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Convergence of solutions of the equations of dynamic linear dipolar elasticity to the solutions of classical elastodynamics

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THE problem of convergence of solutions of the equations of dynamic linear dipolar elasticity to the solutions of the equations of linear elastodynamics as the dipolar terms tend to zero. For bounded regions, it is established that, provided the solutions satisfy an a *priori* bound is studied there is L_2-convergence in the sense of Hölder. No definiteness assumptions are required on the respective elasticities. For unbounded regions, convergence exists in a specified L_2-sense, but now the monopolar elasticities must be positive semi-definite and the dipolar displacement suitably restricted at infinity.

W pracy badane jest zachowanie się rozwiązania początkowo-brzegowego problemu dla liniowej dipolarnej sprężystości w przypadku, kiedy współczynniki reprezentujące wyrazy "dipolarne" mogą zdążać do zera. Głównym celem pracy jest określenie w jakim sensie rozwiązanie (jeżeli istnieje) problemu "dipolarnego" iest zbieżne z rozwiązaniem odpowiedniego początkowo-brzegowego problemu dla anizotropowego liniowo sprężystego materiału, jeżeli współczynniki "dipolarne" dążą do zera.

Данная работа исследует поведение решения начально-краевой задачи для линейной диполярной упругости в случае, когда коэффициенты представляющие „диполярные" члены могут стремиться к нулю. Главной целью работы является определение в каком смысле решение (если существует) „диполярной" задачи сходится к решению соответствующей начально-краевой задачи для анизотропного, линейно упругого материала, если „диполярные" коэффициенты стремятся к нулю.

1. Introduction

THE behaviour of an anisotropic linear elastic material occupying a bounded or unbounded domain Ω of Euclidean 3-space is governed by the following system of equations,

$$\varrho\ddot{v}_i = (a_{ijkh}v_{k,h})_{,j}+\varrho f_i, \quad \text{in } \Omega\times(0, T), \qquad a_{ijkh} = a_{khij}, \tag{1.1}$$

where v_i are the components of displacement about a reference configuration, ϱ is the density, a_{ijkh} are the elasticities, f_i is the body force per unit mass and $T\ (<\infty)$ is a constant. Here and throughout the paper standard Cartesian indicial notation is employed, a superposed dot denotes partial differentiation with respect to time, and a subscript comma followed by a Latin letter, j say, denotes partial differentiation with respect to the spatial variable x_j.

Let $\mathscr{B}$ denote the boundary initial value problem formed by (1.1) together with the following boundary and initial conditions,

$$v_i = q_i(x, t) \quad \text{on } \Sigma\times[0, T], \tag{1.2}$$

$$v_i(x, 0) = g_i(x), \quad \dot{v}_i(x, 0) = h_i(x), \quad x\in\Omega, \tag{1.3}$$

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where Σ denotes the boundary (interior, exterior or otherwise) of Ω and q_i, g_i and h_i are prescribed functions of the indicated arguments. Σ is supposed smooth enough to justify our use of the divergence theorem.

Several studies of qualitative properties of solutions to $\mathscr{B}$ for *indefinite* elasticities have appeared in recent literature; in particular, we mention the stability analyses of KNOPS and PAYNE [8, 9], which employ logarithmic convexity arguments.

The present work again involves $\mathscr{B}$; however, we shall consider the behaviour of the solution to an analogous boundary initial value problem for a linear dipolar, elastic material when the coefficients representing the "dipolar" terms are allowed to approach zero. We wish to determine in what sense, if any, the solution of the "dipolar" problem converges to the solution of $\mathscr{B}$, as the "dipolar" coefficients tend to zero.

2. Basic equations

The dipolar theory we consider is a special case of a general multipolar theory developed by GREEN and RIVLIN [6]. An application of the linear dipolar theory of GREEN and RIVLIN [6] to the torsion of a circular cylinder was given by GREEN and NAGHDI [4] whose notation we now use.

The equations of motion are

$$\varrho\ddot{u}_i-\varrho\Gamma_{ji,j}-\tau_{ji,j}+\Sigma_{(kj)i,kj} = -\varrho F_{ij}+\varrho f_i, \tag{2.1}$$

where u_i are the components of displacement about a reference configuration. Γ_{ij} are dipolar inertia terms, τ_{ij} represent a symmetric tensor (associated with the stress), $\Sigma_{(kj)i}$ are the components of dipolar stress which are symmetric in the first two indices, F_{ij} is the dipolar body force and ϱ and f_i are given by (1.1).

[In fact, τ_{ij} is related to the monopolar stress, σ_{ij}, as follows

$$\tau_{ij} = \Sigma_{(kj)i,k}-\sigma_{ji}+\varrho(F_{ji}-\Gamma_{ji})].$$

The inertia coefficient satisfies (see GREEN and NAGHDI [5]),

$$\Gamma_{ji} = m_{jk}(x)\ddot{u}_{i,k}, \tag{2.2}$$

where m_{ij} is a symmetric tensor such that $m_{ij}\xi_{ik}\xi_{jk} \geqslant m^{-1}\,\xi_{ij}\xi_{ij}$ for a positive constant m.

We assume the Helmholtz free energy function, A, can be written in the following form

$$\varrho A = \frac{1}{2}a_{ijkh}u_{i,j}u_{k,h}+b_{ijkhm}u_{i,j}u_{k,hm}+\frac{1}{2}c_{ijkhmn}u_{i,jk}u_{h,mn}, \tag{2.3}$$

where a_{ijkh} are defined in (1.1) and b_{ijkhm} and c_{ijkhmn} are functions of x, such that

$$c_{ijkhmn} = c_{hmnijk}. \tag{2.4}$$

Constitutive equations for τ_{ji} and $\Sigma_{(kj)i}$ then follow from Eqs. (8) and (9) of GREEN and NAGHDI [4] and are:

$$\tau_{ij} = a_{kmij}u_{k,m}+b_{ijmnq}u_{m,nq}, \tag{2.5}$$

and

$$\Sigma_{(kj)i} = b_{pqijk}u_{p,q}+c_{pqmijk}u_{p,qm}. \tag{2.6}$$

To enable the convergence problem to be concisely stated, we introduce some further notation. Let $\mathscr{L}$ and $\mathscr{L}_\varepsilon$ be the operators

$$\mathscr{L}v_i = \varrho\ddot{v}_i - (a_{ijkh}v_{k,h})_{,j} \tag{2.7}$$

and

$$\mathscr{L}_\varepsilon u_i = \varrho\ddot{u}_i - \varepsilon\varrho(m_{jk}\ddot{u}_{i,k})_{,j} + {}_\varepsilon\Sigma_{(kj)i,kj} - {}_\varepsilon\tau_{ij,j}, \tag{2.8}$$

where

$$_\varepsilon\Sigma_{(kj)i} = \varepsilon b_{pqijk}u_{p,q} + \varepsilon c_{pqmijk}u_{p,qm}, \tag{2.9}$$

$$_\varepsilon\tau_{ji} = a_{kmji}u_{k,m} + \varepsilon b_{jimnq}u_{m,nq}, \tag{2.10}$$

and $\varepsilon(>0)$ is a constant. (In (2.8)–(2.10), a linear dependence on ε is introduced in the "dipolar" terms. The coefficients m_{ij}, b_{ijkhm} and c_{ijkhmn} are different from the corresponding ones in (2.2) and (2.3), therefore, by a factor of ε.

To define a boundary initial value problem for the dipolar elastic material we suppose the displacement, u_i, and the dipolar tractions, $T_{ij} = n_k\Sigma_{kij}$, are specified on Σ, where n_k are the components of the unit outward normal on Σ. (Other boundary conditions are considered, by e.g. GREEN and NAGHDI [4].) The boundary initial value problem so obtained, which we denote by $\mathscr{A}$, can then be represented as follows:

Problem $\mathscr{A}$

$$\begin{aligned} &\mathscr{L}_\varepsilon u_i = \varrho f_i && \text{in } \Omega\times(0,T], \\ &\left.\begin{aligned} u_i &= q_i(x,\varepsilon,t) \\ T_{ij} &= q_{ij}(x,\varepsilon,t) \end{aligned}\right\} && \text{on } \Sigma\times[0,T], \\ &u_i(x,0) = g_i(x), && \dot{u}_i(x,0) = h_i(x), \end{aligned} \tag{2.11}$$

where q_{ij} are prescribed and we have taken the dipolar body force to be zero.

$\mathscr{B}$ can be conveniently rewritten as follows:

Problem $\mathscr{B}$

$$\begin{aligned} &\mathscr{L}v_i = \varrho f_i && \text{in } \Omega\times(0,T], \\ &v_i = q_i(x,0,t) && \text{on } \Sigma\times[0,T], \\ &v_i(x,0) = g_i(x), && \dot{v}_i(x,0) = h_i(x). \end{aligned} \tag{2.12}$$

Clearly, when $\varepsilon = 0$ the differential equation in (2.11) becomes the differential equation for a classical linear elastic material, as in (2.12). (However, problem $\mathscr{A}$ is not equivalent to problem $\mathscr{B}$ when $\varepsilon = 0$). The object of the paper is to investigate in what sense, if any, the solution to $\mathscr{A}$ approaches the solution to $\mathscr{B}$, as $\varepsilon \to 0$. Since *no definiteness* of the elastic coefficients is assumed this, therefore, is an improperly posed singular perturbation problem which arises naturally in solid mechanics. Improperly posed singular perturbation problems of a somewhat different character have recently been studied by PAYNE and SATHER [17] and ADELSON [1].

We do not always expect solutions to problem $\mathscr{B}$ to behave as solutions to problem $\mathscr{A}$, no matter how small ε may be. For example, even when the potential energy in problem $\mathscr{B}$ is a coercive bilinear form, the zero solution need not be stable in the C^0 norm. (This is discussed in detail by KNOPS and WILKES [11], section 6, who use an example of

R. T. SHIELD and A. E. GREEN [18], to demonstrate the so-called focusing effect). However, when the potential energy in problem $\mathscr{A}$ is a coercive form it can be shown that the zero solution to $\mathscr{A}$ *is* stable in the C^0 norm, and so the local "peak" behaviour exhibited in the example of SHIELD and GREEN is not possible, no matter how small ε may be. (Similar remarks were made by KOITER [12] who based his analysis on a different model which, however, also included strain gradients). It is of interest, therefore, to know how solutions to $\mathscr{A}$ behave as $\varepsilon \to 0$. In fact, by imposing a suitable *a priori* bound on the solutions to $\mathscr{A}$ and $\mathscr{B}$, we shall show how to establish convergence in an appropriate sense as $\varepsilon \to 0$ of solutions to the improperly posed problem $\mathscr{A}$ to those of the improperly posed problem $\mathscr{B}$.

The remainder of the paper is divided into two parts. In the first part, Sect. 3, we study problems $\mathscr{A}$ and $\mathscr{B}$ when Ω is a bounded domain of R^3. In this case our approach to convergence employs logarithmic convexity arguments and is based on work by PAYNE and SATHER [17] who studied an elliptic equation which in its reduced form is the backward heat equation. The second part, comprising Sect. 4, is devoted to a similar convergence study when Ω is exterior to a compact set in R^3. The approach here essentially relies on a method originally developed by GRAFFI [3].

3. Convergence when Ω is a bounded set

In this section Ω is a bounded domain of R^3. Let now D_t be the space-time domain $\Omega \times (0, t)$ and let $\|\cdot\|_{D_t}$ be defined by

$$\|q\|^2_{D_t} = \int_{D_t} \varrho q_i q_i \, dx \, d\eta, \tag{3.1}$$

for functions $q_i \in L^2(D_t)$, $i = 1, 2, 3$.

Suppose now there exist classical solutions, u_i and v_i to problems $\mathscr{A}$ and $\mathscr{B}$, respectively, and in addition, these solutions are smooth enough to justify the analysis which follows. We define w_i by

$$w_i = u_i - v_i \tag{3.2}$$

and introduce the constant P given by

$$\begin{aligned} P^2 = \int_{D_t} &\{\varrho m_{jk}([a_{iqrs}v_{r,s}]_{,q} + \varrho f_i)_{jk} + b_{ijmnq}u_{m,nq} - \varepsilon^{-1}{}_{\varepsilon}\Sigma_{(kj)i,k}\} \times \\ &\times \{\varrho m_{jl}([a_{iphd}v_{h,d}]_{,p} + \varrho f_i)_{,l} + b_{ijphd}u_{p,hd} - \varepsilon^{-1}{}_{\varepsilon}\Sigma_{(pj)i,p}\} \, dx \, dt \\ &+ 2T^2 \int_{D_T} \frac{\partial}{\partial t} \{\varrho m_{jk}([a_{iqrs}v_{r,s}]_{,q} + \varrho f_i)_{,k} + b_{ijmnq}u_{m,nq} - \varepsilon^{-1}{}_{\varepsilon}\Sigma_{(kj)i,k}\} \\ &\times \frac{\partial}{\partial t} \{\varrho m_{jl}([a_{iphd}v_{h,d}]_{,p} + \varrho f_i)_{,l} + b_{ijphd}u_{p,hd} - \varepsilon^{-1}{}_{\varepsilon}\Sigma_{(pj)i,d}\} \, dx \, dt. \end{aligned} \tag{3.3}$$

We may now state the main theorem of this section.

THEOREM 1. *Let* w_i *be defined by* (3.2) *and suppose* u_i *and* v_i *belong to the class of functions for which* P *(defined by* (3.3)) *is finite. Then there are positive constants* M, α, δ, *with* $0 < \alpha \leqslant 1$, *such that*

$$\int_{D_t} \varrho(w_i w_i + \varepsilon m_{jk} w_{i,j} w_{i,k}) dx d\eta \leqslant M\varepsilon^{\alpha}, \tag{3.4}$$

provided $0 \leqslant t \leqslant T-\delta$. *In particular, as* $\varepsilon \to 0$,

$$||u-v||_{D_t} \to 0,$$

provided $0 \leqslant t \leqslant T-\delta$, *where* u_i *and* v_i *are classical solutions to* $\mathscr{A}$ *and* $\mathscr{B}$, *respectively.*

Theorem 1 shows that, provided P is bounded *a priori* and t belongs to a compact subinterval of $[0, T)$, then as $\varepsilon \to 0$ a solution to the dipolar problem $\mathscr{A}$ converges to a solution to the classical elasticity problem $\mathscr{B}$ in the D_t norm defined by (3.1).

Before proving the theorem we establish an auxiliary lemma.

LEMMA 1. *The quantity* $E_\varepsilon(t)$ *defined by*

$$E_\varepsilon(t) \equiv \frac{1}{2} \int_{\Omega} \{\varrho(\varepsilon m_{jk} \dot{w}_{i,j} \dot{w}_{i,k} + \dot{w}_i \dot{w}_i) + a_{ijkh} w_{i,j} w_{k,h}\} dx, \tag{3.5}$$

satisfies the following equation

$$E_\varepsilon(t) = \int_{D_t} \{\varepsilon \dot{w}_i[\varrho(m_{jk}\ddot{v}_{i,k})_{,j} + (b_{ijmnq} u_{m,nq})_{,j}] - \dot{w}_i\, {}_{\varepsilon}\Sigma_{(kj)i,kj}\} dx d\eta. \tag{3.6}$$

P r o o f. The proof of the lemma follows immediately from the following equation which, in turn, follows directly from (2.11) and (2.12),

$$\mathscr{L} w_i - \varepsilon\varrho(m_{jk}\ddot{w}_{i,k})_{,j} = \varepsilon\varrho(m_{jk}\ddot{v}_{i,k})_{,j} + \varepsilon(b_{ijmnq} u_{m,nq})_{,j} - {}_{\varepsilon}\Sigma_{(kj)i,kj}. \tag{3.7}$$

P r o o f o f T h e o r e m 1. We define the quantities $F(t)$, $\mathscr{F}(t)$ and $\mathscr{G}(t)$ as follows

$$F(t) = \int_{D_t} \varrho(w_i w_i + \varepsilon m_{jk} w_{i,j} w_{i,k}) dx d\eta, \tag{3.8}$$

$$\mathscr{F}(t) = F(t) + \varepsilon P^2, \tag{3.9}$$

$$\mathscr{G}(t) = \log \mathscr{F}(t) + ct^2, \tag{3.10}$$

where c is a constant to be specified.

Our object is now to show that $\mathscr{G}$ is a convex function of t on $(0, T)$. Hence, we compute the first and second derivatives of F, with respect to t, denoted by $F'(t)$ and $F''(t)$, respectively. Now,

$$\mathscr{F}' \equiv F'(t) = 2 \int_{D_t} \varrho(w_i \dot{w}_i + \varepsilon m_{jk} w_{i,j} \dot{w}_{i,k}) dx d\eta, \tag{3.11}$$

and

$$\mathscr{F}'' \equiv F''(t) = 2 \int_{D_t} \varrho(w_i \ddot{w}_i + \dot{w}_i \dot{w}_i + \varepsilon m_{jk} w_{i,j} \ddot{w}_{i,k} + \varepsilon m_{jk} \dot{w}_{i,j} \dot{w}_{i,k}) dx d\eta, \tag{3.12}$$

where the symmetry of m_{ij} and the initial conditions have been employed.

The kinetic energy, $K(t)$, is defined by

$$K(t) = \frac{1}{2} \int_{\Omega} \varrho(\dot{w}_i \dot{w}_i + \varepsilon m_{jk} \dot{w}_{i,j} \dot{w}_{i,k}) dx, \tag{3.13}$$

and we use this in (3.12) after using the divergence theorem and substituting for the first and third terms on the right of (3.12) from (3.7) to see that

$$F'' = 4 \int_0^t K(\eta) d\eta + 2 \int_{D_t} w_i [(a_{ijkh} w_{k,h})_{,j} + \varepsilon \varrho (m_{jk} \ddot{v}_{i,k})_{,j} + \varepsilon (b_{ijmnq} u_{m,nq})_{,j} - {}_{\varepsilon}\Sigma_{(kj)i,kj}] dx\, d\eta, \tag{3.14}$$

$$= 8 \int_0^t K(\eta) d\eta + 2 \int_{D_t} \{w_i \varepsilon [\varrho (m_{jk} \ddot{v}_{i,k})_{,j} + (b_{ijmnq} u_{m,nq})_{,j}] - w_i {}_{\varepsilon}\Sigma_{(kj)i,kj}\} dx\, d\eta$$
$$-4 \int_0^t (t-\eta) \int_{\Omega} \{\varepsilon \dot{w}_i [\varrho (m_{jk} \ddot{v}_{i,k})_{,j} + (b_{ijmnq} u_{m,nq})_{,j}] - \dot{w}_i {}_{\varepsilon}\Sigma_{(kj)i,kj}\} dx\, d\eta, \tag{3.15}$$

where we have integrated by parts in the second term on the right in (3.14) and then used the lemma to substitute for the $E_\varepsilon(t)$ terms.

We next integrate the last term in (3.15) by parts with respect to t and obtain

$$F'' = 8 \int_0^t K(\eta) d\eta - 2 \int_{D_t} \{\varepsilon w_i [\varrho (m_{jk} \ddot{v}_{i,k})_{,j} + (b_{ijmnq} u_{m,nq})_{,ij}] - w_i {}_{\varepsilon}\Sigma_{(kj)i,kj}\} dx\, d\eta$$
$$+4 \int_0^t (t-\eta) \int_{\Omega} w_i \frac{\partial}{\partial \eta} \{\varepsilon [\varrho (m_{jk} \ddot{v}_{i,k})_{,j} + (b_{ijmnq} u_{m,nq})_{,j}] - {}_{\varepsilon}\Sigma_{(kj)i,kj}\} dx\, d\eta. \tag{3.16}$$

Then, we integrate by parts with respect to the spatial variables in the last two terms in (3.16) and use the arithmetic-geometric mean inequality to see that

$$F'' \geq 8 \int_0^t K(\eta) d\eta - 3\varepsilon \int_{D_t} w_{i,j} w_{i,j} dx\, d\eta - \varepsilon \int_{D_t} \{[\varrho m_{jk} \ddot{v}_{i,k} + b_{ijmnq} u_{m,nq}]$$
$$- \varepsilon^{-1} {}_{\varepsilon}\Sigma_{(kj)i,k}\} \{[\varrho m_{jp} \ddot{v}_{i,p} + b_{ijprs} u_{p,rs}] - \varepsilon^{-1} {}_{\varepsilon}\Sigma_{(pj)i,p}\} dx\, d\eta - 2T^2 \varepsilon \int_{D_t} \frac{\partial}{\partial \eta} \{[\varrho m_{jk} \ddot{v}_{i,k}$$
$$+ b_{ijmnq} u_{m,nq}] - \varepsilon^{-1} {}_{\varepsilon}\Sigma_{(kj)i,k}\} \times \frac{\partial}{\partial \eta} \{[\varrho m_{jp} \ddot{v}_{i,p} + b_{ijprs} u_{p,rs}] - \varepsilon^{-1} {}_{\varepsilon}\Sigma_{(pj)i,p}\} dx\, d\eta. \tag{3.17}$$

Finally, we use (3.3), (3.8), (3.9) and the lower bound for m_{ij} in (3.17) to arrive at

$$F'' \geq 8 \int_0^t K(\eta) d\eta - a\mathscr{F}, \tag{3.18}$$

where

$$a = \max\{3m\varrho^{-1}, 1\}. \tag{3.19}$$

To establish the convexity of $\mathcal{G}$, we observe that

$$\mathcal{F}^2\mathcal{G}'' = \mathcal{F}''\mathcal{F} - (\mathcal{F}')^2 + 2c\mathcal{F}^2, \tag{3.20}$$

and so using (3.18),

$$\mathcal{F}^2\mathcal{G}'' \geqslant \left\{8F(t)\int_0^t K(\eta)d\eta - 4\left[\int_{D_t} \varrho(w_i\dot{w}_i + \varepsilon m_{jk} w_{i,j}\dot{w}_{i,k})dx\,d\eta\right]^2\right\} + 8\varepsilon\int_0^t K(\eta)d\eta P^2 + (2c-a)\mathcal{F}^2. \tag{3.21}$$

The first term on the right of (3.21) is non-negative by virtue of the Cauchy-Schwarz inequality. Hence, if we choose $c = a/2$, it follows that

$$\mathcal{G}'' \geqslant 0, \tag{3.22}$$

and so $\mathcal{G}$ is a convex function of t on $(0, T)$. Jensen's inequality may then be used to show that (see PAYNE and SATHER [17], p. 222, for details)

$$F(t) \leqslant M\varepsilon^{1-t/(T-\delta)}, \quad 0 \leqslant t \leqslant T-\delta, \tag{3.23}$$

where M is a positive constant independent of ε and δ is an arbitrary positive constant.

The theorem now follows with the aid of (3.8)

Remarks. 1. If the potential energies in problems $\mathcal{A}$ and $\mathcal{B}$ are coercive bilinear forms then it is possible to establish stronger convergence than that of Theorem 1. Indeed, making use of a Sobolev embedding theorem, convergence can be established in L^6 rather than L^2 measure. In this case a logarithmic convexity approach is unnecessary and a standard energy technique is sufficient.

2. It is possible to present an analogous development of the results in this paper for the more general dipolar theory of GREEN and RIVLIN [7] (see also MINDLIN [14, 15]).

3. It may be possible to use problem $\mathcal{A}$ to establish existence of solutions to problem $\mathcal{B}$ when the elasticities are merely non-negative. This would then be a *naturally occurring* example of the method of quasi-reversibility. See Lattes and Lions [13] and Payne [16].

4. Convergence in the exterior problem

In Sect. 3 we have shown that a solution to the dipolar problem which belongs to a suitably restricted class will converge, in an L^2 sense, to the solution to the analogous problem for a classical linear elastic material. No definiteness assumptions were imposed on the potential energies although the (spatial) domain was restricted to be bounded. In this section we shall establish a similar convergence result when the domain is exterior to a compact set in $\mathbb{R}^3$. However, we find it necessary to impose a semi-definiteness condition on the potential energy appearing in the classical elasticity model and to also restrict the class of dipolar solutions by imposing a weak requirement on the behaviour for large spatial distances, although, again, no definiteness is assumed of the corresponding dipolar potential energy.

The domain Ω is here taken to be the complement in R^3 of the closure of a bounded region Ω_0. The boundary of Ω_0 is denoted by Σ and it is assumed that Σ is smooth enough to permit applications of the divergence theorem.

Let O be a *fixed* origin of coordinates in R^3. We denote by Ω_R^* the open ball, with centre O, radius R, and we denote by Γ_R the boundary of Ω_R^*, $R > 0$ $(R^2 = x_i x_i)$. Moreover, let $\tilde{R}$ be the smallest value of R such that $(\Omega_{\tilde{R}}^*)^- \supseteq \Omega_0^-$. Then, for $R \geqslant \tilde{R}$, we define Ω_R by

(4.1)
$$\Omega_R = \Omega_R^* \setminus \Omega_0^-, \quad R \geqslant \tilde{R}.$$

Finally, we suppose the elasticities of the classical theory, a_{ijkh}, satisfy the following boundedness and definiteness conditions,

(4.2)
$$|a_{ijkh}(x)| \leqslant M, \quad \forall x \in \Omega,$$

(4.3)
$$a_{ijkh}\zeta_{ij}\zeta_{kh} \geqslant 0, \quad \forall \zeta_{ij},$$

where $M(< \infty)$ is a prescribed constant, and the density is bounded below on Ω, i.e., there is a constant $\varrho_m(> 0)$ such that

(4.4)
$$\varrho(x) \geqslant \varrho_m, \quad \text{for all } x \text{ in } \Omega.$$

To establish convergence of solutions of problem $\mathscr{A}$ to solutions of problem $\mathscr{B}$ when Ω is an exterior region we find it necessary to impose a condition on the "extra" terms introduced by the dipolar theory. To state this condition, which essentially restricts the behaviour of the solution to $\mathscr{A}$ as $R \to \infty$, we define ϕ_i by

(4.5)
$$\phi_i = \varrho(m_{pq}\ddot{u}_{i,q})_{,p} + (b_{ijmnq}u_{m,nq})_{,j} - \varepsilon^{-1}{}_{\varepsilon}\Sigma_{(kj)i,kj},$$

and then for each fixed T introduce the function $\mathscr{I}(R)$ where

(4.6)
$$\mathscr{I}(R) = \Big(\int_0^T (T-t) \int_{\Omega_R} e^{-2t}\phi_i\phi_i\,dx\,dt\Big)^{1/2}.$$

We then require $\mathscr{I}$ to satisfy the following condition,

(4.7)
$$\frac{\mathscr{I}(2R)}{R} \to 0 \quad \text{as } R \to \infty.$$

We shall show that (4.7) may be regarded as representing a class of solutions to $\mathscr{A}$ which converge to solutions of $\mathscr{B}$, as $\varepsilon \to 0$ (Theorem 2).

Before stating our main theorem we present a preliminary lemma.

LEMMA 2. *Let H be a non-decreasing function on* $(0, \infty)$ *and let G be a non-decreasing continuously differentiable function on* $(0, \infty)$, *which satisfies the following inequality,*

(4.8)
$$G'(R) - kG(R) + \varepsilon H(R)G^{1/2}(R) \geqslant 0,$$

for constants ε, k (> 0), where $G' \equiv dG/dR$. Then, G satisfies the following estimate,

(4.9)
$$G(R) \leqslant \exp\left(\frac{-kR}{2}\right) + \frac{H^2(2R)\varepsilon^2}{k^2}\left(1 - \exp\left(\frac{-kR}{2}\right)\right).$$

The proof of the lemma is omitted, since it follows easily from the proofs of similar lemmas in [10] and [19], (see also [2]).

The domain $\Omega_R \times [0, T]$ is denoted by D_R.

THEOREM 2. *Let u_i and v_i be classical solutions to problems $\mathcal{A}$ and $\mathcal{B}$, respectively, with a_{ijkh} satisfying* (4.2) *and* (4.3). *Let w_i be defined by*

$$w_i = u_i - v_i. \tag{4.10}$$

Then, if condition (4.7) *holds, $u_i \to v_i$ as $\varepsilon \to 0$, in the following precise sense*

$$\int\limits_{D_{1/\varepsilon}} e^{-2t} \varrho w_i w_i dx dt \to 0, \quad \varepsilon \to 0. \tag{4.11}$$

Proof. The proof of this theorem uses a technique due to GRAFFI [3], which was adapted to elasticity in [10]. Graffi's technique was also further developed by CANNON and KNIGHTLY [2], and the present work employs ideas similar to those used by the above writers.

We start by computing $\mathcal{L} w_i$, using (2.11), (2.12) and (4.5):

$$\mathcal{L} w_i = \varepsilon \phi_i. \tag{4.12}$$

Introducing ψ_i by $\psi_i = e^{-t} w_i$, (4.12) may be rewritten as

$$\mathcal{L} \psi_i + \varrho(2\dot{\psi}_i + \psi_i) = e^{-t} \varepsilon \phi_i. \tag{4.13}$$

These equations are now multiplied by $\dot{\psi}_i$, integrated over Ω_R and the boundary conditions together with the divergence theorem are used to find

$$2 \int\limits_{\Omega_R} \varrho \dot{\psi}_i \dot{\psi}_i dx + \frac{1}{2} \frac{d}{dt} \int\limits_{\Omega_R} \{\varrho(\dot{\psi}_i \dot{\psi}_i + \psi_i \psi_i) + a_{ijkh} \psi_{i,j} \psi_{k,h}\} dx = \int\limits_{\Gamma_R} n_j a_{ijkh} \dot{\psi}_i \psi_{k,h} dS + \varepsilon \int\limits_{\Omega_R} e^{-t} \phi_i \dot{\psi}_i dx. \tag{4.14}$$

The first term on the right of (4.14) is bounded by means of the Cauchy-Schwarz and arithmetic-geometric mean inequalities together with (4.2), (4.3) and (4.4). Thus, we have (cf. [10]),

$$\int\limits_{\Gamma_R} n_j a_{ijkh} \dot{\psi}_i \psi_{k,h} dS \leqslant \frac{3M}{2\varrho_m^{1/2}} \int\limits_{\Gamma_R} (a_{ijkh} \psi_{i,j} \psi_{k,h} + \varrho \dot{\psi}_i \dot{\psi}_i) dS. \tag{4.15}$$

On inserting this estimate into (4.14) and integrating the resulting equation twice with respect to time, after having applied the Cauchy-Schwarz inequality to the last term on the right of (4.14), we obtain

$$\frac{1}{2} \int\limits_{D_R} \{\varrho(\dot{\psi}_i \dot{\psi}_i + \psi_i \psi_i) + a_{ijkh} \psi_{i,j} \psi_{k,h}\} dx dt \leqslant \frac{3TM}{2\varrho_m^{1/2}} \int\limits_0^T \int\limits_{\Gamma_R} (a_{ijkh} \psi_{i,j} \psi_{k,h} + \varrho \dot{\psi}_i \dot{\psi}_i) dS dt + \frac{\varepsilon T^{1/2}}{\varrho_m^{1/2}} \Big(\int\limits_{D_R} \varrho \dot{\psi}_i \dot{\psi}_i dx dt \Big)^{1/2} \Big(\int\limits_0^T (T-t) \int\limits_{\Omega_R} e^{-2t} \phi_i \phi_i dx dt \Big)^{1/2}. \tag{4.16}$$

We define $F(R)$ by

$$F(R) = \int\limits_{D_R} \{\varrho(\psi_i \psi_i + \dot{\psi}_i \dot{\psi}_i) + a_{ijkh} \psi_{i,j} \psi_{k,h}\} dx dt. \tag{4.17}$$

Then, recognizing the term on the right of (4.16) involving ϕ_i is $\mathscr{I}(R)$, as defined by (4.6), we may deduce from (4.16) that

$$(4.18) \qquad F(R) \leqslant \frac{3TM}{\varrho_m^{1/2}} F'(R) + \left(\frac{2T^{1/2}}{\varrho_m^{1/2}}\right) \mathscr{I}(R) \left(F(R)\right)^{1/2},$$

where $F' \equiv dF/dR$.

Lemma 2 is now applied with $k = \varrho_m^{1/2}/3TM$ and $H(R) = (2/3T^{1/2}M)\mathscr{I}(R)$. We therefore obtain

$$(4.19) \qquad F(R) \leqslant \exp\left(\frac{-kR}{2}\right) + \left[1 - \exp\left(\frac{-kR}{2}\right)\right] \frac{4\varepsilon^2}{9TM^2k^2} \mathscr{I}^2(2R).$$

Using (4.17), (4.3) and rewriting (4.19) in terms of w_i, it is easily seen that

$$(4.20) \qquad \int_{D_R} e^{-2t} \varrho w_i w_i \, dx\, dt \leqslant \exp(-\alpha R) + \varepsilon^2 [1 - \exp(-\alpha R)] \beta \mathscr{I}^2(2R),$$

where $\alpha = \varrho_m^{1/2}/6TM$ and $\beta = 4T/\varrho_m$.

Finally, we set $R = 1/\varepsilon$, $(\varepsilon^{-1} \geqslant \tilde{R})$, and (4.11) follows with the aid of (4.7).

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