Archives of Mechanics • Archiwum Mechaniki Stosowanej • 28, 2, pp. 231-236, Warszawa 1976

BRIEF NOTES

A note on the force of interaction between external loads and a Griffith crack

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The problem of action of a field of external loads on a Griffith crack in an infinite elastic medium is discussed. Using the definition of the force exerted on a defect by the external field given in [1], the formula for the horizontal component of this force in terms of the stress intensity factors is derived in the static and quasi-static cases. The result shows the notion of the force of interaction to be useful in the analysis of fracture phenomena due to its interpretation and the relation to the well-known Irvin's potential energy release rate.

1. Statement of the problem : crack geometry, assumptions relative to external loads, method of solution

Let us consider an infinite, linear-elastic and isotropic medium containing a Griffith crack which is situated in the field of arbitrary external forces [symbolically denoted by (EXT) in Fig. 1].

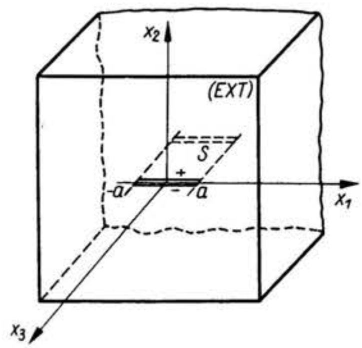

FIG. 1.

The Griffith crack is a plane, two-sided surface S (positive and negative sides denoted by + and −) described in a rectangular Cartesian coordinate system (x_1, x_2, x_3) as follows

$$-a < x_1 < a, \quad x_2 = 0, \quad -\infty < x_3 < \infty.$$

The discontinuity of the displacement vector u_i on the surface S is characterized by the jump vector $U_i(x_1)$

$$
(1.1)\qquad \begin{aligned} &U_i(x_1) = u_i(x_1, 0^+) - u_i(x_1, 0^-) \quad \text{for} \quad |x_1| < a,\\ &U_i(\pm a) = 0. \end{aligned}
$$

The medium is assumed to be loaded by a system of forces applied outside the crack and independent of x_3 in such a manner that the crack deformation can be split into three fundamental modes; thus the problem may be regarded as a problem of a crack in the plane (Mode I and II) and antiplane states of strain (Mode III) and the considerations can be related to the cross-section $x_3 = 0$ of the crack representing a line segment of length $2a$ and tips at $x_1 = \pm a$. It is known from the classical theory of cracks that

in the opening mode (Mode I) $U_2 \neq 0$, $U_1 = 0$, $u_3 = 0$ on S,
in the sliding mode (Mode II) $U_1 \neq 0$, $U_2 = 0$, $u_3 = 0$ on S,
in the tearing mode (Mode III) $U_3 \neq 0$, $u_1 = u_2 = 0$ on S.

Each mode is characterized by the singular crack-tip stresses:

$$\sigma_{22} \text{ for Mode I}, \quad \sigma_{21} \text{ for Mode II}, \quad \sigma_{23} \text{ for Mode III}.$$

Finally, we assume that the problem is static, i.e., the crack is at rest and the external loading is independent of time. Generalization to the quasi-static case will be carried out at the end of this paper.

In order to solve the elastic problem presented, a two-stage method is used. Making use of the superposition principle and taking into account the boundary condition requiring the crack edges to be free of tractions, our problem M can be represented in the form of the sum of two problems (Fig. 2). The first one $M^{(0)}$ concerns the distribution of

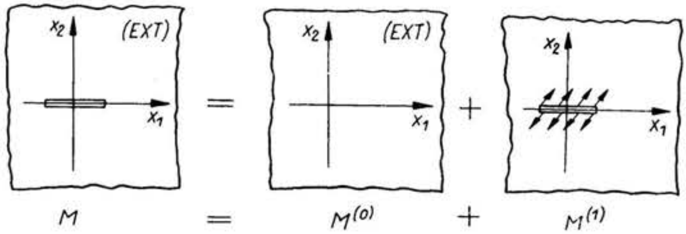

FIG. 2.

displacements and stresses in a continuous crack-free medium loaded by external forces (EXT), while the second problem $M^{(1)}$ concerns the medium with a crack $|x_1| < a$ loaded exclusively by forces $-h_i(x)$ applied to the crack faces; it is assumed that

$$
(1.2)\qquad h_i(x_1) = \sigma_{2i}^{(0)}(x_1, 0) \quad \text{for} \quad |x_1| < a, \quad i = 1, 2, 3,
$$

and the corresponding solutions of $M^{(0)}$ i $M^{(1)}$ are denoted by $\sigma_{ij}^{(0)}$, $u_i^{(0)}$ and $\sigma_{ij}^{(1)}$, $u_i^{(1)}$, $U_i^{(1)}$, respectively.

The simple problem $M^{(0)}$ may be solved by any method known in elasticity (Green functions, potentials, etc.), whereas the solution of $M^{(1)}$ can be obtained by means of the methods usually employed in the case of problems with discontinuous boundary conditions (Cauchy integrals, integral equations, etc.).

2. Force of the external loads on the crack

In order to describe the interaction between the external loads and the crack let us make use of the definition given in [1] of the force exerted on a defect by the "complementary" field. The definition introduced in connection with general problems of dynamics of defects (e.g. dislocations) may also be applied to the crack considered here. This force is formally defined as a functional derivative of the interaction part of the action functional with respect to the defect position and in the present case is given by the fromula

$$F_j = \int_{-a}^{a} \frac{\partial \sigma_{2i}^{(0)}(x_1, x_2)}{\partial x_j}\bigg|_{x_2=0} U_i^{(1)}(x_1)\,dx_1. \tag{2.1}$$

Here F_j denotes the force components, summation over $i = 1, 2, 3$ should be performed, and the symbols $\sigma^{(0)}$ and $U^{(1)}$ have the meaning explained in Sect. 1. It follows that under the crack geometry and external loads assumed, the force $\mathbf{F}$ has at most two non-vanishing components, F_1 and F_2, since $\partial\sigma_{2i}^{(0)}/\partial x_3 = 0$. Moreover, each of the fundamental modes of crack deformation reduces the sum (2.1) to a single term.

Let us observe that, in spite of linearity of the entire formulation, the formula (2.1) does not allow for superposition, the integrand representing a product of forces and deformation (like e.g. in the formula for strain energy).

In order to investigate the action of external field on the crack by means of the relation (2.1), the values of the jumps of displacements should be expressed in terms of external forces. Making use of the results of [2] we observe that the solution of $M^{(1)}$ is common for all three modes and takes the form

$$\sigma_{2i}^{(1)}(x_1, 0) = \begin{cases} \dfrac{(\operatorname{sgn} x_1) H_i(x_1)}{\sqrt{x_1^2 - a^2}} & \text{for} \quad |x_1| > a, \\ -h_i(x_1 & \text{for} \quad |x_1| < a, \end{cases} \tag{2.2}$$

$$u_i^{(1)}(x_1, 0^{\pm}) = \mp \frac{c_i}{\mu} \int_{-a}^{x_1} \frac{H_i(\xi)\,d\xi}{\sqrt{a^2 - \xi^2}} \quad \text{for} \quad |x_1| < a, \tag{2.3}$$

$$U_i^{(1)}(x_1) = -\frac{2c_i}{\mu} \int_{-a}^{x_1} \frac{H_i(\xi)\,d\xi}{\sqrt{a^2 - \xi^2}} \quad \text{for} \quad |x_1| < a, \tag{2.4}$$

where

$$H_i(x_1) = \frac{1}{\pi} \int_{-a}^{a} \frac{h_i(t)\sqrt{a^2 - t^2}}{x_1 - t}\,dt \tag{2.5}$$

(for $|x_1| < a$ these integrals are interpreted in the sense of Cauchy principal values),

$$c_1 = c_2 = 1 - \nu, \quad c_3 = 1, \tag{2.6}$$

μ — shear modulus, ν — Poisson's ratio.

The functions $h_i(x_1)$ are assumed to be bounded for $|x_1| \leqslant a$ to ensure the existence of all integrals appearing in the paper.

3. Horizontal component of the force of interaction

Let us introduce three additional functions

(3.1) $G_i(x_1) = \frac{1}{\pi} \int\limits_{-a}^{a} \frac{h_i(t)\,dt}{\sqrt{a^2-t^2}\,(x_1-t)}$ (for $|x_1| < a$ — Cauchy principal values) and six coefficients

$$e_i = \frac{1}{\pi} \int\limits_{-a}^{a} \frac{h_i(t)\,dt}{\sqrt{a^2-t^2}}, \qquad d_i = \frac{1}{\pi a} \int\limits_{-a}^{a} \frac{t h_i(t)\,dt}{\sqrt{a^2-t^2}}. \tag{3.2}$$

The following relations are easily obtained:

$$H_i(x_1) = e_i x_1 + a d_i + (a^2-x_1^2) G_i(x_1) \quad \text{and} \lim_{x_1 \to \pm a} (a^2-x_1^2) G_i(x_1) = 0, \tag{3.3}$$

$$H_i(\pm a) = a(d_i \pm e_i), \tag{3.4}$$

$$(K_i^R)^2 - (K_i^L)^2 = 4 a d_i e_i. \tag{3.5}$$

Here the stress intensity factors are expressed as follows:

$$\begin{aligned} K_i^R &= \lim_{x_1 \to a^+} \sqrt{2(x_1-a)}\, \sigma_{2i}^{(1)}(x_1, 0) = \frac{H_i(a)}{\sqrt{a}} = \sqrt{a}\,(e_i + d_i), \\ K_i^L &= \lim_{x_1 \to -a^-} \sqrt{2(-x_1-a)}\, \sigma_{2i}^{(1)}(x_1, 0) = -\frac{H_i(-a)}{\sqrt{a}} = \sqrt{a}\,(e_i - d_i). \end{aligned} \tag{3.6}$$

The horizontal (longitudinal) component of the force (2.1) ($j = 1$) is equal to

$$F_1 = \int\limits_{-a}^{a} \frac{d\big(h_i(x_1)\big)}{dx_1} U_i^{(1)}(x_1)\,dx_1. \tag{3.7}$$

Integrating by parts, taking into account $(1.1)_2$ and (2.4), we obtain

$$F_1 = \frac{2c_i}{\mu} \int\limits_{-a}^{a} h_i(t) \frac{H_i(t)}{\sqrt{a^2-t^2}}\,dt. \tag{3.8}$$

Making use of (3.3) and (3.2) we have

$$F_1 = \frac{2c_i}{\mu} (2\pi a d_i e_i + J), \tag{3.9}$$

where

$$J = \int\limits_{-a}^{a} h_i(t) \sqrt{a^2-t^2}\, G_i(t)\,dt = \frac{1}{\pi} \int\limits_{-a}^{a} h_i(t) \sqrt{a^2-t^2} \left(\int\limits_{-a}^{a} \frac{h_i(\xi)\,d\xi}{\sqrt{a^2-\xi^2}(t-\xi)} \right) dt. \tag{3.10}$$

Changing the order of integrating in the last integral we note that

$$\frac{2c_i}{\mu} J = -F_1. \tag{3.11}$$

Relation (3.9) is now rewritten in the form

$$F_1 = \frac{4\pi}{\mu} c_i a d_i e_i - F_1 \tag{3.12}$$

which yields the simple formula for F_1

$$F_1 = \frac{2\pi}{\mu} c_i a d_i e_i. \tag{3.13}$$

Let us now compare this result with (3.5). It is evident that the component F_1 may be expressed in terms of the stress intensity factors

$$F_1 = \frac{\pi}{2\mu} c_i [(K_i^R)^2 - (K_i^L)^2] \tag{3.14}$$

with c_i defined by (2.6).

Thus it has been proved that in each crack deformation mode the component F_1 of the force exerted on the crack by external loads is proportional to the difference of squares of the stress intensity factors at the right and left crack tips, respectively. These factors are, of course, certain functionals of the externally applied forces.

A similar result may be obtained in the same way for a semi-infinite crack:

$$F_1 = \frac{\pi}{2\mu} c_i K_i^2. \tag{3.15}$$

It is interesting to note the relation between F_1 and Irvin's strain (or potential) energy release rate G[4] connected with the well-known Griffith fracture criterion:

$$\begin{aligned} F_1 &= G = \frac{\pi(1-\nu)}{2\mu}(K_1^2 + K_2^2) + \frac{\pi}{2\mu} K_3^2 \quad \text{for a semi-infinite crack,} \\ F_1 &= G^R - G^L \quad \text{for a finite crack.} \end{aligned} \tag{3.16}$$

In order to give the interpretation of this force let us consider the medium with a crack deforming according to the Mode I. The corresponding load is so selected that $\sigma_{21}^{(0)}(x_1, 0) = 0$ implying $F_2 = 0$ (e.g. the system of two concentrated forces is symmetric with respect to x_1-axis [2]). It follows from (3.14) that if $F_1 > 0$ or $F_1 < 0$, then the crack has a tendency to propagate in the positive or negative directions of the x_1-axis, respectively. In the case of equal SIFactors, $K^R = K^L$, the component $F_1 = 0$ and the crack has equal abilities to propagate in the directions.

As far as the vertical component F_2 of the force acting on the crack is concerned, no formulae of the type of (3.14) can be derived and the problem of its possible physical interpretation is not clear as yet.

Our considerations may easily be extended to quasi-static case when the crack moves together with the field of external forces along x_1-axis with a constant speed v. Making use of the results obtained in [3] we observe that all the relations given so far hold true also in the quasi-static problem provided the following substitution of constants $\tilde{c}_i$ for c_i is made:

$$\tilde{c}_1 = \frac{\beta(1-\beta^2)}{\varkappa_R}, \quad \tilde{c}_2 = \frac{\gamma(1-\beta^2)}{\varkappa_R}, \quad \tilde{c}_3 = \frac{1}{\beta}, \tag{3.17}$$

where

$$\beta^2 = 1 - v^2/c_s^2, \quad \gamma^2 = 1 - v^2/c_d^2, \quad \varkappa_R = 4\beta\gamma - (1+\beta^2)^2, \tag{3.18}$$

$c_d = \sqrt{(\lambda + 2\mu)/\varrho}$ — velocity of dilatational longitudinal wave,
$c_s = \sqrt{\mu/\varrho}$ — velocity of shear transverse wave,
ϱ — density of the medium,
λ, μ — Lamé constants.

It is assumed that $v < c_s$ (the motion of the crack and loads is subsonic). It is easily seen that if $v \to 0$, then $\tilde{c}_i \to c_i$, since $\lim\limits_{v \to 0} (1-\beta^2)/\varkappa_R = 1 - \nu$.

The expressions (3.16) remain also valid in the quasi-static case provided G is replaced by its dynamic counterpart as derived in [5]. Taking into account (3.17) and the behaviour of $\varkappa_R$ we arrive at the conclusion that for $v = c_R$ (velocity of Rayleigh waves) in the plane problems (Mode I, II) and for $v = c_S$ in antiplane problems (Mode III), F_1 tends to infinity which sets the practical upper limits on crack velocity in these cases.

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Received September 19, 1975.