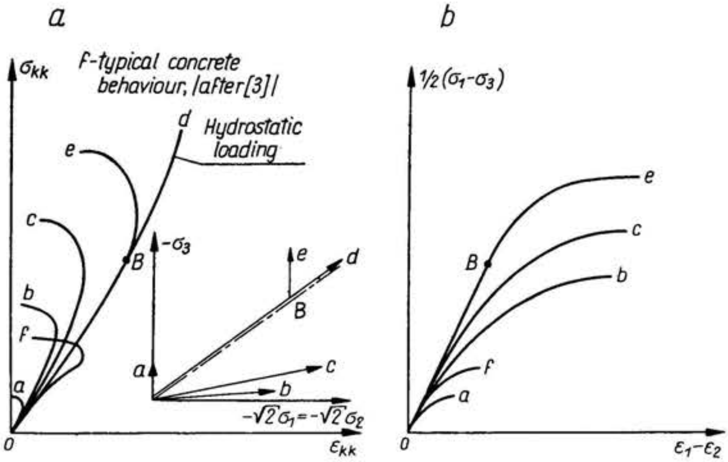

FIG. 1.

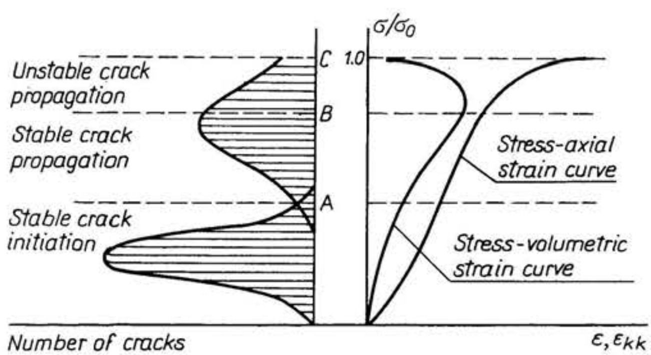

FIG. 2.

the dilatancy effect [2]. An extensive study of this phenomenon and its connection with progressive fracturing is given experimentally by BRACE and co-workers [4].

Examination of loading and unloading curves for rocks and concrete subjected to cyclic loading (see for example [5]) leads to the conclusion that elastic moduli change in the course of progressive fracture, see Fig. 3. It is the result of brittle effects developed on each cycle, and of the influence of residual (plastic) deformation which modifies elastic properties. Hence, both brittle-elastic and elastic-plastic coupling effects occur. The second effect exists rather in "ductile" rocks, like marble, sandstone. In crystalline rocks, like granite, elastic-brittle coupling prevails i.e. actual compressibility is affected by brittle rearrangement growing on each cycle. It may be expected that both effects exist together in most of cases.

The description of elastic-plastic coupling for granular media was discussed by HUECKEL and DRESCHER [6] and HUECKEL [7]. In the present theory, since the irreversible strains are neglected, the variability of material moduli induced by brittle effects is considered.

For specifying the parameter characterizing structural rearrangements, the phenomenon of brittle cracking must be treated analytically. Consequently, phenomenological rule for crack distribution and growth related to stress ought to be determined.

We are especially interested in the actual formation and growth of crack-field during deformation. We base our analysis on some available experimental works containing

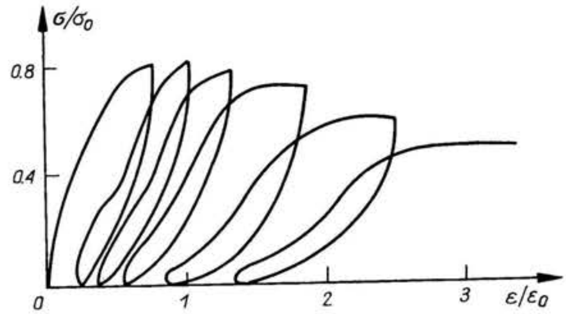

FIG. 3.

valuable results concerning distribution and growth of cracks in samples [8, 9]. Some conclusions may also be done from study of final fracture patterns [10].

Generally, it is possible to distinguish two types of cracking, which will be called the "shear" and "brittle" cracks.

The shear type shown schematically in Fig. 4a is the prevailing mode of final failure of rock specimens loaded triaxially. As reported by PENG and JOHNSON [8] the resulting

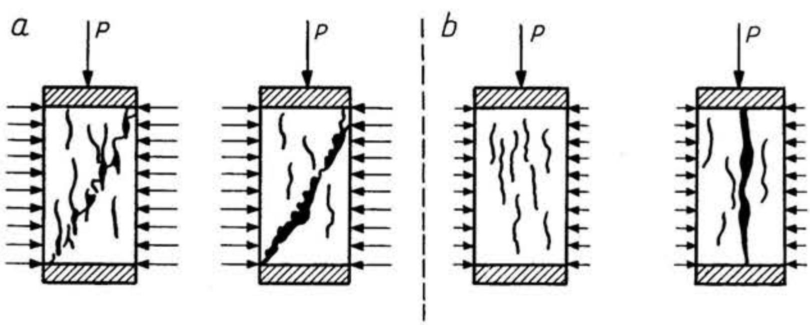

FIG. 4.

failure surface is the fault surface which consists of many steps arranged as in a staircase. This shape is the result of interaction of two mechanisms, i.e. (i) cracking propagating parallel to the direction of axial loading (this will be named brittle mechanism), and (ii) final faultings creating rough cone-in-cone or "shear" surface inclined to the axial directions. (The steplike character of the final-rupture surface is the most visible in crystalline rock specimens. In sedimentary rocks, like sandstone, it is not so distinct). The final pattern resembles that of Coulomb materials, like soils, if detailed shape of fault surfaces is neglected. It is observed that, before failure, most cracks are oriented roughly parallel to the long axes of the specimens, provided $\sigma_3 < \sigma_2 = \sigma_1$ (1) and σ_3 is the axial stress.

As noted previously, the brittle mode is represented by cracks parallel to the principal directions of the stress tensor. The appearances of cylindrical specimens are of the type shown in Fig. 4b. There are researchers suggesting that brittle mode of failure is the only true one in rocks and concretes, while the shear failure is a result of frictional restraint on contacts of the sample with the loading plates.

On the basis of experiments [9, 10, 11] it seems to be reasonable to assume that the mode of cracking in rocks and concretes is the brittle one for general stress states, although the final failure mechanism may be composed of shear fracture planes in specimen subjected to triaxial loading when high (compressive) hydrostatic pressures are applied. For

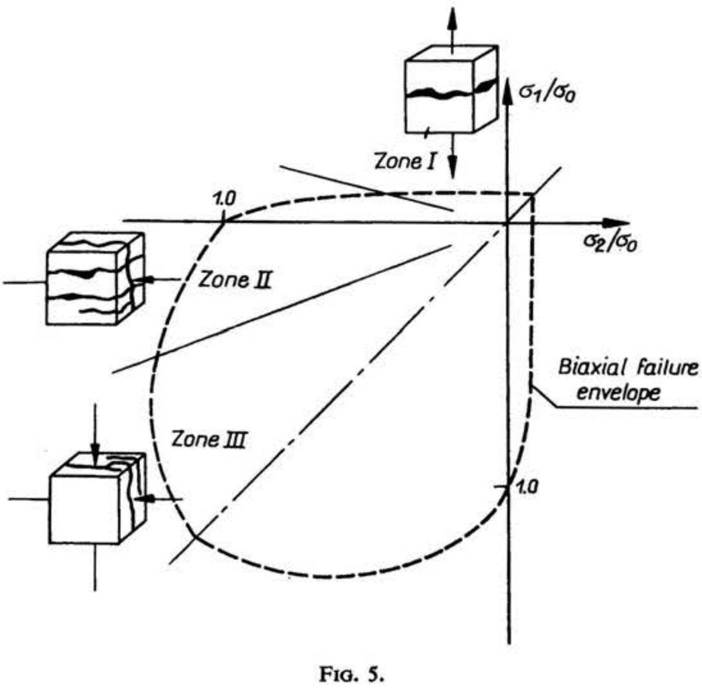

FIG. 5.

moderate pressures and in all remaining stress conditions (for instance, plane stress states) the brittle mode prevails during the process before failure and in the stage of formation of final failure surface. Fig. 5 illustrates typical patterns of brittle fracture in plane stress conditions, after [10].

2.2. Formulation of the kinetics of brittle cracking

Consider for simplicity a specimen subjected to the uniaxial compression. Physically, non-linear behaviour of specimen is observed when it becomes cracked on a macroscale.

(1) Tensile stresses and strains are assumed as positive.

Formation and growth of the axial cracks is observed, so that their directions are generally parallel to the axis of stress applied. Also, no relative sliding in the formed fissures exists, but progressive splitting, i.e. crack opening perpendicular to the crack direction together with increasing of crack length, accompanies stress increment. It is evident that the opening of cracks occurs in some peculiar principal directions of stress. In particular, the development of fracturing is related to the increments of the positive stress-deviator components:

$$s_1 = -\frac{\sigma}{3} > 0, \quad s_2 = -\frac{\sigma}{3} > 0 \quad \left(\text{while } s_3 = \frac{2}{3}\sigma \text{ is negative (compressive)}\right).$$

It is interesting to re-examine this scheme for biaxial and triaxial stress conditions on the basis of some patterns of cracks distribution from experiments like, for instance, these in Fig. 5. It can easily be checked that the same rule for crack growth and direction of crack opening, as described above, remains valid within the brittle mode for all stress-states.

As stated above, the geometry of crack-field developing in mechanical process is restricted by the actual stress-state. The phenomenon seems to be complicated, since brittle cracks open in actual principal stress-deviator directions of positive-valued components, while they do not open in remaining principal directions. This connection of brittle cracking with stress tensor suggests that the process can adequately be modelled using also tensorial internal parameter simulating fracturing. Moreover, the above discussion points at "switching" type of relationship between fracture and stress tensors, available to permit crack growth related only to positive-valued principal deviator components. Of course, there are stress-paths for which no cracks appear and consequently there is no increment of any fracture tensor component for such stresses.

To give more precise description of discussed phenomena, we utilize some concepts similar to those of plasticity. They will be adopted here for treating brittle cracking material.

First, it is assumed that actual stage of brittle fracturing can be characterized by the local tensor parameter, denoted ϕ_{ij}. We also postulate that increment of this parameter may be related to the stress and stress increment by the local relation for processes in which irreversible brittle effects develop. Reversible processes, like crack opening and subsequent closure and so on, are not taken into account. Hence, the tensor ϕ_{ij} characterizes rather permanent damage of initial material.

Secondly, we postulate the brittle-process-onset condition under combined stresses; the concept is analogous to the yield condition of plasticity. It is here the function of stress tensor being the condition permitting brittle response, i.e. the kinetics of brittle cracking together with its consequences like history dependence and non-linearity. Geometrically, we assume the existence of the surface in stress-space limiting the domain of linear elastic response. Since the compressive hydrostatic pressures inhibit the onset of cracking and, on the other hand, the departure from the hydrostatic axis in the principal stress-space accelerates it, we assume our stress condition to be the function of both the first stress and the second deviatoric invariants, i.e.,

$$f(\sigma_{ij}) = f(I_1, I_2') = 0, \tag{2.1}$$

where

$$I_1 = \sigma_{kk}, \quad I'_2 = -\frac{1}{2} s_{ij} s_{ij}.$$

In particular, the conical surface in the principal stress space may be proposed for simplicity. Hence the Eq. (2.1) may be written in the form

$$f(I_1, I'_2) = a_0 I_1 + c_0 \sqrt{-I'_2} = 0, \tag{2.2}$$

where both a_0 and c_0 are positive constants. As it was assummed previously, no irreversible brittle cracking occurs within the domain bounded by the surface (2.2).

Generally we assume that the kinetics of brittle fracture proceeds if the following inequality holds

$$f(I_1, I'_2) \geqslant 0. \tag{2.3}$$

The latter is necessary but not sufficient condition. On the basis of previous analysis we must define precisely the stress-paths on which brittle fracturing occurs (in a manner which is analogous to the loading conditions formulated in plasticity, which define the stress-paths on which plastic strain occurs). The consequence of considered crack-formation and growth rule is the following restriction:

The kinetics of brittle fracture occurs if the actual stress-state satisfies (2.3) *and the stress-path involves the positive increment of at least one, non-negative principal deviator component.*

Hence, we are now in a position to establish the analytical relationship of the fact that crack-opening occurs along the directions of the principal, positive-valued deviator components. This may be written as the coaxial, incremental rule for fracturing increment represented by the crack-tensor ϕ_{ij}. For principal directions of both crack and stress tensors we have:

$$(2.4)\quad \begin{array}{lllll} \text{(a)} & \left\{\begin{array}{l} d\phi_i = g(I_1, I'_2)ds_i \\ \text{and } d\phi_i = 0 \\ \end{array}\right. & \begin{array}{l}\text{if} \\ \\ \\ \end{array} & \begin{array}{l} f(I_1, I_2) \geqslant 0 \\ \\ \\ \end{array} & \begin{array}{l} \text{for those } i \text{ for which both } s_i \geqslant 0, \\ ds_i > 0, \\ \text{for remaining } i, \end{array} \\ \text{(b)} & \quad d\phi_i = 0 & \text{if} & f(I_1, I'_2) \geqslant 0 & \begin{array}{l}\text{for all } i, \text{ if there is not any } i \text{ for} \\ \text{which } s \geqslant 0,\ ds > 0, \end{array} \\ & & \text{or if} & f(I_1, I'_2) < 0 & \text{in any case,} \end{array}$$

where s_i is the principal component of stress deviator. In the Eq. $(2.4)_a$, $g(I_1, I_2)$ is a positive-valued function of its arguments. The coefficient $g(I_1, I'_2)$ involves the influence of actual stresses on fracturing, on particular stress-path. As the function $f(I_1, I_2)$ is the stress condition of the onset of brittle cracking, so $g(I_1, I_2)$ incorporates inhibiting or accelerating influence of the actual stress-state in the course of fracturing process being advanced.

In view of above assumptions, for restricted stress-paths, the ϕ_i increments (crack opening) are taking place in such principal stress directions, for which conditions $(2.4)_{a,1}$ are satisfied. For other principal directions we have $d\phi_i = 0$. This is just "switching" mechanism — *only selected principal stress directions are directions of crack opening.*

As an illustration of the rule (2.1) the uniaxial compression may be again considered. If the inequality $f(I_1, I_2') \geqslant 0$ is satisfied, we have cracking increments if compressive stress increases, since

$$s_1 = -\frac{\sigma}{3} > 0, \quad ds_1 > 0; \quad s_2 = -\frac{\sigma}{3} > 0, \quad ds_2 > 0; \quad s_3 = \frac{2}{3}\sigma, \ \sigma < 0, ds_3 < 0.$$

Thus $d\phi_1 = g(I_1, I_2')ds_1$, $d\phi_2 = g(I_1, I_2')ds_2$ and $d\phi_3 = 0$ because $s_3 < 0, ds_3 < 0$. This type of fracture pattern, previously considered, is illustrated in Fig. 5 (scheme II).

The Eqs. (2.4) must be discussed more fully now in connection with mathematical admissibility. We must prove that fracture tensor may be interpreted as a second-order tensor possessing three orthogonal, principal directions, etc..., and, consequently, that the formulae (2.4) are permissible. At this point we may refer to the paper by VAKULENKO and KACHANOV jr. [12], who discussed the concept of crack-tensor. They proposed the parameter which may be interpreted as the may density of crack-field, the single crack being characterized by tensor product $b_i n_j$ of crack-opening vector b_i and unit normal vector n_j orthogonal to initial discontinuity surface $S_{(k)}$ (crack surface before crack opening). This dyad is integrated over $S_{(k)}$; (index k means that k^{th} crack is analysed). Crack-opening vector is defined as one connecting every two points M', M'' being a unit point M before the crack formation. The formula of average crack-tensor density is then the following one

$$\bar{\phi}_{ij} = \frac{1}{V} \sum_k \int_{S_{(k)}} b_i^{(k)} n_j^{(k)} dS_{(k)}. \tag{2.5}$$

It is shown by KACHANOV jr. [13] that tensor $\bar{\phi}_{ij}$ so interpreted may be additively decomposed into slipping part and crack-opening part on the basis of appropriate decomposition of vector b_i into components $b_j^{(s)}$ perpendicular to n_i (so parallel to $S_{(k)}$) and $b_j^{(n)}$ parallel to n_i (so normal to discontinuity surface $S_{(k)}$).(²)

Since we do not consider slipping in crack, the second-crack-opening part may be recognized to simulate brittle fissuration, as defined by us. For this crack-opening part, it can be proved that: $\phi_{ij}^{(n)} = \phi_{ji}^{(n)}$.

Hence, three real principal components in three orthogonal directions can be found for $\phi_{ij}^{(n)}$. Further, we shall leave symbol (n) when writing ϕ_{ij}. It is seen now that the interpretation (2.5)′ may be ascribed to non-zero ϕ_{ij}-components determined by (2.4). The restriction that we deal with "normal" part only must be taken into account. It is, however, necessary to remark that ϕ_{ij} is the phenomenological parameter determined by the incremental relation (2.4). The interpretation (2.5)′ serves only as a mathematical concept making the derivation of (2.4) permissible. It is now more clear that $d\phi_{ij}$ calculated from the kinetics-relationship (2.4) characterizes both intensity and geometry of crack-growth, associated with principal stress directions, as pointed earlier.

Clearly, starting from principal directions one can obtain any arbitrary components

$$d\phi_{ij} = Q_{i1}Q_{j1}d\phi_1 + Q_{i2}Q_{j2}d\phi_2 + Q_{i3}Q_{j3}d\phi_3$$

(²) Here we use the notion of the local crack-density tensor associated with material points of continuous medium:

$$\phi_{ij} = b_i n_j|_x. \tag{2.5′}$$

Doing so, we replace in fact the real cracked body continuum on which the tensor field (2.5′) is defined.

for any proper orthogonal transformation defined by Q_{mn}, while the kinetics equation for $d\phi_i$ is given by (2.4) in principal axes. The latter can also be rewritten in arbitrary basis in the form:

$$d\phi_{ij} = g(I_1, I_2')d\bar{\sigma}_{ij} \tag{2.6}$$

for non-zero increments of ϕ_{ij}. In Eq. (2.6) $d\bar{\sigma}_{ij}$ is not the full stress-tensor increment, but only its part obtained from the transformation of artificial stress-tensor (i.e. deviator in which negative principal components are neglected) from the principal to arbitrary basis. In uniaxial compression it is, for instance

$$s = \begin{bmatrix} s_1 & 0 & 0 \\ \cdot & s_2 & 0 \\ \cdot & \cdot & s_3 \end{bmatrix}, \quad \bar{s} = \begin{bmatrix} s_1 & 0 & 0 \\ \cdot & s_2 & 0 \\ \cdot & \cdot & 0 \end{bmatrix}$$

and $\bar{\sigma}_{ij}$ is obtained from s_{ij} by transformation to arbitrary basis. Notice that $\bar{s}_{ii}$ is no longer of the zero value.

It is seen, however, that the physical sense of the form (2.4) is not so distinct in arbitrary basis representation (2.6).

2.3. Additional remarks. Progressive cracking in cyclic processes

It must be noted additionally that the kinetic relation is incomplete since it does not take into consideration the fact that in uniform triaxial tension we observe very speedy crack growth. Generally the fact that increment of I_1 accelerates the brittle fracture is closed in coefficient $g(I_1, I_2')$ for positive deviator-fracturing formula for compression and compression-tension stress-states. For purely tensile stresses where cracking is released and goes yet more quickly, the additional "releasing" term proportional to the increment of the first stress-invariant should be added in (2.4) (for $I_1 > 0$). Alternatively, in tension, principal deviator components in the formula (2.4) may be substituted by principal stress components. In this paper, such more complicated kinetics formulae are not discussed.

The important feature to be discussed is the variability of the domain of linear behaviour $f(I_1, I_2^l) < 0$; it is connected with the lack of fracture progress. We have assumed previously that specific geometries of cracking are related to particular stress-paths on the basis of experiments, see Fig. 5. There are no experiments for rock-like materials in which loading cycles were prescribed with different stress-paths on each cycle (what implies consequently different crack patterns). However, on the basis of experimental studies on the nature of fracture in rocks, concretes [5, 8, 9, 10], it seems that the hypothesis of "localized brittle hardening" may be proposed. It postulates localized change of initial elastic domain boundary $f(I_1, I_2') = 0$ in the loading process when brittle fracturing occurs. This is illustrated in Fig. 7. The current region of elastic linear response is bounded by the part of initial surface $f(I_1, I_2^l) = 0$ together with surfaces $s_i = \text{const} > 0$ traced by the stress vector in process on the actual cycle which crossed the initial surface. Consequently it means that unloading is linear; ϕ_{ij} remains constant on the level $\phi_{ij}^{(M)}$ (Fig. 7). But this "hardening" is not permanent. On the next loading cycle, new portion of cracks will develop immediately after the condition $f(I_1, I_2^l) = 0$ is satisfied, etc. If the second cycle stress-path is different from the first one, other deviator components may act and different crack-growth geometry is realized.

3. The form of constitutive equation

It is recalled from Sect. 2.2 that any stress-path produces no irreversible brittle rearrangements if stress-state satisfies the inequality:

$$f(I_1, I_2) < 0.$$

By assumptions made previously, the material is assumed to behave as linear-elastic in this case. The question of just how this behaviour is modified during continued loading when brittle cracking developes, is yet to be established mathematically. We have discussed in Sect. 2.1 how complex are the effects of brittle rearrangement of material. On the other hand, in order to formulate useful model of discussed phenomenon, we must preserve its essential physical features together with making necessary simplifications. One of the possible assumptions now is to neglect irreversible deformation and proceed the quasi-elastic approach also for the processes involving brittle effects. Since the progressive, permanent destruction is then observed due to ϕ_{ij}-increment, in dealing with such a procedure, the existence of variable potential function is postulated which depends on stress and, moreover, on the current crack-tensor value:

$$W = W(\sigma_{ij}, \phi_{ij}). \tag{3.1}$$

The consequences of this step are very important. As it was found, ϕ_{ij} is the function of the stress trajectory by setting the incremental relation of kinetics (2.4). Hence, the model postulated, although based on the quasi-elastic framework, takes into account the stress-history influence on the material response, owing to the internal parameter concept involved. Thus, having in mind the Eq. (2.4), we must write:

$$W = W[\sigma_{ij}, \phi_{ij}(\text{history of } \sigma_{kl})]. \tag{3.2}$$

In fact, some aspects of inelastic behaviour can be described if non-zero ϕ_{ij}-increment exists. In order to do it, the incremental form of strain-stress relation is assumed as follows:

$$\mathrm{d}\varepsilon_{ij} = \left.\frac{\partial^2 W[\sigma_{ij}, \phi_{ij}(\sigma_{ij})]}{\partial\sigma_{ij}\,\partial\sigma_{kl}}\right|_{(\Phi=\text{const})} d\sigma_{kl} + \frac{\partial^2 W[\sigma_{ij}, \phi_{ij}(\sigma_{ij})]}{\partial\sigma_{ij}\,\partial\phi_{kl}} d\phi_{kl}(\sigma_{ij}, d\sigma_{ij}), \tag{3.3}$$

where

$$d\phi_{kl} = g(I_1, I_2')\,d\bar{\sigma}_{kl}$$

as it was established previously.

Note, that the relation (3.3) is of the type $d\varepsilon_{ij} = C_{ijkl}(\sigma_{ij}, \phi_{ij})d\sigma_{kl}$ throughout the formula for ϕ_{ij}-increment. Using the latter we obtain

$$d\varepsilon_{ij} = \frac{\partial^2 W}{\partial\sigma_{ij}\,\partial\sigma_{kl}} d\sigma_{kl} + \frac{\partial^2 W}{\partial\sigma_{ij}\,\partial\phi_{kl}} g(I_1, I_2')\,d\bar{\sigma}_{kl}.$$

If the difference between the actual stress-increment $d\sigma_{ij}$ and the reduced stress-increment $d\bar{\sigma}_{kl}$ involving the kinetics equation is defined as

$$d\sigma^*_{kl} = d\sigma_{kl} - d\bar{\sigma}_{kl},$$

then (3.3) may be written in the form

$$d\varepsilon_{ij} = \left[\frac{\partial^2 W}{\partial\sigma_{ij}\,\partial\sigma_{kl}} + \frac{\partial^3 W}{\partial\sigma_{ij}\,\partial\phi_{kl}} g(I_1, I_2')\right] d\sigma_{kl} - \frac{\partial^2 W}{\partial\sigma_{ij}\,\partial\phi_{kl}} g(I_1, I_2')\,d\sigma^*_{kl}. \tag{3.4}$$

Generally, incorporating the crack-tensor into the potential W must lead to anisotropy of the relation (3.3) which is related to anisotropy of material response induced by brittle cracking. The specification of material coefficients may be done using the formula (3.3) or equivalent form of (3.4). For simple stress-paths, the constant principal directions of stress imply regular crack-patterns and known, clear types of anisotropy may appear. The problem is more involved for non-simple loading involving rotation of principal stress directions. According to the kinetics relationship, also the principal directions of ϕ_{ij} will

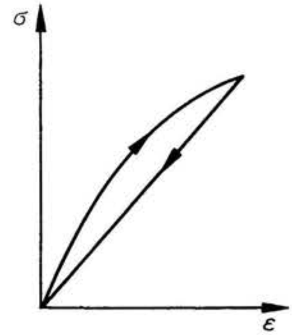

FIG. 6.

rotate and very complicated anisotropy may appear as a result of many increments $d\phi_{ij}$. It may be still modified by every next step of loading. Here we shall not consider anisotropy induced by incrementally defined fracturing of the material([3]). Furthermore, we consider very simplified form of (3.3), neglecting anisotropy but considering non-linearity together with simulation of volume changes and elastic-brittle coupling effects.

We assume that for any stress-path within the initial elastic domain only the first term of (3.3) is valid, since there is no crack-tensor increment. The same is for all stress-paths which — according to the loading restrictions of Sect. 2.2 — do not involve brittle cracking even for $f(I_1, I_2') \geqslant 0$. So, the tensor $\dfrac{\partial^2 W}{\partial\sigma_{ij}\partial\sigma_{kl}}$ is the tensor of actual moduli functions dependent on the current value of the crack tensor ϕ_{ij}. It may be restricted to be the tensor of actual secant material moduli and assumed to define the unloading (since the unloading stress-paths are also such ones with no brittle cracking on them). Under those assumptions we have a model with quasi-elastic-brittle coupling because unloading moduli change on every stage of process, depending on actual value of the tensor ϕ_{ij}, see Fig. 6 and 7.

Now, we postulate $W(\sigma_{ij}, \phi_{ij})$ to be apparently the quadratic homogeneous form in σ_{ij}, remembering that it is furthermore dependent on σ_{ij} through ϕ_{kl}. According our isotropy restriction, we proceed with the form

$$W = \alpha(\phi_{kl})I_1^2 + \beta(\phi_{kl})I_2'. \tag{3.5}$$

We now introduce the intensity of tensor ϕ_{ij} defined as follows:

$$c_{(i)}(\phi_{ij}) = [(\phi_{kk})^2 + \phi_{ij}\phi_{ij}]^{\frac{1}{2}}. \tag{3.6}$$

([3]) For total $\sigma - \varepsilon$ elastic material without crack-kinetics, some remarks on this subject are given by KACHANOV jr [13].

For some stage of brittle cracking, the final splitting on the macroscale takes place (the final failure). It is assumed that it occurs under constant intensity named c_{ult} independent of stress-state in compression. This assumption will be discussed in Sect. 4. We also set

$$\alpha(\phi_{kl}) = \alpha\left[\frac{\dot{c}(\phi_{kl})}{c_{ult}}\right], \quad \beta(\phi_{kl}) = \beta\left[\frac{c(\phi_{kl})}{c_{ult}}\right].$$

We now propose particular representation of W as the function of crack-tensor ϕ_{kl}, throughout specifying functions $\alpha(\phi_{kl})$, $\beta(\phi_{kl})$ related simply to variable moduli K_s (the

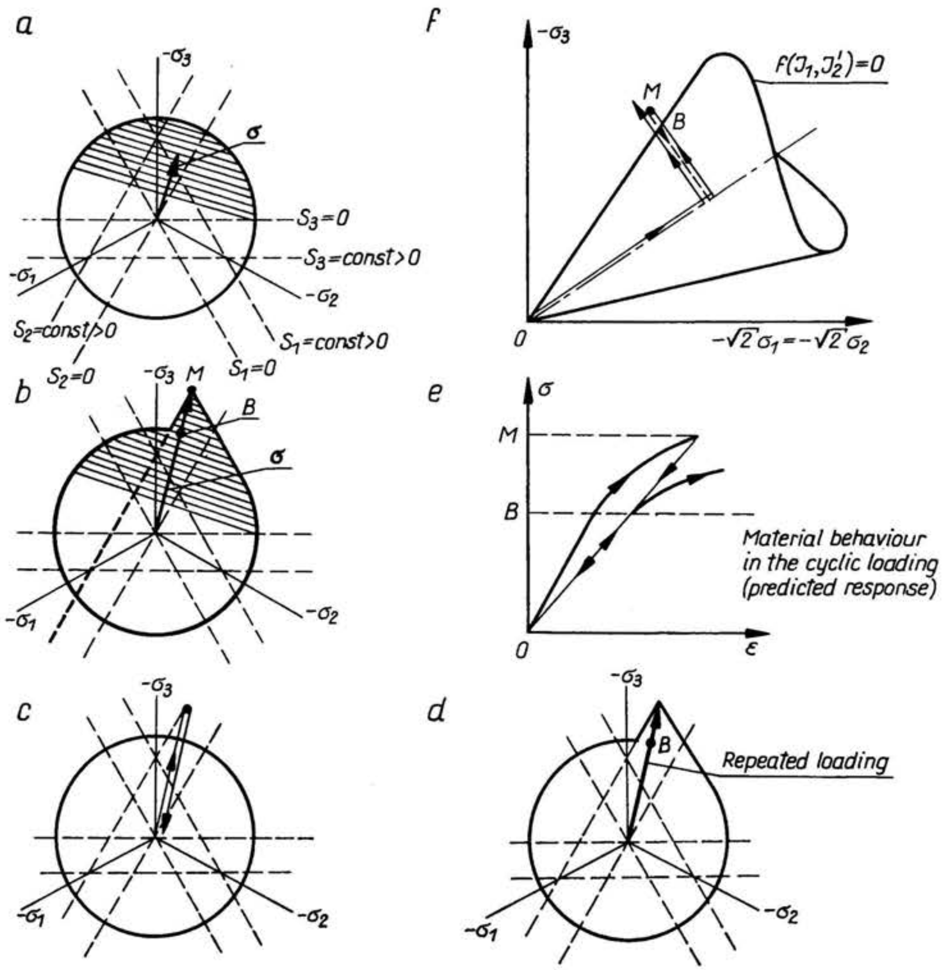

FIG. 7.

volumetric one) and G_s (the shear one). The form of functions proposed here is motivated mainly by particular non-linear behaviour of rock-like materials in simple load-paths. These functions simulate progressively developing the non-linearity due to cracking, as it was discussed in Sect. 2.1 on the basis of experiments.

Detailed forms of functions for isotropic behaviour are proposed as

(3.7)
$$\beta(\phi_{kl}) = -\frac{1}{2G_s(\phi_{ij})} = \frac{-1}{2G_0\left[1-b\cdot\left(\frac{c(\phi)}{c_{ult}}\right)^n\right]}, \quad 6\alpha[\phi_{kl}] = \frac{1}{3K_s[\phi_{ij}]} = \frac{1}{3K_0 h\left(\frac{c(\phi)}{c_{ult}}\right)^2},$$

or, alternatively

$$6\alpha(\phi_{kl}) = \frac{1}{3K_0\left\{h\cdot\frac{c(\phi)}{c_{ult}}\left[\frac{c(\phi)}{c_{ult}}-1\right]+1\right\}},$$

where b, n, h are material constants. G_0 and K_0 are initial moduli. The formula $(3.7)_3$ describes concrete behaviour where characteristic inflection of volumetric curves is observed (see Fig. 1, the curve f). Throughout $\alpha(\phi_{ij})$, $\beta(\phi_{ij})$ the formulae (3.7) describe the irreversible variation of material potential function $W(\sigma_{ij}, \phi_{ij})$ due to brittle cracking.

Now, after substituting (3.7) into (3.5), using (3.3) we obtain the particular form of constitutive equation; we decompose it into shear and volumetric parts. It becomes:

(3.8)
$$d\varepsilon_{kk} = \frac{1}{3K_s(\phi_{ij})}d\sigma_{kk} + \sigma_{kk}\frac{\frac{hK_0}{c_{ult}}\left(\frac{1}{c(\phi)}-\frac{2}{c_{ult}}\right)(\phi_{kk}\delta_{ij}+\phi_{ij})d\phi_{ij}}{3K_s^2(\phi_{ij})},$$
$$de_{ij} = \frac{1}{2G_s(\phi_{ij})}ds_{ij} - s_{ij}\frac{G_0 bn\frac{c(\phi)^{n-2}}{c_{ult}}(\phi_{kk}\delta_{ij}+\phi_{ij})d\phi_{ij}}{2G_s^2(\phi_{ij})},$$

while $d\phi_{ij}$ is determined by (2.4) or (2.6).

In (3.8) the $(3.7)_3$ form of $\alpha(\phi_{ij})$ is incorporated.

Consider now two triaxial programmes of compression illustrated in Fig. 8, and assume

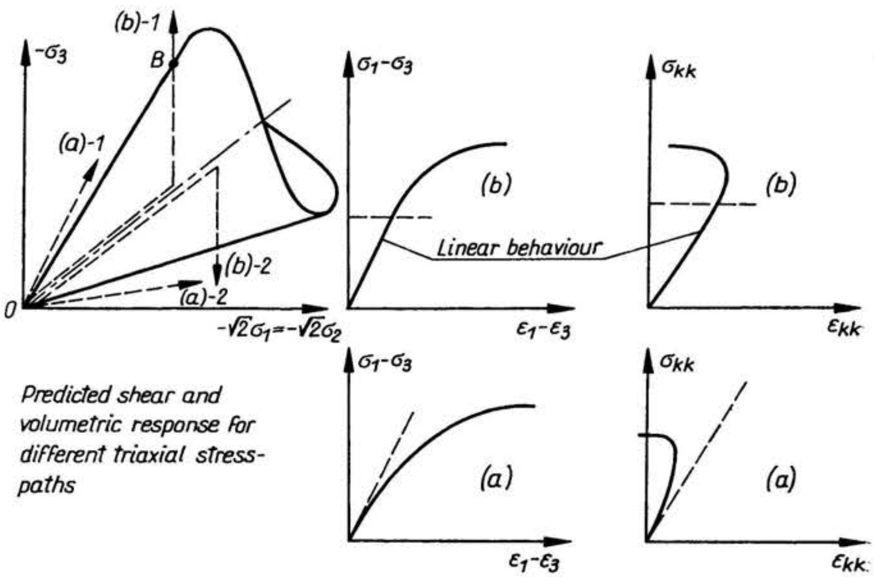

FIG. 8.

for simplicity that the surface of the brittle-crack-onset condition in the stress space is of the conical shape as given by the Eq. (2.2).

We conclude for the second programme (b) that the linear behaviour is predicted until point B on the surface (2.2) is reached. At this stage ϕ_{ij} remains equal to the initial value $\Phi_{ij}^{(0)}$ while formally the first term of (3.3) is governing the material behaviour, with the initial moduli K_0, G_0.

After passing B and setting $\sigma_1 = \sigma_2$, while $\sigma_3 < \sigma_1 = \sigma_2$, the process of brittle cracking proceeds with ϕ_1 and ϕ_2 active components of the crack tensor. There is $\phi_1 = \phi_2$ because $s_1 = s_2 > 0$ and positive ds_1 and ds_2 increments are applied. Thus, the non-linearity follows as a consequence of ϕ_{ij}-increments. Suitable curves (b) for shear and volumetric response are now of the form, as shown in Fig. 8(b).

Turning to the radial path (a) it is noted that now progressive cracking of the same geometry as in (b) with $\phi_1 = \phi_2$ non-zero components characterizes the process from the beginning. The full equation (3.3) is adopted on entire stress-path (not on its part as previously), together with the kinetic equation (2.4) and the curves are non-linear in the complete range of stresses appplied (Fig. 8(a)).

4. The failure envelopes dependent on the stress-history

The results developed so far include the possibility of deriving the failure envelopes dependent on the stress trajectory applied in particular process. Indeed, by integrating the kinetic equation (2.4) one can compute the actual value of the intensity $c_{(i)}(\phi_{ij})$ defined by the formula (3.6). By determining the critical intensity value c_{ult} for some reference test, for instance for uniaxial compression test, we obtain the measure of failure. It is calculated by integrating the Eq. (2.4); the value of c_{ult} is the function of maximum uniaxial stress. Having established the function $g(I_1, I_2)$ as the linear function of stresses:

$$g = a_0 I_1 + b_0 \tau_{\mathrm{oct}}, \quad \text{where } \tau_{\mathrm{oct}} = \left(-\frac{2}{3} I_2'\right)^{\frac{1}{2}}, \text{ and } a_0 = 1.5\left(\frac{\mathrm{cm}^2}{\mathrm{kG}}\right),$$

$$b_0 = 10\left(\frac{\mathrm{cm}^2}{\mathrm{kG}}\right),$$

we obtain for uniaxial compression

$$c_{\mathrm{ult}}^{(\mathrm{uniax.})} = 1.30\sigma_0^2, \tag{4.1}$$

where σ_0 is the maximum uniaxial compressive stress (uniaxial strength). We have assumed that intensity of crack-field developed, measured by the scalar $c(\phi_{ij})$ at the stage of final splitting is the same for different stress-states (and consequently for different crack-geometries). In other words, we assert that intensity (density) of cracks in elementary volume referred to the final failure is established in spite of different geometries in different stress-states. This seems to be true at least for compressive stress-states associated with "crushing"-brittle cracking.

Now, we can compare $c_{(i)}(\phi_{ij})$ for any stress trajectory to the reference value (4.1) and calculate the failure stress for this process related to uniaxial strength from the following equality:

$$c_{(i)\mathrm{ult}}(\phi_{ij}) = c_{\mathrm{ult}}^{(uniax.)}. \tag{4.2}$$

The possibility of such prediction of failure level of stress, without no *a priori* postulating, seems to be important. It has been reported by some investigators that the final fracture surface, in general, may not be fixed in the stress-space [14]. In fact, it is dependent on the stress-history for rock-like materials.

The failure envelope in the principal stress-space for radial paths in plane states is plotted in Fig. 10. Similarly, the curves for triaxial compression, when the lateral principal stresses are equal, are shown in Fig. 9.

The results presented are in good agreement with those found experimentally. It is shown in Fig. 9 where the domains of scattering of different experimental data, taken from [9, 14, 15, 16], are plotted (limited by dotted lines). The resulting theoretical curves

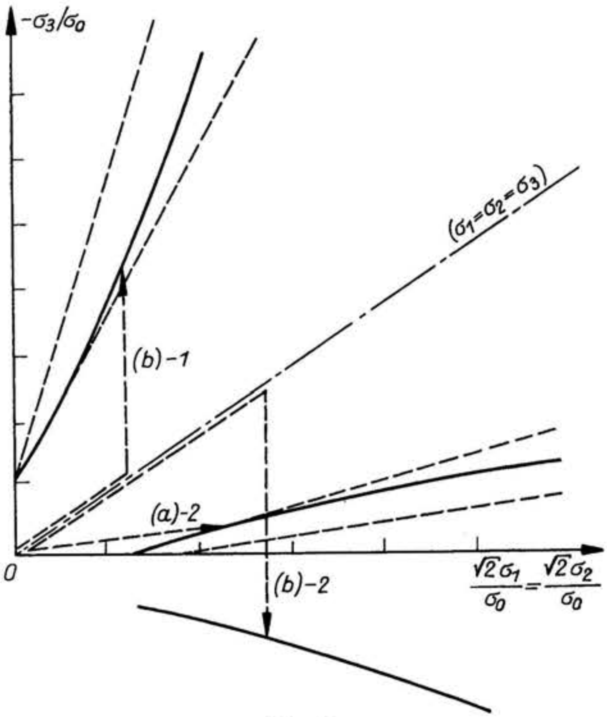

FIG. 9.

for triaxial conditions are hyperbolas in the principal stress-space. The equations of envelopes for different programmes are given below:

Programme (b-1); the non-zero valued components of ϕ_{ij} are (in principal directions) $\phi_1, \phi_2; \phi_1 = \phi_2;\ c_{\text{ult}} = P\sigma_0^2;\ (P = 1.30)$

$$k^2\left(\frac{\sqrt{2}}{3}\frac{b_0}{2}-\frac{a_0}{2}\right)-nk\left(2a_0+\frac{\sqrt{2b_0}}{3}\right)+n^2\left(\frac{5}{2}a_0+\frac{\sqrt{2}}{3}\frac{b_0}{2}\right)-\frac{3P}{\sqrt{6}}=0,$$

where

$$k=\frac{\sigma_3}{\sigma_0},\quad n=\frac{\sigma_1}{\sigma_0}=\frac{\sigma_2}{\sigma_0},\ \text{moreover, in the formula (2.2)}\ c_0=b_0\sqrt{\frac{2}{3}}.$$

Programme (*b*-2); $\phi_3 \neq 0$

$$k^2\left(\frac{1}{2}a_0+\frac{\sqrt{2}}{6}b_0\right)+nk\left(2a_0-\frac{\sqrt{2}}{3}b_0\right)+bn^2\left(\frac{\sqrt{2}}{3}\frac{b_0}{2}-\frac{5}{2}a_0\right)-\frac{3p}{2\sqrt{2}}=0.$$

Programme (*a*-2); also $\phi_3 \neq 0$ (radial path)

$$(s\cdot n)^2\left(a_0+\frac{\sqrt{2}}{3}b_0\right)+n\cdot(5\cdot n)\left(a_0-\frac{2\sqrt{2}}{3}b_0\right)-n^2\left(2a_0-\frac{\sqrt{2}}{3}b_0\right)-\frac{3P}{\sqrt{2}}=0,$$

where

$$\frac{\sigma_3}{\sigma_0}=\frac{s\sigma_1}{\sigma_0}=\frac{s\sigma_2}{\sigma_0}=s\cdot n;\qquad n=\frac{\sigma_1}{\sigma_0}=\frac{\sigma_2}{\sigma_0}.$$

The invariant δ (determinant of coefficients of second-order terms of above equations) is the same for all curves:

$$\delta=-\frac{9}{4}a_0^2<0.$$

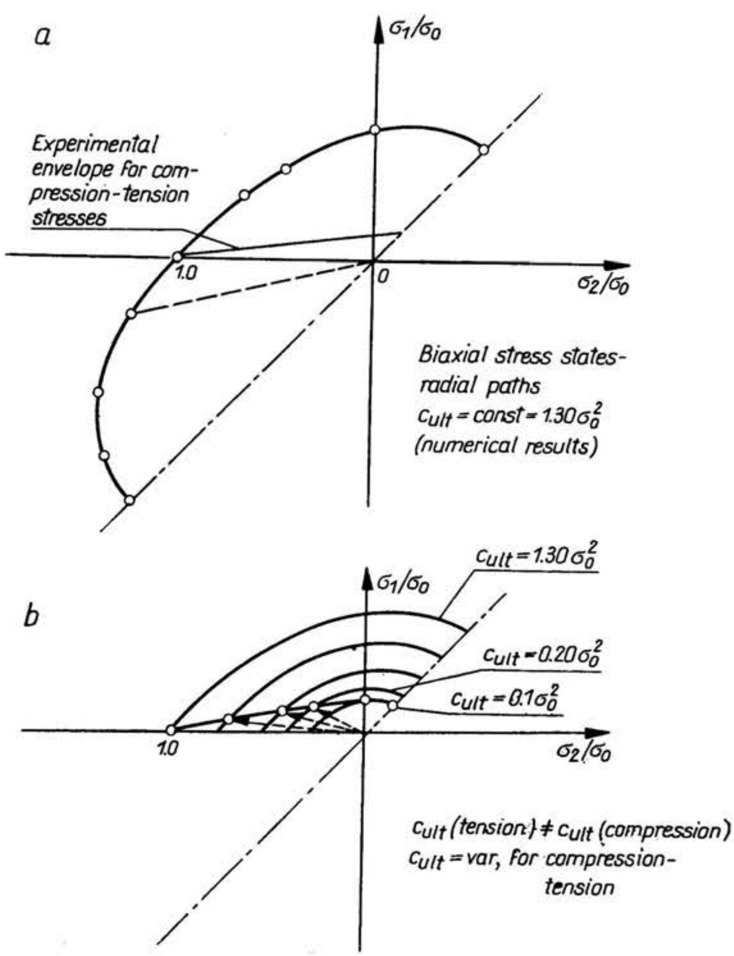

FIG. 10.

Discussion of results for stress-path denoted by (*b*-2) is difficult because of lack of many tests carried out for this regime. It may be possible that differences between the form of envelopes (*a*-2) and (*b*-2) are too great, but they are qualitatively correct.

In the plane stress, the experimental data [4, 10] confirm theoretically computed envelope in compressive domain. The significant discrepancy between observed and predicted envelope in tension and compression-tension takes place because of the simplification made by setting c_{ult} as constant value, independent of stress state. This is true in compression. Under tensile stresses, however, only few cracks develop. Sometimes in specimens under tension the single crack quickly grows until final failure. On the contrary, in compression the crack field is more "dense" until failure occurs, within the same brittle mechanism defined in Sect. 2.2. The direction of cracks in tension strongly decreases the working cross-section of specimen and consequently no more cracks growth can occur before failure. Taking it into account, c_{ult} variable in tensile and compressive-tensile zone was assumed (its value growing from tension to compression). Doing so, good agreement with experiments, also in transient states of compression-tension is obtained. It is shown schematically in Fig. 10b.

5. Concluding remarks

Here our concern has been exclusively with theoretical model of physically non-linear behaviour of rock-like materials. Macroscopic response is treated within the framework developed, so that the process of progressive fracturing is incorporated.

The approach presented may be regarded as a convenient representation of non-linear behaviour when complex volumetric deformations and elastic-brittle coupling effects are observed. The incremental stress-strain relationship, traced to the change of internal cracking parameter, is introduced. Such description exhibits important advantage, namely, the theory is history-dependent, valid for any loading path including failure. The failure criteria depend on stress-history. They are not *a priori* postulated, but derived using the kinetic equation for brittle cracking and the concept of the critical intensity of cracking parameter. The paper presents some applications of the model for moderate hydrostatic pressures.

Acknowledgement

The author is greatly indebted to Professor Z. Mróz for his inspiration and helpful criticism in the preparation of this paper.

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Received November 10, 1974.