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On the uniqueness of solutions of the stress equations of elastostatics

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As it is known, the system of differential stress equations of elastostatics consists of the set of equilibrium equations and the set of compatibility equations. We give the proof of uniqueness of solution of this system, being a modification of classical Kirchhoff proof.

In elastostatics it is assumed that a set of stress equilibrium equations together with the set of stress equations of deformation compatibility, the so-called Beltrami-Michell equations, with external forces given on the boundary of the medium, allows to determine uniquely the field of stress [1–4]. In so far as we know, however, no proof of the uniqueness of the solutions of these equations has been given hitherto. We are doing this below for the two and three-dimensional problems. Our proof is a modification of the classical Kirchhoff proof: an additional step is the derivation of dependences from the stress equations which are essentially relationships between the deformations and the displacements (although we do not employ these conceptions in the proof). We shall consider a medium of the linear theory of elasticity with Poisson's ratio ν, in a rectangular straight line system of coordinates x_1, x_2, x_3.

1. The two-dimensional problem

Let us consider the problem of planar state of stress. The procedure in the case of a planar state of deformation is similar. Let the elastic medium have the shape of a not necessarily regular cylinder, with the generating line parallel to the x_3-axis. The cylinder cross-section by the $x_3 =$ const plane will be designated by Ω, and the cross-section contour by Γ. The versor of the normal external to the contour Γ will be denoted by $n_\alpha = n_\alpha(x_\gamma)$, $\alpha, \gamma = 1, 2$, $x_\gamma \in \Gamma$, and the given vector of the field of forces applied from the exterior to the contour by $q_\alpha = q_\alpha(x_\gamma)$, $x_\gamma \in \Gamma$. Thus we assume that q_α does not depend on x_3 and on the time. Moreover, we accept that body forces act on the medium, described by the vectorial field $X_\alpha = X_\alpha(x_\gamma)$, $x_\gamma \in \Omega$. Then, if the end faces of the cylinder are without load, and the height of the cylinder is not large, it can be assumed that in the medium prevails a state of plane stress

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

independent of x_3. Thus

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x_\gamma), \quad \alpha, \beta, \gamma = 1, 2.$$

THEOREM 1. *The set of differential equations*

$$\sigma_{\alpha\beta,\beta}(x_\gamma)+X_\alpha(x_\gamma) = 0, \tag{1.1}$$

$$\sigma_{\alpha\alpha,\beta\beta}(x_\gamma)+(1+\nu)X_{\alpha,\alpha}(x_\gamma) = 0, \quad x_\gamma \in \Omega \tag{1.2}$$

with the condition

$$\sigma_{\alpha\beta}(x_\gamma)n_\beta(x_\varepsilon) = q_\alpha(x_\gamma), \quad x_\gamma \in \Gamma \tag{1.3}$$

defines uniquely the field

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x_\gamma), \quad x_\gamma \in \Omega. \tag{1.4}$$

Proof. The proof will be carried out if we show that the homogeneous set of equations

$$\sigma_{\alpha\beta,\beta}(x_\gamma) = 0, \quad x_\gamma \in \Omega, \tag{1.1'}$$

$$\sigma_{\alpha\alpha,\beta\beta}(x_\gamma) = 0, \quad x_\gamma \in \Omega, \tag{1.2'}$$

$$\sigma_{\alpha\beta}(x_\gamma)n_\beta(x_\varepsilon) = 0, \quad x_\gamma \in \Gamma \tag{1.3'}$$

has only be a zero-solution

$$\sigma_{\alpha\beta}(x_\gamma) = 0, \quad x_\gamma \in \Omega. \tag{1.4'}$$

From the set of Eqs (1.1') it follows that

$$2\sigma_{12,12} = -\sigma_{11,11}-\sigma_{22,22} \tag{1.5}$$

whereas Eq (1.2') can be written in the form

$$(\sigma_{11}-\bar{\nu}\sigma_{22})_{,22}+(\sigma_{22}-\bar{\nu}\sigma_{11})_{,11} = -(1+\bar{\nu})\,(\sigma_{11,11}+\sigma_{22,22}), \tag{1.6}$$

where $\bar{\nu}$ is a constant of the type of Poisson's ratio.

Combining (1.5) and (1.6) we obtain

$$(\sigma_{11}-\bar{\nu}\sigma_{22})_{,22}+(\sigma_{22}-\bar{\nu}\sigma_{11})_{,11} = 2(1+\bar{\nu})\sigma_{12,12}. \tag{1.7}$$

Let us introduce two functions $u_1^0 = u_1^0(x_\gamma)$, $u_2^0 = u_2^0(x_\gamma)$, without consideration of their physical sense, in the following manner

$$u_{1,1}^0 = \sigma_{11}-\bar{\nu}\sigma_{22}, \quad u_{2,2}^0 = \sigma_{22}-\bar{\nu}\sigma_{11}. \tag{1.8}$$

Thus, in accordance with (1.7),

$$[2(1+\bar{\nu})\sigma_{12}-u_{1,2}^0-u_{2,1}^0]_{,12} = 0 \tag{1.9}$$

and upon integration

$$2(1+\bar{\nu})\sigma_{12}(x_\gamma) = u_{1,2}^0(x_\gamma)+u_{2,1}^0(x_\gamma)+f_1(x_1)+f_2(x_2), \tag{1.10}$$

where f_α are certain unknown functions of the indicated arguments.

Introducing new designations

$$u_1(x_\gamma) = u_1^0(x_\gamma)+\int f_2(x_2)dx_2, \quad u_2(x_\gamma) = u_2^0(x_\gamma)+\int f_1(x_1)dx_1 \tag{1.11}$$

we may write, in accordance with (1.8), (1.10),

$$\sigma_{\alpha\beta} = \frac{1}{1+\bar{\nu}}\left[u_{(\alpha,\beta)}+\frac{\bar{\nu}}{1-\bar{\nu}}u_{\gamma,\gamma}\delta_{\alpha\beta}\right]. \tag{1.12}$$

Let us note that for each point $x_\gamma \in \Omega$

$$\sigma_{\alpha\beta} u_{(\alpha,\beta)} = \frac{1}{1+\bar{\nu}} \left[u_{(\alpha,\beta)} u_{(\alpha,\beta)} + \frac{\bar{\nu}}{1-\bar{\nu}} u_{\alpha,\alpha} u_{\gamma,\gamma} \right] \geqslant 0, \tag{1.13}$$

if only $0 \leqslant \bar{\nu} < 1$.

On the strength of Green theorem and Eqs. (1.1′), (1.3′)

$$\int_\Omega \sigma_{\alpha\beta} u_{(\alpha,\beta)} d\Omega = \int_\Gamma \sigma_{\alpha\beta} u_\alpha n_\beta d\Gamma = 0. \tag{1.14}$$

Wanting this result to be consistent with (1.13) we must assume

$$u_{(\alpha,\beta)}(x_\gamma) = 0, \quad x_\gamma \in \Omega, \quad -0 \leqslant \bar{\nu} < 1$$

and thus on the strength of (1.12)

$$\sigma_{\alpha\beta}(x_\gamma) = 0, \quad x_\gamma \in \Omega, \quad \text{Q.E.D.}$$

It is noteworthy that for the proof it was essential to use a Poisson type number, although it does not appear in the Eqs. (1.1′)–(1.3′).

2. The three-dimensional problem

An elastic medium occupying the region V is surrounded by the surface S. To this surface are applied external forces described by the vectorial function $q_i = q_i(x_k)$, $i, k = 1, 2, 3$, $x_k \in S$. Body forces are described by the vectorial field $X_i = X_i(x_k)$, $x_k \in V$.

THEOREM 2. *The set of equations*

$$\sigma_{ij,j}(x_k) + X_i(x_k) = 0, \tag{2.1}$$

$$\sigma_{ij,kk}(x_l) + \frac{1}{1+\nu} \sigma_{kk,ij}(x_l) + X_{(i,j)}(x_l) + \frac{\nu}{1-\nu} X_{k,k}(x_1) \delta_{ij} = 0, \tag{2.2}$$

$$\sigma_{ij}(x_k) n_j(x_l) = q_i(x_k), \quad x_k \in S \tag{2.3}$$

for $0 < \nu \leqslant 1/2$ *determines uniquely the field*

$$\sigma_{ij} = \sigma_{ij}(x_k), \quad x_k \in V. \tag{2.4}$$

Proof. For the proof it is sufficient to show that the set of equations

$$\sigma_{ij,j}(x_k) = 0, \quad x_k \in V, \tag{2.1′}$$

$$\sigma_{ij,kk}(x_l) + \frac{1}{1+\nu} \sigma_{kk,ij}(x_l) = 0, \quad x_l \in V \tag{2.2′}$$

$$\sigma_{ij}(x_k) n_j(x_l) = 0, \quad x_k \in S \tag{2.3′}$$

can have only zero solutions

$$\sigma_{ij}(x_k) = 0, \quad x_k \in V. \tag{2.4′}$$

Part I of the proof: $0 < \nu < 1/2$.

Instead of the function $\sigma_{ij}(x_k)$ we shall introduce the functions $\varepsilon_{ij}(x_k)$ in the following manner:

$$\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}. \tag{2.5}$$

Otherwise

$$\sigma_{ij} = \frac{1}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{ll}\delta_{ij}\right). \tag{2.6}$$

From the contraction of Eqs. (2.2′) it results that

$$\sigma_{ll,kk} = 0 \tag{2.7}$$

and, since $\varepsilon_{ll} = (1-2\nu)\sigma_{ii}$, therefore

$$\varepsilon_{ll,kk} = 0. \tag{2.8}$$

Hence Eqs. (2.2′) can be written in the form

$$\varepsilon_{ij,kk} + \frac{1}{1-2\nu}\varepsilon_{kk,ij} = 0. \tag{2.9}$$

From Eqs. (2.1′) upon using (2.6) and suitable differentiation, it follows

$$\varepsilon_{ik,kj} + \varepsilon_{jk,ki} + \frac{2\nu}{1-2\nu}\varepsilon_{kk,ij} = 0. \tag{2.10}$$

Subtracting by sides (2.10) from (2.9) we obtain

$$\varepsilon_{ij,kk} + \varepsilon_{kk,ij} - \varepsilon_{ik,kj} - \varepsilon_{jk,ki} = 0. \tag{2.11}$$

The Eqs. (2.11) for $i \neq j$ appear as follows

$$\begin{aligned} \varepsilon_{11,23} + \varepsilon_{23,11} &= (\varepsilon_{12,3} + \varepsilon_{13,2})_{,1}, \\ \varepsilon_{22,13} + \varepsilon_{31,22} &= (\varepsilon_{21,3} + \varepsilon_{23,1})_{,2}, \\ \varepsilon_{33,12} + \varepsilon_{12,33} &= (\varepsilon_{32,1} + \varepsilon_{31,2})_{,3}, \end{aligned} \tag{2.12a}$$

whereas the Eqs. (2.11) for $i = j$, without summation:

$$\begin{aligned} \varepsilon_{11,22} + \varepsilon_{11,33} + (\varepsilon_{22} + \varepsilon_{33})_{,11} &= 2(\varepsilon_{12,21} + \varepsilon_{13,31}), \\ \varepsilon_{22,11} + \varepsilon_{22,33} + (\varepsilon_{11} + \varepsilon_{33})_{,22} &= 2(\varepsilon_{21,12} + \varepsilon_{23,32}), \\ \varepsilon_{33,11} + \varepsilon_{33,22} + (\varepsilon_{11} + \varepsilon_{22})_{,33} &= 2(\varepsilon_{31,13} + \varepsilon_{32,23}), \end{aligned} \tag{2.13}$$

in the way of algebraic transformations can be reduced to the form[1]:

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[1] E. g. from Eqs. $(2.13)_{1,3}$

$$\varepsilon_{11,22} + \varepsilon_{11,33} + (\varepsilon_{22} + \varepsilon_{33})_{,11} - 2\varepsilon_{12,21} = \varepsilon_{33,11} + \varepsilon_{33,22} + (\varepsilon_{11} + \varepsilon_{22})_{,33} - 2\varepsilon_{32,23},$$

and upon utilizing $(2.13)_2$

$$\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,21} - \varepsilon_{33,22} - \varepsilon_{22,33} = -\varepsilon_{22,11} - \varepsilon_{22,33} - (\varepsilon_{11} + \varepsilon_{33})_{,22} + 2\varepsilon_{21,12};$$

hence

$$2(\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,21}) = 0.$$

(2.12b)
$$\begin{aligned}\varepsilon_{11,22}+\varepsilon_{22,11} &= 2\varepsilon_{12,12},\\ \varepsilon_{33,11}+\varepsilon_{11,33} &= 2\varepsilon_{13,13},\\ \varepsilon_{33,22}+\varepsilon_{22,33} &= 2\varepsilon_{23,23}.\end{aligned}$$

Subsequently we introduce the functions $u_i^0 = u_i^0(x_k)$ using the relations

(2.14)
$$\varepsilon_{11} = u^0_{1,1}, \quad \varepsilon_{22} = u^0_{2,2}, \quad \varepsilon_{33} = u^0_{3,3}.$$

Substituting (2.14) into (2.12b) we obtain

(2.15)
$$\begin{aligned}(2\varepsilon_{12}-u^0_{1,2}-u^0_{2,1})_{,12} &= 0,\\ (2\varepsilon_{13}-u^0_{3,1}-u^0_{1,3})_{,13} &= 0,\\ (2\varepsilon_{23}-u^0_{3,2}-u^0_{2,3})_{,23} &= 0,\end{aligned}$$

hence upon integration

(2.16)
$$\begin{aligned}\varepsilon_{12} &= u^0_{(1,2)}+f_1(x_1, x_3)+f_2(x_2, x_3),\\ \varepsilon_{13} &= u^0_{(1,3)}+g_1(x_1, x_2)+g_2(x_2, x_3),\\ \varepsilon_{23} &= u^0_{(2,3)}+h_1(x_1, x_2)+h_2(x_1, x_3),\end{aligned}$$

where f_α, g_α, h_α are certain functions of the indicated arguments.

Putting (2.16) together with (2.14) into (2.12a) gives

(2.17)
$$\begin{aligned}(h_1+h_2)_{,11} &= f_{1,31}+g_{1,21},\\ (g_1+g_2)_{,22} &= f_{2,32}+h_{1,12},\\ (f_1+f_2)_{,33} &= h_{2,13}+g_{2,23},\end{aligned}$$

which, considering the dependence of the particular functions upon the variables x_1, x_2, x_3, can be written in the form

(2.18)
$$\begin{aligned}h_{1,11} &= g_{1,21}, & h_{2,11} &= f_{1,31},\\ g_{1,22} &= h_{1,12}, & g_{2,22} &= f_{2,32},\\ f_{1,33} &= h_{2,13}, & f_{2,33} &= g_{2,23}.\end{aligned}$$

Hence

(2.19)
$$\begin{aligned}h_{1,1} &= g_{1,2}+b_1(x_2), & h_{2,1} &= f_{1,3}+b_2(x_3),\\ g_{1,2} &= h_{1,1}+c_1(x_1), & g_{2,2} &= f_{2,3}+c_2(x_3),\\ f_{1,3} &= h_{2,1}+d_1(x_1), & f_{2,3} &= g_{2,2}+d_2(x_2),\end{aligned}$$

where b_α, c_α, d_α are certain functions of the indicated arguments.

Comparing the particular expressions in the set of Eqs. (2.19) we obtain

(2.20) $$b_1(x_2) = -c_1(x_1) = 0, \quad d_1(x_1) = -b_2(x_3) = 0, \quad c_2(x_3) = -d_2(x_2) = 0.$$

Therefore

(2.21) $$h_1 = \int g_{1,2}dx_1+e_1(x_2), \quad h_2 = \int f_{1,3}dx_1+e_2(x_3), \quad g_2 = \int f_{2,3}dx_2+e_3(x_3),$$

where e_i — certain functions.

If we now define the new functions

$$u_1 = u_1^0+2\int f_2(x_2, x_3)dx_2+2\int e_3(x_3)dx_3,$$
$$u_2 = u_2^0+2\int f_1(x_1, x_3)dx_1+2\int e_2(x_3)dx_3, \tag{2.22}$$
$$u_3 = u_3^0+2\int g_1(x_1, x_2)dx_1+2\int e_1(x_2)dx_2,$$

we may write, on the strength of (2.14), (2.21) and (2.16)

$$\varepsilon_{ij} = u_{(i,j)}. \tag{2.23}$$

We shall note that on the strength of Gauss theorem, [5], and Eqs. (2.1′), (2.3′),

$$\int_V \sigma_{ij}\varepsilon_{ij}dV = \int_V \sigma_{ij}u_{i,j}dV = \int_S \sigma_{ij}u_i n_j dS = 0. \tag{2.24}$$

On the other hand, in accordance with (2.6), for $1/2 > \nu \geqslant 0$, everywhere for $(x_k \in V)$;

$$\sigma_{ij}\varepsilon_{ij} = \frac{1}{1+\nu}\left(\varepsilon_{ij}\varepsilon_{ij}+\frac{\nu}{1-2\nu}\varepsilon_{ll}\varepsilon_{ii}\right) \geqslant 0. \tag{2.25}$$

Comparison of (2.24) and (2.25) signifies that

$$\varepsilon_{ij}(x_k) = 0, \quad x_k \in \mathrm{V}, \tag{2.26}$$

and hence, according to (2.6), (2.4′) occurs. Q.E.D.

Part II *of the proof*: $\nu = 1/2$.

If $\nu = 1/2$ (incompressible body, $\varepsilon_{kk} = 0$, though in general $\sigma_{kk} \neq 0$), then instead of (2.6)

$$\sigma_{ij} = \frac{2}{3}\varepsilon_{ij}+\frac{1}{3}\sigma_{kk}\delta_{ij}. \tag{2.6′}$$

Substituting (2.6′) into (2.2′) and making use of the fact that $\varepsilon_{kk} = 0$ and that (2.7) occurs, we obtain

$$\varepsilon_{ij,kk}+\sigma_{ll,ij} = 0. \tag{2.9′}$$

Substituting in turn (2.6′) into (2.1′) we obtain: $\varepsilon_{ik,k}+\frac{1}{2}\sigma_{kk,i} = 0$, whence following suitable differentiations

$$\varepsilon_{ik,kj}+\varepsilon_{jk,ki}+\sigma_{kk,ij} = 0; \tag{2.10′}$$

subtracting by sides (2.10′) from (2.9′) we obtain

$$\varepsilon_{ij,kk}-\varepsilon_{ik,kj}-\varepsilon_{jk,ki} = 0$$

or, since $\varepsilon_{kk} = 0$,

$$\varepsilon_{ij,kk}+\varepsilon_{kk,ij}-\varepsilon_{ik,kj}-\varepsilon_{jk,ki} = 0. \tag{2.11′}$$

We can thus repeat successively the steps from (2.12a) up to (2.24). Instead of (2.25) we have for $\nu = 1/2$

$$\sigma_{ij}\varepsilon_{ij} = \frac{2}{3}\varepsilon_{ij}\varepsilon_{ij}-\frac{1}{3}\sigma_{kk}\varepsilon_{ii} = \frac{2}{3}\varepsilon_{ij}\varepsilon_{ij} \geqslant 0, \tag{2.25′}$$

whence, upon comparing with (2.24),

$$\varepsilon_{ij} = 0, \tag{2.17}$$

and thus in accordance with (2.6′)

$$\sigma_{ij}(x_k) = \frac{1}{3}\sigma_{ll}(x_k)\,\delta_{ij}. \tag{2.28}$$

This result put into (2.1′) gives

$$\sigma_{ll,i} = 0. \tag{2.29}$$

Integrating (2.29) consecutively for $i = 1, 2, 3$ we obtain

$$\sigma_{ll} = C_1(x_2, x_3), \quad \sigma_{ll} = C_2(x_1, x_3), \quad \sigma_{ll} = C_3(x_1, x_2), \tag{2.30}$$

where C_i are temporarily unknown functions of the indicated arguments.

Comparing with each other (2.30) we find that

$$\sigma_{ll} = C,$$

where C is a constant independent of x_k. Thus in conformity with (2.28)

$$\sigma_{ij}(x_k) = \frac{1}{3}C\delta_{ij}. \tag{2.31}$$

Putting (2.31) into the boundary condition (2.3′) we obtain

$$Cn_i(x_l) = 0.$$

Since $n_i n_i = 1 \neq 0$, therefore

$$C = 0, \tag{2.32}$$

and this on strength of (2.31) again leads to (2.4′), Q.E.D.

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