On the averaged-equation approach to conduction through a suspension

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THIS paper derives the group expansion for the average heat flux through a suspension using the averaged-equation approach to the problem. The equation governing heat conduction is expressed in a form which reflects the structure of the suspension and an ensemble average is then taken. The resulting hierarchy of equations is solved by truncation and the familiar group-expansion solution emerges. This shows that the averaged-equation approach is consistent with earlier approaches.

W pracy niniejszej wyprowadzono rozwinięcie względem grupy dla uśrednionego strumienia ciepła przepływającego przez zawiesinę, wykorzystując dla rozwiązania zagadnienia podejście uśrednionego równania. Równanie rządzące przewodnictwem ciepła wyrażono w postaci odzwierciedlającej strukturę zawiesiny, a następnie utworzono zbiór uśredniony. Otrzymany w ten sposób układ równań rozwiązano przez odrzucenie odpowiednich wyrazów rozwinięcia i uzyskano znane rozwiązanie rozwinięcia względem grupy. Pokazano, że metoda uśrednionego równania jest zgodna z podejściami wcześniejszymi.

В настоящей работе выведено разложение по отношению к группе для усредненного потока тепла протекающего через взвесь, используя для решения задачи подход усредненного уравнения. Уравнение описывающее теплопроводность выражено в виде отображающем структуру взвеси, а затем создано усредненное множество. Полученная таким образом система уравнений решена путем отбрасывания соответствующих членов разложения и получено известное решение разложения по отношению к группе. Показано, что метод усредненного уравнения совпадает с более ранними подходами.

$\overline{\mathbf{F}}$ average heat flux,
$\overline{\mathbf{G}}$ average temperature gradient,
$\overline{\mathbf{S}}$ average dipole strength,
$\mathbf{x}$ point in suspension,
$\mathbf{r}$ position of sphere centre,
$\mathscr{C}_k$ set of vectors $\mathbf{r}_1 \ldots \mathbf{r}_k$,
$\mathscr{C}_{0,k}$ set of vectors $\mathbf{r}_0 \ldots \mathbf{r}_k$,
$\mathscr{P}(\mathscr{C}_k|r_0)$ probability density,
T temperature,
$\langle T \rangle_{0 \ldots k}$ average temperature (angle bracket with subscripts),
$\langle T \rangle_{\mathbf{r}}$ average temperature (angle bracket with $\mathbf{r}$ (bold) subscript),
$\langle T \rangle_{0 \ldots k}^{(N-k)}$ approximate solution,
$\delta(f_i)$ Delta-function on $f_i = 0$.

1. Introduction

THE problem to be considered is the conduction of heat through a homogeneous suspension in which the volume fraction of the particles, which for simplicity are assumed to

be spherical, is low. Two approaches to this problem exist; the first one, due to BATCHELOR [1], has been used by JEFFREY [8] to find complete group expansion for the average heat flux through the suspension. The second one, the averaged-equation approach, has been slower to develop and the object of this paper is to show that it is consistent with the first one by using averaged equations to derive the group expansion for the average heat flux. The reason for doing this is that it seems that the averaged-equation approach is the more powerful and versatile of the two and the first step in proving this is to show that it reproduces existing results. Other work has shown that the averaged-equation approach can be applied to problems for which the group expansions break down (see [2, 5, 6]), thus giving further support to the contention.

2. The hierarchy of averaged equations

Let the particles have conductivity λ_2 and the matrix (fluid) have conductivity λ_1. In the suspension there is a temperature field T which is the result of an applied average gradient $\overline{\mathbf{G}}$; the average heat flux $\overline{\mathbf{F}}$ is required. The averages here are ensemble averages, however, because of the homogeneous statistics, the volume averages of [8] will equal their ensemble-average counterparts here. The aim is to find an equation for $\langle T \rangle$, the ensemble average of T, and more important $\langle T \rangle_0$, the average when a particle (a sphere) is fixed with its centre at $\mathbf{r}_0$. In fact $\langle T \rangle$ must equal $\overline{\mathbf{G}} \cdot \mathbf{x}$ by definition and it is actually $\langle T \rangle_0$ which determines the flux.

The governing equation for any one realisation of the suspension is

(2.1) $$\nabla \cdot (\lambda \nabla T) = 0.$$

This form is not convenient for taking an ensemble average, however, and so some preliminary manipulation must be done. From (2.1),

$$\lambda \nabla^2 T + \nabla \lambda \cdot \nabla T = 0.$$

The $\nabla \lambda$ term is zero except at the surface of a sphere when it has to be represented by a δ-function concentrated on the surface of the sphere. If the surface of a sphere with radius a and with centre at $\mathbf{r}_i$ is given by

$$f_i(\mathbf{x}) = |\mathbf{x} - \mathbf{r}_i| - a = 0,$$

then the δ-function concentrated on $f_i = 0$, written $\delta(f_i)$, is defined by [4]

$$\int g(\mathbf{x}) \delta(f_i) dV = \int_{f_i=0} g(\mathbf{x}) dA,$$

where g is an arbitrary good function and the volume integration is over all space. In terms of this δ-function, (2.1) becomes [9]

(2.2) $$\nabla^2 T = \sum_{i=0}^{\infty} (\alpha - 1) \frac{\partial T}{\partial n} \delta(f_i),$$

where $\alpha = \lambda_2 / \lambda_1$ and $\partial T / \partial n$ is the normal derivative of T calculated from the field inside the sphere.

A few words about this equation. It should be noticed that the left-hand side now contains only T and not, as in (2.1), a product of T and λ. Because of this separation

of T and λ, an equation for $\langle T \rangle$ follows directly from the average of (2.2), in contrast to the equations obtained from averaging (2.1) [3]. It is also possible to look at (2.2) as a Laplace equation; after all the equation for T could equally well have been written as the Laplace equation together with the saltus of the temperature gradient at the surfaces of the particles. Looked at in this way, the δ-functions on the right-hand side of (2.2) have been used to modify the Laplace equation so that a single equation replaces equation-plus-boundary-conditions; each term in the summation represents the presence of one particle.

The ensemble average can now be taken. In the manipulations that follow a convention on the use of $\mathbf{x}$ and $\mathbf{r}$ is observed: $\mathbf{x}$ is used to denote a point in the suspension and $\mathbf{r}$ to denote the position of the centre of a sphere. Averaging (2.2) gives

$$\nabla^2 \langle T \rangle = \left\langle \sum_{i=0}^{\infty} (\alpha - 1) \frac{\partial T}{\partial n} \delta(f_i) \right\rangle.$$

Both sides of this equation remain functions of $\mathbf{x}$. At any point $\mathbf{x}$, the term being averaged on the right-hand side is only non-zero when a sphere is touching that point. Thus the average can be written as an average over the positions of all particles except one, followed by an average over all positions in which this particle touches $\mathbf{x}$. The first part of the average introduces $\langle T \rangle_{\mathbf{r}}$, the average of T with a particle fixed at $\mathbf{r}$, and the second part introduces an average over $\mathbf{r}$. The resulting equation is [9]

$$(2.3) \qquad \nabla^2 \langle T \rangle = (\alpha - 1) \int \frac{\partial}{\partial n} \langle T \rangle_{\mathbf{r}} \mathscr{P}(\mathbf{r}) \delta(f) dV = (\alpha - 1) \int \frac{\partial}{\partial n} \langle T \rangle_{\mathbf{r}} \mathscr{P}(\mathbf{r}) dA(\mathbf{r}),$$

where $\mathscr{P}(\mathbf{r})$ is the probability density for any sphere centre being at $\mathbf{r}$, $f(\mathbf{x}) = 0$ is the surface of the sphere and $dA(\mathbf{r})$ is an element of area on the surface of the sphere. It is obvious from (2.3) that an equation for $\langle T \rangle_{\mathbf{r}}$ will introduce an average of T with two spheres fixed and hence that (2.3) is the first member of an infinite hierarchy of equations. The general equation can be written down by adapting the notation used in [8]. Let the fixed particles be at $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_k$ and let $\mathscr{C}_{0,k}$ be the set of these position vectors. Let $\langle T \rangle_{0,1,\ldots,k}$ be the corresponding average of T, then

$(2.4) \qquad \nabla^2 \langle T \rangle_{0,1,\ldots,k}$

$$= \sum_{i=0}^{k} (\alpha - 1) \frac{\partial}{\partial n} \langle T \rangle_{0,1,\ldots,k} \delta(f_i) + (\alpha - 1) \int \frac{\partial}{\partial n} \langle T \rangle_{0,1,\ldots,k,\mathbf{r}} \mathscr{P}(\mathbf{r}|\mathscr{C}_{0,k}) dA(\mathbf{r}),$$

where the integral is over $f = 0$ as before. The boundary condition at infinity for (2.3) and (2.4) is $\langle T \rangle_{0,1,\ldots,k} \to \langle T \rangle \to \bar{\mathbf{G}} \cdot \mathbf{x}$ as $x \to \infty$. The terms in the summation in (2.4) serve the same purpose as those in (2.2).

3. The expression for the flux and the group-expansion solution

The expression derived in [7] and [8] for the average flux $\bar{\mathbf{F}}$ carries over, with some adaptation, to the present calculation. The expression is

$$(3.1) \qquad\qquad\qquad \bar{\mathbf{F}} = \lambda_1 \bar{\mathbf{G}} + n\bar{\mathbf{S}},$$

where n is the number density of the particles and $\mathbf{S}$ is defined for each particle as

$$\mathbf{S} = (\lambda_2 - \lambda_1) \int \nabla T dV,$$

the integration being over the particle volume. The original approach to the calculation of $\bar{\mathbf{S}}$ proceeded by choosing a reference particle and averaging the value of $\mathbf{S}$ for this particle over all realisations of the suspension while holding the reference particle fixed. In the present notation this is expressed by holding the reference particle fixed at $\mathbf{r}_0$ and calculating $\langle \mathbf{S} \rangle_0$. Thus (3.1) can be used with the present approach, provided $\bar{\mathbf{S}}$ is re-defined as

(3.2) $$\bar{\mathbf{S}} = (\lambda_2 - \lambda_1) \int \nabla \langle T \rangle_0 dV,$$

the integration being over the sphere at $\mathbf{r}_0$. This equation is the bridge between the two approaches. What must now be shown is that the Eqs. (2.4) can be solved for $\langle T \rangle_0$ in such a way that (3.2) yields the group expansion given in [8] as

(3.3) $$\bar{\mathbf{S}} = \sum_{k=0}^{\infty} \int \mathbf{S}_k^*(\mathscr{C}_k; \bar{G}) d\mathscr{C}_k,$$

where

(3.4) $$\mathbf{S}_k^* = \mathbf{S}_k(\mathscr{C}_k; \bar{G}) \mathscr{P}(\mathscr{C}_k|\mathbf{r}_0) - \sum_{i=0}^{k-1} \mathbf{S}_i^*(\mathscr{C}_i; G_{k-i}(\mathscr{C}_{k-i})) \mathscr{P}(\mathscr{C}_{k-i}).$$

The notation is explained in [8], but note that $\mathscr{C}_k$ here and $\mathscr{C}_{0,k}$ in (2.4) are connected by $\mathscr{C}_{0,k} = \mathscr{C}_k \cup \{\mathbf{r}_0\}$. The important point to be remembered for the present is that the $\mathbf{S}_k$ term in (3.4) comes directly from the definition of $\bar{\mathbf{S}}$ and the other terms were introduced through the condition that the average field must be $\bar{G}$ and are needed to make the integrals in (3.3) absolutely convergent.

4. Closure and solution of hierarchy of equations

It turns out that the simplest closure of the hierarchy is all that is required to obtain the group expansion: if, for some N, the $k = N$ equation is closed by omitting the integral term completely, the resulting solution for $\bar{\mathbf{S}}$ is equal to the first N terms of (3.3). The truncated $k = N$ equation is

$$\nabla^2 \langle T \rangle_{0...N} = \sum_{i=0}^{N} (\alpha - 1) \frac{\partial}{\partial n} \langle T \rangle_{0...N} \delta(f_i),$$

which is identical to (2.2) save that the sum stops at N instead of continuing to infinity. Thus the solution of this equation is simply the temperature field around $N+1$ spheres when there is a gradient $\bar{G}$ at infinity; this can be assumed to be known, as it was in [8], and the problem left is to work back down the hierarchy to $\langle T \rangle_0$.

The general approach can be illustrated by truncating the equations at $k = 0$. The equations are then

(4.1) $$\nabla^2 \langle T \rangle = (\alpha - 1) \int \frac{\partial}{\partial n} \langle T \rangle_\mathbf{r} \mathscr{P}(\mathbf{r}) dA(\mathbf{r}),$$

and

$$\nabla^2 \langle T \rangle_0 = (\alpha - 1) \frac{\partial}{\partial n} \langle T \rangle_0 \, \delta(f_0).$$

Multiplying the second equation by $\mathscr{P}(\mathbf{r}_0)$ and integrating with respect to $\mathbf{r}_0$ gives

$$\nabla^2 \int \langle T \rangle_0 \mathscr{P}(\mathbf{r}_0) \, d\mathbf{r}_0 = (\alpha - 1) \int \frac{\partial}{\partial n} \langle T \rangle_0 \mathscr{P}(\mathbf{r}_0) \, \delta(f_0) \, d\mathbf{r}_0.$$

This equation will be the same as (4.1), provided

(4.2) $$\langle T \rangle = \int \langle T \rangle_0 \mathscr{P}(\mathbf{r}_0) \, d\mathbf{r}_0.$$

This procedure of solving one equation by comparing it with another seems the perfect tool until it is realised that the integral with respect to r_0 does not converge ($\langle T \rangle_0$ is $0(|\mathbf{x} - \mathbf{r}_0|^{-2})$ far from $\mathbf{r}_0$). To see this another way, consider the formal solution of the Poisson equation for $\langle T \rangle$,

(4.3) $$\langle T \rangle = \bar{\mathbf{G}} \cdot \mathbf{x} + \int (\alpha - 1) \int \frac{\partial}{\partial n} \langle T \rangle_{\mathbf{r}} \mathscr{P}(\mathbf{r}) \, dA(\mathbf{r}) \frac{dV(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

where the inner integral is now over all spheres touching $\mathbf{x}'$. The essential difference between (4.2) and (4.3) is the order of integration: in (4.2) the $\mathbf{r}$ integral is done last and in (4.3) the $\mathbf{x}'$ integral is done last. It would seem that reversing the order of integration in (4.3) — which is effectively what is done to obtain (4.2) — leaves the integrals non-convergent. The order has to be reversed, however, if the group expansion is going to be derived, because a solution in the form of integrals over $\mathbf{r}$ is needed.

The reason for considering this example first, besides the fact that it lays bare the convergence difficulty, is that (4.1) provides the identity which allows the equations to be modified so that only convergent integrals appear in the solution. It was stated in Sect. 1 that $\langle T \rangle = \bar{G} \cdot \mathbf{x}$ and if this is substituted into (4.1), the equation becomes

$$(\alpha - 1) \int \frac{\partial}{\partial n} \langle T \rangle_{\mathbf{r}} \mathscr{P}(\mathbf{r}) \, dA = 0.$$

This identity is now subtracted from (2.4):

(4.4) $$\nabla^2 \langle T \rangle_{0,\dots,k} = \sum_{i=0}^{k} (\alpha - 1) \frac{\partial}{\partial n} \langle T \rangle_{0,\dots,k} \, \delta(f_i)$$

$$+ (\alpha - 1) \int \left\{ \frac{\partial}{\partial n} \langle T \rangle_{0,\dots,k,\mathbf{r}} \mathscr{P}(\mathbf{r}|\mathscr{C}_{0,k}) - \frac{\partial}{\partial n} \langle T \rangle_{\mathbf{r}} \mathscr{P}(\mathbf{r}) \right\} dA.$$

The Eq. (2.3) is no longer required. In the terminology of [4], (4.4) has been renormalised. In line with the remarks in the last section, the condition that the average gradient is $\bar{\mathbf{G}}$ has been the source of the terms which correct the divergence difficulties. The systematic solution of (4.4) proceeds by considering the hierarchy to be truncated at $k = 0$, $k = 1, \dots k = N$ and $k = N+1$. The notation used for the solutions of the successive systems of equations is

(i) $k = 0$:   the solution for $\langle T \rangle_0$ is $\langle T \rangle_0^{(0)}$;

(ii) $k = 0, 1$:   the solution for $\langle T \rangle_0$ is $\langle T \rangle_0^{(0)} + \langle T \rangle_0^{(1)}$,

and for $\langle T \rangle_{0,1}$ is $\langle T \rangle_{0,1}^{(0)}$;

(iii) $k = 0 \ldots N$: the solution for $\langle T \rangle_0$ is $\langle T \rangle_0^{(0)} + \langle T \rangle_0^{(1)} + \ldots + \langle T \rangle_0^{(N)}$,

for $\langle T \rangle_{0,1}$ is $\langle T \rangle_{0,1}^{(0)} + \ldots + \langle T \rangle_{0,1}^{(N-1)}$,

for $\langle T \rangle_{0 \ldots k}$ is $\langle T \rangle_{0 \ldots k}^{(0)} + \ldots + \langle T \rangle_{0 \ldots k}^{(N-k)}$,

for $\langle T \rangle_{0 \ldots N}$ is $\langle T \rangle_{0 \ldots N}^{(0)}$.

The equation for $\langle T \rangle_{0 \ldots k}^{(N-k)}$ is

$$\nabla^2 \langle T \rangle_{0 \ldots k}^{(N-k)} = \sum_{i=0}^{k} (\alpha - 1) \frac{\partial}{\partial n} \langle T \rangle_{0 \ldots k}^{(N-k)} \delta(f_i)$$

$$+ (\alpha - 1) \int \left\{ \frac{\partial}{\partial n} \langle T \rangle_{0 \ldots k, \mathbf{r}}^{(N-k-1)} \mathscr{P}(\mathbf{r} | \mathscr{C}_{0,k}) - \frac{\partial}{\partial n} \langle T \rangle_{\mathbf{r}}^{(N-k-1)} \mathscr{P}(\mathbf{r}) \right\} dA,$$

with $\langle T \rangle_{0 \ldots k}^{(N-k)} \to 0$ as $x \to \infty$ provided $k < N$.

What has to be shown is

$$(\lambda_2 - \lambda_1) \int \nabla \langle T \rangle_0^{(N)} dV = \int \mathbf{S}_N^*(\mathscr{C}_N; \bar{\mathbf{G}}) d\mathscr{C}_N.$$

The proof is by induction. Suppose $\mathbf{S}_N^*$ is given by (3.4) (the $\mathbf{S}_0^*$ and $\mathbf{S}_1^*$ cases are trivial but have been given in [9]) and suppose now that the equations are truncated at $k = N+1$. All equations retain the same form for $k < N$ with the superscripts increased by 1; the $k = N$ equation gains an integral. The $k = N$ equation can be solved by the comparison method described above

$$\langle T \rangle_{0 \ldots N}^{(0)} + \langle T \rangle_{0 \ldots N}^{(1)} = \int \left\{ \langle T \rangle_{0 \ldots N+1}^{(0)} \mathscr{P}(\mathbf{r}_{N+1} | \mathscr{C}_{0,N}) - \langle T \rangle_{0 \ldots N}^{(0)} (\mathbf{G}_1(\mathbf{r}_{N+1})) \mathscr{P}(\mathbf{r}_{N+1}) \right\} d\mathbf{r}_{N+1}.$$

The notation is adapted from [8]; $\langle T \rangle_{0 \ldots N}^{(0)} (\mathbf{G}_1(\mathbf{r}_{N+1}))$ is the temperature field around the spheres in $\mathscr{C}_{0,N}$ when the field at infinity is $\mathbf{G}_1$. Then since $\langle T \rangle_{0 \ldots N}^{(0)}$ produces the term $\mathbf{S}_N \mathscr{P}(\mathscr{C}_N | \mathbf{r}_0)$ in the expression for $\mathbf{S}_N^*$, $\langle T \rangle_{0 \ldots N}^{(1)}$ will produce terms

$$\mathbf{S}_{N+1} \mathscr{P}(\mathscr{C}_{N+1} | \mathbf{r}_0) - \mathbf{S}_N(\mathscr{C}_N; \mathbf{G}_1(\mathbf{r}_{N+1})) \mathscr{P}(\mathscr{C}_N | \mathbf{r}_0) \mathscr{P}(\mathbf{r}_{N+1})$$

in the expression for $\mathbf{S}_{N+1}^*$. The other contribution to $\mathbf{S}_{N+1}^*$ comes from the replacing of the term $\frac{\partial}{\partial n} \langle T \rangle_{\mathbf{r}}^{(N-k-1)}$ by $\frac{\partial}{\partial n} \langle T \rangle_{\mathbf{r}}^{(N-k)}$ throughout the hierachy. Thus each term $\mathbf{S}^*(\mathbf{G}_{N-i}) \mathscr{P}(\mathscr{C}_{N-i})$ must be replaced by

$$\mathbf{S}_i^*(\mathbf{G}_{N-i+1}) \mathscr{P}(\mathscr{C}_{N-i+1}) - \mathbf{S}_i^*(\mathbf{G}_{N-i}(\mathbf{G}_1(\mathbf{r}_{n+1}))) \mathscr{P}(\mathscr{C}_{N-i}) \mathscr{P}(\mathbf{r}_{N+1}).$$

When these two expressions are added together, the two subtracted terms combine to give $-\mathbf{S}_N^*(\mathscr{C}_N; \mathbf{G}_1) \mathscr{P}(\mathbf{r}_{N+1})$ and $\mathbf{S}_{N+1}^*$ has the same form as $\mathbf{S}_N^*$.

5. Concluding remarks

The properties of the group expansion have been discussed in [8]. The main purpose of this paper has been to establish the averaged-equation approach as an alternative to existing methods. The approach has great flexibility both in the way the equations are

closed and in the way the closed set is closed. The present lucubration has hardly made best use of the possibilities but this was not its purpose. I think that averaged equations have great potential for further development.

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