XVIII.

THE AUXILIARY FUNCTION T FOR TWO THIN LENSES CLOSE TOGETHER IN VACUO, AND FOR A SINGLE THIN LENS IN VACUO

July, 1833.

[Note Book 28, pp. 29–39 (back).]

For a small spheric cap at origin, with curvature $= r_i$,

$$z_i = \tfrac{1}{2} r_i (x_i^2 + y_i^2) + (\tfrac{1}{2} r_i)^3 (x_i^2 + y_i^2)^2;$$

and

$$z_i - p_i x_i - q_i y_i = r_i^{-1} \{1 - \sqrt{1 + p_i^2 + q_i^2}\}$$
$$= -\frac{p_i^2 + q_i^2}{2r_i} + \frac{(p_i^2 + q_i^2)^2}{8r_i}.$$

Therefore for a series of n spheric refracting or reflecting surfaces close together at origin,*

$$T = -\tfrac{1}{2} \Sigma_{(i)1}^{\ n} \frac{(\mu_i \alpha_i - \mu_{i-1}\alpha_{i-1})^2 + (\mu_i \beta_i - \mu_{i-1}\beta_{i-1})^2}{r_i (\mu_i \gamma_i - \mu_{i-1}\gamma_{i-1})}$$
$$+ \tfrac{1}{8} \Sigma_{(i)1}^{\ n} \frac{\{(\mu_i \alpha_i - \mu_{i-1}\alpha_{i-1})^2 + (\mu_i \beta_i - \mu_{i-1}\beta_{i-1})^2\}^2}{r_i (\mu_i \gamma_i - \mu_{i-1}\gamma_{i-1})^3};$$

and

$$V = \mu_n (x_n \alpha_n + y_n \beta_n + z_n \gamma_n) - \mu_0 (x_0 \alpha_0 + y_0 \beta_0 + z_0 \gamma_0) - T.$$

For two thin refracting lenses close together *in vacuo*, with four curved surfaces,†

$$V = x_5 \alpha_4 + y_5 \beta_4 + z_5 \gamma_4 - (x_0 \alpha_0 + y_0 \beta_0 + z_0 \gamma_0)$$
$$+ \frac{(\mu_1 \alpha_1 - \alpha_0)^2 + (\mu_1 \beta_1 - \beta_0)^2}{2r_1 (\mu_1 \gamma_1 - \gamma_0)} + \frac{(\mu_3 \alpha_3 - \alpha_2)^2 + (\mu_3 \beta_3 - \beta_2)^2}{2r_3 (\mu_3 \gamma_3 - \gamma_2)}$$
$$+ \frac{(\alpha_2 - \mu_1 \alpha_1)^2 + (\beta_2 - \mu_1 \beta_1)^2}{2r_2 (\gamma_2 - \mu_1 \gamma_1)} + \frac{(\alpha_4 - \mu_3 \alpha_3)^2 + (\beta_4 - \mu_3 \beta_3)^2}{2r_4 (\gamma_4 - \mu_3 \gamma_3)}$$
$$- \frac{\{(\mu_1 \alpha_1 - \alpha_0)^2 + (\mu_1 \beta_1 - \beta_0)^2\}^2}{8r_1 (\mu_1 \gamma_1 - \gamma_0)^3} - \frac{\{(\mu_3 \alpha_3 - \alpha_2)^2 + (\mu_3 \beta_3 - \beta_2)^2\}^2}{8r_3 (\mu_3 \gamma_3 - \gamma_2)^3}$$
$$- \frac{\{(\alpha_2 - \mu_1 \alpha_1)^2 + (\beta_2 - \mu_1 \beta_1)^2\}^2}{8r_2 (\gamma_2 - \mu_1 \gamma_1)^3} - \frac{\{(\alpha_4 - \mu_3 \alpha_3)^2 + (\beta_4 - \mu_3 \beta_3)^2\}^2}{8r_4 (\gamma_4 - \mu_3 \gamma_3)^3};$$

$$\gamma_0 = \sqrt{1 - \alpha_0^2 - \beta_0^2} = 1 - \tfrac{1}{2}(\alpha_0^2 + \beta_0^2) - \tfrac{1}{8}(\alpha_0^2 + \beta_0^2)^2;$$

$$\gamma_1 = \&c.; \quad \mu_1 \gamma_1 - \gamma_0 = \mu_1 - 1 - \tfrac{1}{2}\mu_1 (\alpha_1^2 + \beta_1^2) + \tfrac{1}{2}(\alpha_0^2 + \beta_0^2);$$

$$(\mu_1 \gamma_1 - \gamma_0)^{-1} = (\mu_1 - 1)^{-1} + \frac{\mu_1 (\alpha_1^2 + \beta_1^2) - (\alpha_0^2 + \beta_0^2)}{2(\mu_1 - 1)^2};$$

* [The thickness of the lenses, although in general of the order of $\alpha_0^2 + \beta_0^2$, is here neglected. The argument used here is that of (K^7) of the Third Supplement (p. 216).]

† [The point x_5, y_5, z_5, is any point on the final ray.]

(July 22, 1833) the approximate conditions of the stationary value* with respect to α_1, α_2, α_3 are

$$\frac{\mu_1\alpha_1-\alpha_0}{r_1}+\frac{\alpha_2-\mu_1\alpha_1}{r_2}=0,\quad \frac{\alpha_2-\mu_1\alpha_1}{r_2(\mu_1-1)}+\frac{\mu_3\alpha_3-\alpha_2}{r_3(\mu_3-1)}=0,\quad \frac{\mu_3\alpha_3-\alpha_2}{r_3}+\frac{\alpha_4-\mu_3\alpha_3}{r_4}=0;$$

that is,

$$\left.\begin{aligned}\mu_1(r_1-r_2)\,\alpha_1&=r_1\alpha_2-r_2\alpha_0\\ \mu_3(r_3-r_4)\,\alpha_3&=r_3\alpha_4-r_4\alpha_2\end{aligned}\right\}$$

and

$$\alpha_2\{r_2(\mu_1-1)-r_3(\mu_3-1)\}=\frac{r_2(\mu_1-1)(r_3\alpha_4-r_4\alpha_2)}{r_3-r_4}-\frac{r_3(\mu_3-1)(r_1\alpha_2-r_2\alpha_0)}{r_1-r_2}$$

$$\because\dagger\ \alpha_2\left\{r_2(\mu_1-1)-r_3(\mu_3-1)+\frac{r_2r_4(\mu_1-1)}{r_3-r_4}+\frac{r_1r_3(\mu_3-1)}{r_1-r_2}\right\}=r_2r_3\left\{\frac{\alpha_4(\mu_1-1)}{r_3-r_4}+\frac{\alpha_0(\mu_3-1)}{r_1-r_2}\right\}$$

$$\because\ \alpha_2\left\{\frac{\mu_1-1}{r_3-r_4}+\frac{\mu_3-1}{r_1-r_2}\right\}=\frac{\alpha_4(\mu_1-1)}{r_3-r_4}+\frac{\alpha_0(\mu_3-1)}{r_1-r_2}$$

$$\because\ \alpha_2\{(\mu_1-1)(r_1-r_2)+(\mu_3-1)(r_3-r_4)\}=\alpha_0(\mu_3-1)(r_3-r_4)+\alpha_4(\mu_1-1)(r_1-r_2),$$

$\because$ finally‡

$$\alpha_2=\frac{\alpha_0P_{,}+\alpha_4P}{P+P_{,}}=\frac{\alpha'P_{,}+\alpha P}{P+P_{,}};\quad \beta_2=\frac{\beta P+\beta'P_{,}}{P+P_{,}}:$$

using P, $P_{,}$, to denote the *powers* of the 1st and 2nd lens. That is,

$$P(\alpha_4-\alpha_2)=P_{,}(\alpha_2-\alpha_0):$$

an equation which probably admits of some simple geometrical enunciation.

$$\mu_1\alpha_1=\alpha_2+\frac{r_2(\alpha_2-\alpha_0)}{r_1-r_2}=\alpha_2+\frac{(\mu_1-1)\,r_2(\alpha_2-\alpha_0)}{P}=\alpha_2+\frac{(\mu_1-1)\,r_2(\alpha_4-\alpha_0)}{P+P_{,}}$$

$$=\alpha_0+\frac{r_1(\alpha_2-\alpha_0)}{r_1-r_2}=\alpha_0+\frac{(\mu_1-1)\,r_1(\alpha_2-\alpha_0)}{P}=\alpha_0+\frac{(\mu_1-1)\,r_1(\alpha_4-\alpha_0)}{P+P_{,}};$$

$$\mu_3\alpha_3=\alpha_4+\frac{r_4(\alpha_4-\alpha_2)}{r_3-r_4}=\alpha_4+\frac{(\mu_3-1)\,r_4(\alpha_4-\alpha_2)}{P_{,}}=\alpha_4+\frac{(\mu_3-1)\,r_4(\alpha_4-\alpha_0)}{P+P_{,}}$$

$$=\alpha_2+\frac{r_3(\alpha_4-\alpha_2)}{r_3-r_4}=\alpha_2+\frac{(\mu_3-1)\,r_3(\alpha_4-\alpha_2)}{P_{,}}=\alpha_2+\frac{(\mu_3-1)\,r_3(\alpha_4-\alpha_0)}{P+P_{,}}$$

$$=\frac{\alpha_4P+\alpha_0P_{,}+(\mu_3-1)\,r_3(\alpha_4-\alpha_0)}{P+P_{,}}=\alpha_4+\frac{(\mu_3-1)\,r_4(\alpha_4-\alpha_0)}{P+P_{,}},\text{ as before:}$$

and similarly

$$\mu_1\beta_1=\beta_0+\frac{(\mu_1-1)\,r_1(\beta_4-\beta_0)}{P+P_{,}};\quad \mu_3\beta_3=\beta_4+\frac{(\mu_3-1)\,r_4(\beta_4-\beta_0)}{P+P_{,}}.$$

* [That is, of T; the function V plays no essential part in this investigation.]

† [In the manuscripts of this period, Hamilton used this symbol $\because$ consistently in the sense of "therefore." The symbol was used in this sense in the seventeenth century by W. Oughtred and J. H. Rahn; see F. Cajori, *History of Mathematical Notations*, vol. I (Chicago, 1928), pp. 190, 211.]

‡ [Hamilton here reverts to his customary notation, in which α', β', γ' refer to the incident ray, α, β, γ to the emergent ray.]

These values of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are to be substituted in V or T. Thus,*

$$-T^{(2)} = \frac{(\mu_1\alpha_1-\alpha_0)^2}{2r_1(\mu_1-1)} - \frac{(\alpha_2-\mu_1\alpha_1)^2}{2r_2(\mu_1-1)} + \frac{(\mu_3\alpha_3-\alpha_2)^2}{2r_3(\mu_3-1)} - \frac{(\alpha_4-\mu_3\alpha_3)^2}{2r_4(\mu_3-1)}$$

$$+ \text{ same function of the } \beta\text{'s}$$

$$= \frac{\frac{1}{2}P\{(\alpha_4-\alpha_0)^2+(\beta_4-\beta_0)^2\}}{(P+P_,)^2} + \frac{\frac{1}{2}P_,\{(\alpha_4-\alpha_0)^2+(\beta_4-\beta_0)^2\}}{(P+P_,)^2}$$

$$= \frac{(\alpha-\alpha')^2+(\beta-\beta')^2}{2(P+P_,)}.$$

Also

$$\Delta\sigma_0 = \mu_1\alpha_1-\alpha_0 = \lambda_0(\alpha-\alpha'),\quad \Delta\tau_0 = \lambda_0(\beta-\beta'),\quad \lambda_0 = \frac{(\mu_1-1)r_1}{P+P_,};$$

$$\Delta\sigma_1 = \alpha_2-\mu_1\alpha_1 = \lambda_1(\alpha-\alpha'),\quad \Delta\tau_1 = \lambda_1(\beta-\beta'),\quad \lambda_1 = -\frac{(\mu_1-1)r_2}{P+P_,};$$

$$\Delta\sigma_2 = \mu_3\alpha_3-\alpha_2 = \lambda_2(\alpha-\alpha'),\quad \Delta\tau_2 = \lambda_2(\beta-\beta'),\quad \lambda_2 = \frac{(\mu_3-1)r_3}{P+P_,};$$

$$\Delta\sigma_3 = \alpha_4-\mu_3\alpha_3 = \lambda_3(\alpha-\alpha'),\quad \Delta\tau_3 = \lambda_3(\beta-\beta'),\quad \lambda_3 = -\frac{(\mu_3-1)r_4}{P+P_,};$$

and

$$\Delta v_0 = \mu_1\gamma_1-\gamma_0 = \mu_1-1-\tfrac{1}{2}\mu_1(\alpha_1^2+\beta_1^2)+\tfrac{1}{2}(\alpha_0^2+\beta_0^2)$$

$$= \mu_1-1-\frac{(\alpha_0+\Delta\sigma_0)^2+(\beta_0+\Delta\tau_0)^2}{2\mu_1}+\frac{\alpha_0^2+\beta_0^2}{2}$$

$$= (\mu_1-1)\left\{1+\frac{\alpha_0^2+\beta_0^2}{2\mu_1}-\frac{\alpha_0\Delta\sigma_0+\beta_0\Delta\tau_0}{\mu_1(\mu_1-1)}-\frac{(\Delta\sigma_0)^2+(\Delta\tau_0)^2}{2\mu_1(\mu_1-1)}\right\};$$

$$\Delta v_1 = \gamma_2-\mu_1\gamma_1 = 1-\mu_1-\tfrac{1}{2}(\alpha_2^2+\beta_2^2)+\tfrac{1}{2}\mu_1(\alpha_1^2+\beta_1^2)$$

$$= -(\mu_1-1)\left\{1+\frac{\alpha_2^2+\beta_2^2}{2(\mu_1-1)}-\frac{\mu_1(\alpha_1^2+\beta_1^2)}{2(\mu_1-1)}\right\};$$

$$\Delta v_2 = \mu_3\gamma_3-\gamma_2 = (\mu_3-1)\left\{1-\frac{\mu_3(\alpha_3^2+\beta_3^2)}{2(\mu_3-1)}+\frac{\alpha_2^2+\beta_2^2}{2(\mu_3-1)}\right\};$$

$$\Delta v_3 = \gamma_4-\mu_3\gamma_3 = -(\mu_3-1)\left\{1-\frac{\mu_3(\alpha_3^2+\beta_3^2)}{2(\mu_3-1)}+\frac{\alpha_4^2+\beta_4^2}{2(\mu_3-1)}\right\}:$$

or better thus: change μ_1, μ_3 to $\mu, \mu_,$; and put

$$\alpha_4^2+\beta_4^2 = \epsilon,\quad \alpha_0^2+\beta_0^2 = \epsilon',\quad \alpha_4\alpha_0+\beta_4\beta_0 = \epsilon_,;$$

then

$$(\alpha_4-\alpha_0)^2+(\beta_4-\beta_0)^2 = \epsilon-2\epsilon_,+\epsilon';$$

$$\Delta\sigma_0^2+\Delta\tau_0^2 = \lambda_0^2(\epsilon-2\epsilon_,+\epsilon');\quad \Delta\sigma_2^2+\Delta\tau_2^2 = \lambda_2^2(\epsilon-2\epsilon_,+\epsilon');$$

$$\Delta\sigma_1^2+\Delta\tau_1^2 = \lambda_1^2(\qquad\qquad);\quad \Delta\sigma_3^2+\Delta\tau_3^2 = \lambda_3^2(\qquad\qquad);$$

$$\gamma_0 = 1-\tfrac{1}{2}(\alpha_0^2+\beta_0^2) = 1-\tfrac{1}{2}\epsilon';\quad \gamma_4 = 1-\tfrac{1}{2}(\alpha_4^2+\beta_4^2) = 1-\tfrac{1}{2}\epsilon;$$

* [$T^{(2)}$ denotes the part of the expansion of T which is of the second order in $\alpha, \beta, \alpha', \beta'$.]

$$\mu_1\gamma_1 = \mu - \frac{\mu}{2}(\alpha_1^2 + \beta_1^2) = \mu - \frac{1}{2\mu}\left\{\overline{\alpha' + \lambda_0(\alpha - \alpha')}^2 + \overline{\beta' + \lambda_0(\beta - \beta')}^2\right\}$$

$$= \mu - \frac{1}{2\mu}\{\epsilon' + 2\lambda_0(\epsilon_, - \epsilon') + \lambda_0^2(\epsilon - 2\epsilon_, + \epsilon')\};$$

$$\gamma_2 = 1 - \tfrac{1}{2}(\alpha_2^2 + \beta_2^2) = 1 - \frac{P^2\epsilon + 2PP_,\epsilon_, + P_,^2\epsilon'}{2(P + P_,)^2};$$

$$\mu_3\gamma_3 = \mu_, - \frac{\mu_,}{2}(\alpha_3^2 + \beta_3^2) = \mu_, - \frac{1}{2\mu_,}\{(\alpha - \lambda_3\overline{\alpha - \alpha'})^2 + (\beta - \lambda_3\overline{\beta - \beta'})^2\}$$

$$= \mu_, - \frac{1}{2\mu_,}\{\epsilon - 2\lambda_3(\epsilon - \epsilon_,) + \lambda_3^2(\epsilon - 2\epsilon_, + \epsilon')\}:$$

$$\therefore \left\{\begin{aligned}
\Delta v_0 &= \mu_1\gamma_1 - \gamma_0 = (\mu - 1)\left\{1 + \frac{\epsilon'}{2\mu} - \frac{\lambda_0(\epsilon_, - \epsilon')}{\mu(\mu-1)} - \frac{\lambda_0^2(\epsilon - 2\epsilon_, + \epsilon')}{2\mu(\mu-1)}\right\};\\
\Delta v_1 &= \gamma_2 - \mu_1\gamma_1 = -(\mu - 1)\left\{1 + \frac{P^2\epsilon + 2PP_,\epsilon_, + P_,^2\epsilon'}{2(\mu-1)(P+P_,)^2}\right.\\
&\qquad\left. - \frac{\epsilon' + 2\lambda_0(\epsilon_, - \epsilon') + \lambda_0^2(\epsilon - 2\epsilon_, + \epsilon')}{2\mu(\mu-1)}\right\};\\
\Delta v_2 &= \mu_3\gamma_3 - \gamma_2 = (\mu_, - 1)\left\{1 + \frac{P^2\epsilon + 2PP_,\epsilon_, + P_,^2\epsilon'}{2(\mu_, - 1)(P+P_,)^2}\right.\\
&\qquad\left. - \frac{\epsilon - 2\lambda_3(\epsilon - \epsilon_,) + \lambda_3^2(\epsilon - 2\epsilon_, + \epsilon')}{2\mu_,(\mu_, - 1)}\right\};\\
\Delta v_3 &= \gamma_4 - \mu_3\gamma_3 = -(\mu_, - 1)\left\{1 + \frac{\epsilon}{2\mu_,} + \frac{\lambda_3(\epsilon - \epsilon_,)}{\mu_,(\mu_, - 1)} - \frac{\lambda_3^2(\epsilon - 2\epsilon_, + \epsilon')}{2\mu_,(\mu_, - 1)}\right\}:
\end{aligned}\right.$$

$$T = -\tfrac{1}{2}\Sigma_{(i)}{}_1^4\frac{(\Delta\sigma_{i-1})^2 + (\Delta\tau_{i-1})^2}{r_i\Delta v_{i-1}} + \tfrac{1}{8}\Sigma_{(i)}{}_1^4\frac{\{(\Delta\sigma_{i-1})^2 + (\Delta\tau_{i-1})^2\}^2}{r_i(\Delta v_{i-1})^3};$$

$$r_1(\mu - 1) = \lambda_0(P + P_,); \quad -r_4(\mu_, - 1) = \lambda_3(P + P_,);$$
$$-r_2(\mu - 1) = \lambda_1(P + P_,); \quad r_3(\mu_, - 1) = \lambda_2(P + P_,);$$
$$\Delta v_0 = (\mu - 1)(1 - \xi_0); \quad \Delta v_1 = -(\mu - 1)(1 - \xi_1);$$
$$\Delta v_2 = (\mu_, - 1)(1 - \xi_2); \quad \Delta v_3 = -(\mu_, - 1)(1 - \xi_3);$$
$$r_1\Delta v_0 = \lambda_0(P + P_,)(1 - \xi_0); \quad r_2\Delta v_1 = \lambda_1(P + P_,)(1 - \xi_1);$$
$$r_3\Delta v_2 = \lambda_2(P + P_,)(1 - \xi_2); \quad r_4\Delta v_3 = \lambda_3(P + P_,)(1 - \xi_3):$$

therefore observing that $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1$, we get as before*

$$T^{(2)} = -\frac{(\epsilon - 2\epsilon_, + \epsilon')}{2(P + P_,)},$$

and

$$T^{(4)} = -\frac{(\lambda_0\xi_0 + \lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3)(\epsilon - 2\epsilon_, + \epsilon')}{2(P + P_,)}$$

$$+ \frac{(\epsilon - 2\epsilon_, + \epsilon')^2}{8(P + P_,)}\left\{\frac{\lambda_0^3 + \lambda_1^3}{(\mu - 1)^2} + \frac{\lambda_2^3 + \lambda_3^3}{(\mu_, - 1)^2}\right\}:$$

* [For a justification of the method of approximation employed, see Appendix, Note 24, p. 507.]

in which

$$\frac{\lambda_0^3+\lambda_1^3}{(\mu-1)^2}=\frac{(\mu-1)(r_1^3-r_2^3)}{(P+P_,)^3}=\frac{P(r_1^2+r_1r_2+r_2^2)}{(P+P_,)^3},$$

$$\frac{\lambda_2^3+\lambda_3^3}{(\mu_,-1)^2}=\frac{(\mu_,-1)(r_3^3-r_4^3)}{(P+P_,)^3}=\frac{P_,(r_3^2+r_3r_4+r_4^2)}{(P+P_,)^3};$$

$$-\lambda_0\xi_0=\frac{\lambda_0\epsilon'}{2\mu}-\frac{\lambda_0^2(\epsilon_,-\epsilon')}{\mu(\mu-1)}-\frac{\lambda_0^3(\epsilon-2\epsilon_,+\epsilon')}{2\mu(\mu-1)},$$

$$-\lambda_1\xi_1=-\frac{\lambda_1\epsilon'}{2\mu(\mu-1)}-\frac{\lambda_1\lambda_0(\epsilon_,-\epsilon')}{\mu(\mu-1)}-\frac{\lambda_1\lambda_0^2(\epsilon-2\epsilon_,+\epsilon')}{2\mu(\mu-1)}$$

$$+\frac{\lambda_1}{2(\mu-1)}\{(\lambda_1+\lambda_0)^2\epsilon+2(\lambda_1+\lambda_0)(\lambda_3+\lambda_2)\epsilon_,+(\lambda_3+\lambda_2)^2\epsilon'\},$$

$$-\lambda_2\xi_2=\frac{\lambda_2}{2(\mu_,-1)}\{(\lambda_1+\lambda_0)^2\epsilon+\qquad\qquad+\qquad\qquad\}$$

$$-\frac{\lambda_2\epsilon}{2\mu_,(\mu_,-1)}+\frac{\lambda_2\lambda_3(\epsilon-\epsilon_,)}{\mu_,(\mu_,-1)}-\frac{\lambda_2\lambda_3^2(\epsilon-2\epsilon_,+\epsilon')}{2\mu_,(\mu_,-1)},$$

$$-\lambda_3\xi_3=\frac{\lambda_3\epsilon}{2\mu_,}+\frac{\lambda_3^2(\epsilon-\epsilon_,)}{\mu_,(\mu_,-1)}-\frac{\lambda_3^3(\epsilon-2\epsilon_,+\epsilon')}{2\mu_,(\mu_,-1)};$$

therefore since $\lambda_1=-\lambda_0+\frac{P}{P+P_,}$, and $\lambda_2=-\lambda_3+\frac{P_,}{P+P_,}$, we have

$$-2(\lambda_0\xi_0+\lambda_1\xi_1+\lambda_2\xi_2+\lambda_3\xi_3)=\frac{\lambda_0\epsilon'}{\mu-1}-\frac{2\lambda_0P(\epsilon_,-\epsilon')}{\mu(\mu-1)(P+P_,)}-\frac{\lambda_0^2P(\epsilon-2\epsilon_,+\epsilon')}{\mu(\mu-1)(P+P_,)}$$

$$-\frac{P\epsilon'}{\mu(\mu-1)(P+P_,)}+\left\{\frac{1}{\mu-1}\left(\frac{P}{P+P_,}-\lambda_0\right)+\frac{1}{\mu_,-1}\left(\frac{P_,}{P+P_,}-\lambda_3\right)\right\}\frac{P^2\epsilon+2PP_,\epsilon_,+P_,^2\epsilon'}{(P+P_,)^2}$$

$$-\frac{P_,\epsilon}{\mu_,(\mu_,-1)(P+P_,)}+\frac{\lambda_3\epsilon}{\mu_,-1}+\frac{2\lambda_3P_,(\epsilon-\epsilon_,)}{\mu_,(\mu_,-1)(P+P_,)}-\frac{\lambda_3^2P_,(\epsilon-2\epsilon_,+\epsilon')}{\mu_,(\mu_,-1)(P+P_,)}.$$

If we put

$$T^{(4)}=Q\epsilon^2+Q_,\epsilon\epsilon_,+Q'\epsilon\epsilon'+Q_{,,}\epsilon_,^2+Q_,'\epsilon_,\epsilon'+Q''\epsilon'^2,$$

then

$$Q=\frac{P(r_1^2+r_1r_2+r_2^2)+P_,(r_3^2+r_3r_4+r_4^2)}{8(P+P_,)^4}$$

$$-\frac{1}{2(P+P_,)}\left\{\lambda_0\frac{\delta\xi_0}{\delta\epsilon}+\lambda_1\frac{\delta\xi_1}{\delta\epsilon}+\lambda_2\frac{\delta\xi_2}{\delta\epsilon}+\lambda_3\frac{\delta\xi_3}{\delta\epsilon}\right\};$$

$$Q_,=-\frac{P(r_1^2+r_1r_2+r_2^2)+P_,(r_3^2+r_3r_4+r_4^2)}{2(P+P_,)^4}$$

$$+\frac{1}{P+P_,}\left\{\lambda_0\frac{\delta\xi_0}{\delta\epsilon}+\qquad+\qquad+\qquad\right\}$$

$$-\frac{1}{2(P+P_,)}\left\{\lambda_0\frac{\delta\xi_0}{\delta\epsilon_,}+\qquad+\qquad+\qquad\right\};\ \&c.$$

July 23d.) Passing to a single lens, $r_3=0$, $r_4=0$, therefore $\lambda_2=0$, $\lambda_3=0$, $\lambda_0+\lambda_1=1$, $P_{,}=0$,

$$-2(\lambda_0\xi_0+\lambda_1\xi_1)=\frac{\lambda_0\epsilon'}{\mu-1}-\frac{2\lambda_0(\epsilon_{,}-\epsilon')}{\mu(\mu-1)}-\frac{\lambda_0^2(\epsilon-2\epsilon_{,}+\epsilon')}{\mu(\mu-1)}-\frac{\epsilon'}{\mu(\mu-1)}+\frac{\epsilon\lambda_1}{\mu-1}$$

$$=\frac{r_1}{P\mu}\left\{\mu\epsilon'-2(\epsilon_{,}-\epsilon')-\frac{(\mu-1)r_1(\epsilon-2\epsilon_{,}+\epsilon')}{P}-\frac{\epsilon'P}{(\mu-1)r_1}-\frac{\epsilon r_2\mu}{r_1}\right\};$$

$$Q=\frac{r_1^2+r_1r_2+r_2^2}{8P^3}-\frac{(\mu-1)r_1^2}{4\mu P^3}-\frac{r_2}{4P^2};$$

$$Q_{,}=-\frac{r_1^2+r_1r_2+r_2^2}{2P^3}+\frac{(\mu-1)r_1^2}{2\mu P^3}+\frac{r_2}{2P^2}-\frac{r_1}{2\mu P^2}+\frac{(\mu-1)r_1^2}{2\mu P^3};$$

and at the same time $r_2=r_1-\dfrac{P}{\mu-1}$, therefore*

$$Q=\frac{r_1^2}{4P^3}\left(\tfrac{1}{2}+\frac{1}{\mu}\right)-\frac{r_1}{4P^2}\left(1+\frac{\frac{3}{2}}{\mu-1}\right)+\frac{1+2(\mu-1)}{8P(\mu-1)^2}$$

$$=\frac{(\mu+2)r_1^2}{8\mu P^3}-\frac{r_1(2\mu+1)}{8P^2(\mu-1)}+\frac{2\mu-1}{8P(\mu-1)^2};$$

$$Q_{,}=\frac{r_1^2}{P^3}\left(-\tfrac{3}{2}+\frac{\mu-1}{\mu}\right)+\frac{r_1}{2P^2}\left(\frac{3}{\mu-1}+1-\frac{1}{\mu}\right)-\frac{\mu}{2P(\mu-1)^2}$$

$$=-\frac{(\mu+2)r_1^2}{2\mu P^3}+\frac{r_1(\mu^2+\mu+1)}{2P^2\mu(\mu-1)}-\frac{\mu}{2P(\mu-1)^2};\ \text{\&c.}$$

July 27th.) For this case of a SINGLE THIN LENS μ, IN VACUO, AT ORIGIN,

$$V=x\alpha+y\beta+z\gamma-x'\alpha'-y'\beta'-z'\gamma'-T^{(2)}-T^{(4)},$$

$$T^{(2)}+T^{(4)}=-\frac{(\mu\alpha_1-\alpha')^2+(\mu\beta_1-\beta')^2}{2r_1(\mu\gamma_1-\gamma')}+\frac{(\mu\alpha_1-\alpha)^2+(\mu\beta_1-\beta)^2}{2r_2(\mu\gamma_1-\gamma)}$$
$$+\frac{\{(\mu\alpha_1-\alpha')^2+(\mu\beta_1-\beta')^2\}^2}{8r_1(\mu-1)^3}-\frac{\{(\mu\alpha_1-\alpha)^2+(\mu\beta_1-\beta)^2\}^2}{8r_2(\mu-1)^3};$$

$$\frac{-\mu\alpha_1+\alpha'}{r_1}+\frac{\mu\alpha_1-\alpha}{r_2}=0,\quad \alpha_1=\frac{\dfrac{\alpha}{r_2}-\dfrac{\alpha'}{r_1}}{\mu\left(\dfrac{1}{r_2}-\dfrac{1}{r_1}\right)}=\frac{\alpha r_1-\alpha' r_2}{\mu(r_1-r_2)};$$

$$\mu\alpha_1-\alpha'=\frac{(\alpha-\alpha')r_1}{r_1-r_2},\quad \mu\alpha_1-\alpha=\frac{(\alpha-\alpha')r_2}{r_1-r_2};$$

similarly for the β's; therefore putting as before

$$\epsilon=\alpha^2+\beta^2,\quad \epsilon_{,}=\alpha\alpha'+\beta\beta',\quad \epsilon'=\alpha'^2+\beta'^2,$$

* [These expressions have been corrected: in the first line, the MS. reads $-\frac{\frac{3}{2}}{\mu-1}$ instead of $+\frac{\frac{3}{2}}{\mu-1}$, and, in the second, $(2\mu-5)$ instead of $(2\mu+1)$. The compact method which follows is independent of these results.]

we have

$$\left.\begin{aligned}(\mu\alpha_1-\alpha')^2+(\mu\beta_1-\beta')^2&=\frac{r_1^2(\epsilon-2\epsilon_,+\epsilon')}{(r_1-r_2)^2}\\(\mu\alpha_1-\alpha)^2+(\mu\beta_1-\beta)^2&=\frac{r_2^2(\epsilon-2\epsilon_,+\epsilon')}{(r_1-r_2)^2}\end{aligned}\right\}\ \because\ T^{(2)}=\frac{-(\epsilon-2\epsilon_,+\epsilon')}{2(\mu-1)(r_1-r_2)}:$$

$$\left\{T^{(4)}-\frac{(r_1^2+r_1r_2+r_2^2)(\epsilon-2\epsilon_,+\epsilon')^2}{8(\mu-1)^3(r_1-r_2)^3}\right\}\div\frac{\epsilon-2\epsilon_,+\epsilon'}{4(\mu-1)^2(r_1-r_2)^2}$$

$$=2r_1(\overline{\mu\gamma_1-\gamma'}-\overline{\mu-1})-2r_2(\overline{\mu\gamma_1-\gamma}-\overline{\mu-1})$$

$$=r_1\{\epsilon'-\mu(\alpha_1^2+\beta_1^2)\}-r_2\{\epsilon-\mu(\alpha_1^2+\beta_1^2)\}$$

$$=r_1\epsilon'-r_2\epsilon-\mu(r_1-r_2)(\alpha_1^2+\beta_1^2)$$

$$=r_1\epsilon'-r_2\epsilon-\frac{\epsilon r_1^2-2\epsilon_,r_1r_2+\epsilon' r_2^2}{\mu(r_1-r_2)};$$

$$\therefore\ \frac{8(\mu-1)^3(r_1-r_2)^3T^{(4)}}{\epsilon-2\epsilon_,+\epsilon'}$$

$$=(r_1^2+r_1r_2+r_2^2)(\epsilon-2\epsilon_,+\epsilon')+2(\mu-1)(r_1-r_2)(r_1\epsilon'-r_2\epsilon)$$

$$-2(1-\mu^{-1})(\epsilon r_1^2-2\epsilon_,r_1r_2+\epsilon' r_2^2)$$

$$=\epsilon\{r_1^2(-1+2\mu^{-1})+r_1r_2(-2\mu+3)+r_2^2(2\mu-1)\}-2\epsilon_,\{r_1^2+r_1r_2(-1+2\mu^{-1})+r_2^2\}$$

$$+\epsilon'\{r_1^2(2\mu-1)+r_1r_2(-2\mu+3)+r_2^2(-1+2\mu^{-1})\};$$

therefore putting

$$T^{(4)}=\epsilon^2Q+\epsilon\epsilon_,Q_,+\epsilon\epsilon'Q'+\epsilon_,^2Q_{,,}+\epsilon_,\epsilon'Q_,'+\epsilon'^2Q'',$$

we have*

$$\left.\begin{aligned}Q&=\frac{r_1^2(-\mu+2)+r_1r_2(-2\mu^2+3\mu)+r_2^2(2\mu^2-\mu)}{8\mu(\mu-1)^3(r_1-r_2)^3};\\Q_,&=\frac{r_1^2+r_1r_2(-\mu^2+\mu+1)+r_2^2\mu^2}{-2\mu(\mu-1)^3(r_1-r_2)^3};\\Q'&=\frac{r_1^2(\mu^2-\mu+1)+r_1r_2(-2\mu^2+3\mu)+r_2^2(\mu^2-\mu+1)}{4\mu(\mu-1)^3(r_1-r_2)^3};\\Q_{,,}&=\frac{r_1^2\mu+r_1r_2(-\mu+2)+r_2^2\mu}{2\mu(\mu-1)^3(r_1-r_2)^3};\\Q_,'&=\frac{r_1^2\mu^2+r_1r_2(-\mu^2+\mu+1)+r_2^2}{-2\mu(\mu-1)^3(r_1-r_2)^3};\\Q''&=\frac{r_1^2(2\mu^2-\mu)+r_1r_2(-2\mu^2+3\mu)+r_2^2(-\mu+2)}{8\mu(\mu-1)^3(r_1-r_2)^3}.\end{aligned}\right\}\ \text{Single thin lens } \textit{in vacuo.}$$

Let p be power; $p=(\mu-1)(r_1-r_2)$: then

$$T^{(2)}=-\frac{1}{2p}(\epsilon-2\epsilon_,+\epsilon');$$

[The manuscript ends at this point.]

* [The expression for Q'' has been corrected; the MS. reads "3" instead of "3μ."]