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Interaction of cracks and inclusions in elastic media

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INFLUENCE of the elastic inclusions (nonhomogeneities) on the behaviour of the Griffith cracks in an infinite medium is considered. For this purpose we use the notion of the interaction force of external field on a defect, introduced in paper [1] and applied for a case of crack in paper [2]. The phenomenon of attraction of the crack by the holes and repulsion of the cracks by rigid inclusions is established. The considerations are restricted to the cases of the circular inclusions (holes) and to the antiplane state of strain in a medium.

1. Introduction

THE aim of this paper is to analyze the influence of inclusions (nonhomogeneities) on the behaviour of Griffith cracks located in an infinite elastic medium. The point of departure of the considerations is the notion of a force exerted by external field on a defect as defined in [1], and its relation to the stress intensity factors exposed in [2]. For the sake of simplicity, considerations of the present paper are confined to the simplest case of antiplane state of strain in an infinite medium subject to uniform shear and containing a single Griffith crack and a cylindrical inclusion with the axis coplanar with the crack. Physical interpretation of the force of interaction given in Sect. 4 proves to remain also true in the case of a plane state of stress discussed in [3] and presented in Sect. 5. Certain observations are made as regards the force of interaction in the case when the inclusion is located above the plane of the Griffith crack (Sect. 6).

2. Formulation of the problem

Interaction of a Griffith crack with an arbitrary external field of loading was considered in [2]. Let us apply the results of that paper to the case in which the original stress field is disturbed by a cylindrical elastic inclusion. The inclusion itself might be viewed as a special kind of defect, and its force of interaction with a crack could be evaluated according to the procedure outlined in [1]. However, it proves to be more effective to apply a different approach according to which the inclusion in a stressed medium is replaced with a suitably selected system of generalized forces depending on external loading of the system. This makes it possible to treat the inclusion as an additional external load acting on the body, and to determine the force exerted by that inclusion on the crack according to the formulae derived in [2].

The problem is stated as follows: infinite elastic medium, characterized by the shear modulus μ and Poisson's ratio ν, is loaded at infinity by uniformly distributed shearing forces $\sigma_{23} = \tau$ and contains a Griffith crack of constant width $2a$ (extending from $-\infty$ to $+\infty$ in the x_3-direction), and a cylindrical inclusion of diameter $2b$; its axis is parallel to the x_3-axis and lies in the $x_1 x_3$-plane of the coordinate system (Fig. 1). Elastic properties of the inclusion are characterized by the shear modulus $\mu_{(i)}$ (cf. also [4]).

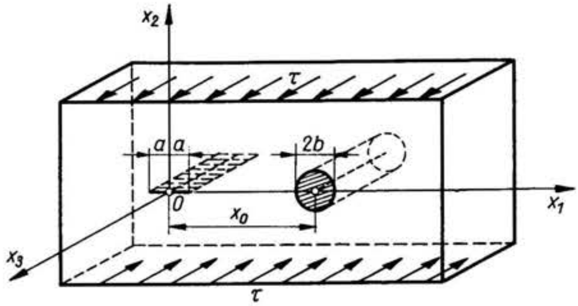

FIG. 1.

According to the formalism introduced in [2], the three fundamental modes of deformation of the medium containing a crack are denoted by $M_j, j = 1, 2, 3$; the present case (antiplane strain) corresponds to Mode III crack deformation and $j = 3$, while $j = 1$ denotes the Mode II crack deformation (sliding mode), and $j = 2$ — the Mode I crack deformation (opening mode).

The force exerted by external field on the crack (under static conditions) is given by the general formula (cf. [1, 2])

$$F_k = \int_{-a}^{a} \frac{\partial \sigma_{2j}^{(0)}(x_1, x_2)}{\partial x_k}\bigg|_{x_2=0} U_j^{(1)}(x_1)\,dx_1, \quad k = 1, 2, \quad j = 1, 2, 3. \tag{2.1}$$

Here $\sigma_{2j}^{(0)}(x_1, x_2)$ are the stresses produced by external loads in the medium without a crack, and $U_j^{(1)}(x_1)$ — components of the displacement u_j jump vector at the crack due to external loading.

The transversal ($k = 2$) component of the force F_k vanishes in this particular case due to the symmetry with respect to the $x_1 x_3$-plane; the longitudinal (horizontal) component F_1 may be written, according to [2], in the general form

$$F_1 = \frac{2\pi}{\mu} c_j a d_j e_j \tag{2.2}$$

with the following notations (cf. [2])

$$\begin{aligned} e_j &= \frac{1}{\pi} \int_{-a}^{a} \frac{h_j(t)\,dt}{\sqrt{a^2-t^2}}, \quad d_j = \frac{1}{\pi a} \int_{-a}^{a} \frac{t h_j(t)\,dt}{\sqrt{a^2-t^2}}, \\ h_j(x_1) &= \sigma_{2j}^{(0)}(x_1, 0), \\ c_1 &= c_2 = \begin{cases} 1-\nu \text{ (plane strain)}, \\ \dfrac{1}{1+\nu} \text{ (plane stress)}, \end{cases} \\ c_3 &= 1 \text{ (antiplane strain)}. \end{aligned} \tag{2.3}$$

The force F_1 may also be expressed in the alternative form

(2.4)
$$F_1 = \frac{\pi}{2\mu} c_j[(K_j^R)^2-(K_j^L)^2],$$

in which K_j^R, K_j^L are the respective right- and left-hand side stress intensity factors at the crack tips $x_1 = \pm a$, $x_2 = 0$.

3. Generalized force systems equivalent to the inclusion

The action of a cylindrical inclusion (centered in I) in a stressed medium may be shown to be formally equivalent (cf. [4, 5]) to a system of generalized (double or multiple) forces applied to the homogeneous medium at the point I; the force intensities depend on the external load acting on the body (Fig. 2). In the particular case of a circular cylindrical

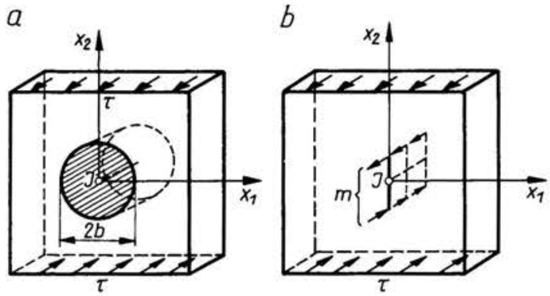

FIG. 2.

inclusion and antiplane, uniform state of strain, the double force distribution shown in Fig. 2b yields exactly the same stresses and strains (outside the shaded region) as those resulting from the original problem presented in Fig. 2a provided the double force intensity m is equal to

(3.1)
$$m = 2\pi\tau\alpha b^2.$$

Here

(3.2)
$$\alpha = \frac{\mu-\mu_{(i)}}{\mu+\mu_{(i)}}.$$

The value $\alpha = 1$ corresponds to the case of a cylindrical cavity ("perfectly deformable" or "soft" inclusion), and $\alpha = -1$ — to a perfectly rigid inclusion: $\mu_{(i)} = 0$ and $\mu_{(i)} \to \infty$, respectively.

In the case of non-uniform original stress distributions or non-circular inclusions, equivalence of the two systems becomes approximate (in the asymptotic sense, at large distance from the inclusion), unless certain higher order multiple force systems are added. A similar situation is encountered in the cases of M_1 or M_2 deformation modes (plane stress or strain states). In spite of that the generalized force intensities remain proportional to the cross-sectional area (volume) of the inclusion and to the external loads applied to the body.

A similar procedure is known from electrostatics [6] in discussing the behaviour of dielectrics in electric fields.

4. Evaluation of the force of interaction

Let us consider the M_3-deformation case of the medium containing a crack and inclusion (Fig. 1) under the assumption that the inclusion is located far enough from the right-hand crack tip, i.e.,

$$x_0 \gg a, \quad x_0 \gg b. \tag{4.1}$$

This assumption makes it possible to avoid higher order terms in the evaluation of the force of interaction between the crack and the inclusion.

In order to determine the contribution $F_1^{(i)}$ of the inclusion to the force F_1 acting on the crack, the whole procedure must be split into two steps. First of all we calculate the force $F_1^{(\tau)}$ which would act on the crack in the medium loaded by τ if the inclusion were absent. Force $F_1^{(i)}$ is then expressed as the difference of F_1 and $F_1^{(\tau)}$,

$$F_1^{(i)} = F_1 - F_1^{(\tau)},$$

and F_1 must be calculated from Eqs. (2.2) or (2.4) under the assumption of simultaneous action of external loads and the inclusion. However, the solution of the first problem yields $F_1^{(\tau)} = 0$ since, due to the symmetry of the system considered, $K_\text{I}^R = K_\text{I}^L$.

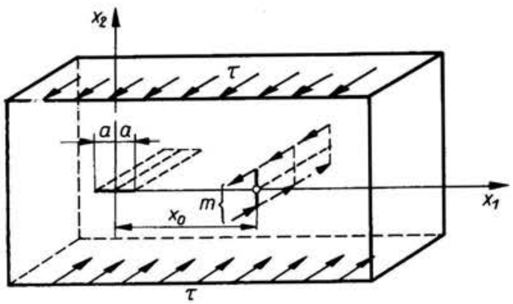

FIG. 3.

The second problem to be solved now is shown in Fig. 3 in which the inclusion is represented by the double force of intensity m calculated from Eq. (3.1). Owing to the former assumptions (4.1), the secondary influence of the crack on the original stress value $\sigma_{23} = \tau$ at the inclusion is disregarded.

The force F_1 is evaluated from the formulae (2.1)–(2.4) in which we assume $j = 3$ (antiplane state of strain). The stresses $\sigma_{23}^{(0)}$ and $\sigma_{13}^{(0)}$ on the x_1-axis are determined from the formulae ([4])

$$\sigma_{23}^{(0)}(x_1, 0) = \tau\left[1 + \frac{\alpha b^2}{(x_0 - x_1)^2}\right], \quad \sigma_{13}^{(0)}(x_1, 0) = 0. \tag{4.2}$$

Using the notation (2.3) we have $\sigma_{23}^{(0)}(x_1, 0) = h_3(x_1)$ and so the coefficients d_3, e_3 in Eqs. (2.3) are expressed in the form

$$d_3 = \frac{\tau}{\pi a}\int_{-a}^{a} \frac{t[1 + \alpha b^2/(x_0 - t)^2]dt}{\sqrt{a^2 - t^2}}, \quad e_3 = \frac{\tau}{\pi}\int_{-a}^{a} \frac{[1 + \alpha b^2/(x_0 - t)^2]dt}{\sqrt{a^2 - t^2}}. \tag{4.3}$$

For our purposes it proves sufficient to simplify the evaluation of integrals (4.3) by expanding the function $(x_0-t)^{-2}$ into a MacLaurin series

$$\frac{1}{(x_0-t)^2} = \frac{1}{x^2}\left(1+\frac{2t}{x_0}+\ldots\right)$$

and retaining the first two terms of that expansion; such procedure is justified in view of the assumption $x_0 \gg a$. This leads to the approximate results

$$d_3 \approx \frac{\pi\alpha a b^2}{x_0^3}, \quad e_3 \approx \tau\left(1+\frac{\alpha b^2}{x_0^2}\right)$$

and, due to the inequalities $x_0 \gg b$ and $|\alpha| \leqslant 1$, we obtain

$$F_1^{(i)} \approx \frac{2a^2\tau^2}{\mu}\alpha\pi b^2 \frac{1}{x_0^3}. \tag{4.4}$$

The following conclusions may be drawn from the above result:

(i) In the case of a cylindrical *cavity* ($\alpha = 1$) the force exerted by it on the crack is positive, $F_1 = F_i^1 > 0$, and may be interpreted as the force of *attraction*;

(ii) In the case of a perfectly *rigid* inclusion ($\alpha = -1$) the force becomes negative, $F_1 < 0$, and may be interpreted as the force of *repulsion*.

The both conclusions seem to have an intuitively obvious physical justification; e.g., by introducing a cavity in the neighbourhood of a crack tip, certain additional forces are directed towards the crack tip; the corresponding stress intensity factor must increase what increases (according to the Griffith fracture criterion) the tendency of the crack to be propagated in the direction of the cavity, i.e., in the positive direction of x_1 in the case considered. A rigid inclusion would create an opposite tendency (crack arrest).

(iii) The force of interaction F_1 is proportional to the cross-sectional area of the inclusion [the term πb^2 in Eq. (4.4)]. This conclusion is confirmed by the analysis of other, non-circular inclusions and by the analogy with the electric polarization phenomena mentioned before [6].

(iv) For sufficiently large distances x_0 the force of interaction is inversely proportional to the third power of the distance between the crack and inclusion.

5. Other deformation modes

The principal conclusion of the present paper concerning the interpretation of forces exerted by "soft" and "rigid" inclusions on a crack as reflecting the tendency of "attraction" or "repulsion" may — at least quantitatively — be confirmed by simple inspection of the formula (2.4) also in the cases involving plane states of strain or stress. It is sufficient to observe the effect of inclusions on the stress distribution at the crack tips, i.e. the changes of the stress intensity factors K_j^R, K_j^L reflecting the perturbations of the original stress distribution caused by inclusions. The results derived in papers [3, 4] which deal with similar bodies subject to different deformation modes ($j = 2$ and $j = 3$) lead to similar results. In both cases the variation of the stress intensity factors may be expressed by the formulae

$$K_j^R = K_j(1+\Delta_j^R), \quad K_j^L = K_j(1+\Delta_j^L) \tag{5.1}$$

in which K_j is the S.I.F. when there is no inclusion (symmetry),

$\Delta_j^R > \Delta_j^L > 0$ in the case of a cavity, and

$\Delta_j^R < \Delta_j^L < 0$ in the case of a rigid inclusion.

For instance, if $x_0/a = 1.2$, $x_0/g = 12$, we have $|\Delta_j^R| < 0.2$ and $|\Delta_j^L| < 0.05$.

From the fundamental formula (2.4) we then obtain by means of Eqs. (5.1) the result

$$F_1^{(i)} = \frac{\pi}{2\mu} c_j(K_j^R+K_j^L)(K_j^R-K_j^L) = \frac{\pi}{2\mu} c_j(K_j)^2(2+\Delta_j^R+\Delta_j^L)(\Delta_j^R-\Delta_j^L) \tag{5.2}$$

which yields the final conclusion that $F_1^{(i)} > 0$ in the case of a cavity, and $F_1^{(i)} < 0$ in the case of a rigid inclusion. Moreover, by substituting the stress intensity factors calculated in [4]

$$K_3 = \tau\sqrt{a}, \quad \Delta_3^R \approx \frac{\alpha b^2 x_0}{(x_0^2-a^2)(x_0-a)}, \quad \Delta_3^L \approx \frac{\alpha b^2 x_0}{(x_0^2-a^2)(x_0+a)}$$

into the formula (2.4), we may directly obtain the result (4.4).

6. Remarks on the transversal component of the interaction force

The physical situation becomes more complicated in the cases in which the stress distribution ceases to be symmetric with respect to the plane of the crack. This is, e.g., the case when inclusions are located not on the x_1-axis. As it was already mentioned in [2], the derivative $\partial\sigma_{2j}^{(0)}/\partial x_2$ is then different from zero and a transversal component F_2 of the force of interaction may appear. Its physical interpretation proves to be not so simple as that of the longitudinal component F_1. Nevertheless, by using an approximate procedure similar to that shown in Sect. 4 of the present paper, the corresponding forces exerted by arbitrarily located circular inclusions have also been calculated (again under the assumption of sufficiently large distances between the crack and inclusion). The results are illustrated by the schematic diagram shown in Fig. 4. Different positions of the inslusion (marked

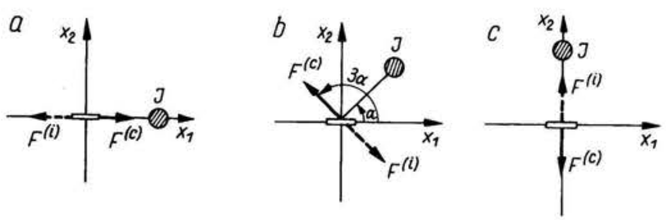

FIG. 4.

by small circles I) lead to different directions of the resulting force $\mathbf{F}$ (marked by arrows) acting on the crack. Solid arrows correspond to the case of a soft inclusion (cavity), dashed arrows — to rigid inclusions. Fig. 4a illustrates the situation discussed in the present paper.

Physical interpretation of the result which states that, in the case of a cavity, the angle of inclination of $\mathbf{F}$ to the x_1-axis is three times the angle IOx_1, has not been established.

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