XVI.

OPTICAL INVESTIGATIONS (1831)*

(*March* 11.) *Review of some sheets of optical investigations, begun Feb.* 2, 1831, *respecting the aberrations and caustic surfaces of the system of rays, produced by ordinary refraction of spheric lenses having a common axis of revolution, the first incident rays having diverged from a point not far from that axis.*

On the First Sheet, I have given an approximate integral of the partial differential equation

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = \mu^2;$$

this integral represents the Characteristic Function of an ordinary refracted system, with index μ, produced by a given refracting spheric surface, the incident rays having been parallel to each other *in vacuo*, perpendicular to the axis of y, and slightly inclined to the axis of z.

In the First Sheet, fourth page, we supposed that the rays of the refracted system emerged into a vacuum through a second refracting spheric surface infinitely near the first, and having, like it, the axis of z for a diameter; and to the end of the Third Sheet we discussed this and similar problems, with a view to obtain the approximate development of the Characteristic Function V, and thereby an approximate integral of the partial differential equation

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = \mu^2$$

applicable to each problem.

We had now found the following general form for the approximate integral, in all problems of the present class, supposing the last $\mu = 1$, and representing by $x_,$, $y_,$, $z_,$ rectangular coordinates such that the axis of $z_,$ is the axis of the lens and $y_,$ is perpendicular to the diametral plane of the lens and system,

$$\begin{aligned} V = z_, + x_,\alpha_0 - \frac{z_,\alpha_0^2}{2} + \frac{(x_, - z_,\alpha_0)^2 + y_,^2}{2(z_, + A)} - \frac{z_,\alpha_0^4}{8} - \frac{z_,\alpha_0^3(x_, - z_,\alpha_0)}{2(z_, + A)} \\ - \frac{3\alpha_0^2(x_, - z_,\alpha_0)^2(z_, + \mathrm{a})}{4(z_, + A)^2} - \frac{\alpha_0^2 y_,^2 (z_, + \mathrm{b})}{4(z_, + A)^2} \\ - \frac{\alpha_0(x_, - z_,\alpha_0)\{(x_, - z_,\alpha_0)^2 + y_,^2\}(z_, + \mathrm{c})}{2(z_, + A)^3} - \frac{\{(x_, - z_,\alpha_0)^2 + y_,^2\}^2(z_, + \mathrm{e})}{8(z_, + A)^4}; \end{aligned}$$

in which α_0 is a small constant, namely the sine of the inclination of the ray through the origin to the axis of the lens; and A, a, b, c, e are constants connected so with the development of the function W,†

$$W = \tfrac{1}{2}A(\alpha_,^2 + \beta_,^2) + \tfrac{1}{8}\mathrm{e}(\alpha_,^2 + \beta_,^2)^2 + \tfrac{1}{4}(3\mathrm{a} - 2A)\alpha_0^2\alpha_,^2 + \tfrac{1}{4}\mathrm{b}\,\alpha_0^2\beta_,^2 + \tfrac{1}{2}(\mathrm{c} - A)\alpha_0\alpha_,(\alpha_,^2 + \beta_,^2).$$

* [See Appendix, Note 23, p. 506.]

† [It must be remembered that at this time the point of view of the Third Supplement had not been developed: that is, W is still to be regarded as a function of final direction, rather than of initial point and final direction. The initial coordinates do not appear explicitly. The form here given for W does not depend on the facts that the surfaces are spherical and close together: it can be deduced from symmetry alone. The expression for V can then be deduced by eliminating $\alpha_,$, $\beta_,$ between $V + W = \alpha_, x'' + \beta_, y'' + \gamma_, z''$ and the two first equations on the next page, and then transforming from x'', y'', z'' to $x_,$, $y_,$, $z_,$.

We had also found rules for computing the values of these constants, for each successive system, and specially for calculating their changes produced by passing through an infinitely thin lens *in vacuo*.* In the expression for $W, \alpha_,$ is the cosine of the angle which a variable ray makes with a perpendicular to the given ray that passes through the origin; $\beta_,$ is the cosine of the angle which the same variable ray makes with the axis of $y_,$, which also is perpendicular to the same given ray; and if we take new rectangular axes of coordinates, x'', y'', z'', having the same origin as $x_,, y_,, z_,$, and such that $\alpha_,, \beta_,, \sqrt{1-\alpha_,^2-\beta_,^2}$ are the cosines of the angles which they make with the variable ray (y'' coinciding with $y_,$, and z'' with the ray from the origin) we shall have for the equations of this variable ray

$$x''=\frac{\alpha_, z''}{\sqrt{1-\alpha_,^2-\beta_,^2}}+\frac{dW}{d\alpha_,}, \quad y''=\frac{\beta_, z''}{\sqrt{1-\alpha_,^2-\beta_,^2}}+\frac{dW}{d\beta_,},$$

which equations contain the whole theory of the aberrations and the caustic surfaces.

The equations of the ray may be thus written,

$$x''=\alpha_,(z''+A)+\tfrac{1}{2}\alpha_,(z''+\mathrm{e})(\alpha_,^2+\beta_,^2)+\tfrac{1}{2}(3\mathrm{a}-2A)\alpha_0^2\alpha_,+\tfrac{1}{2}(\mathrm{c}-A)(3\alpha_,^2+\beta_,^2)\alpha_0,$$

$$y''=\beta_,(z''+A)+\tfrac{1}{2}\beta_,(z''+\mathrm{e})(\alpha_,^2+\beta_,^2)+\tfrac{1}{2}\mathrm{b}\alpha_0^2\beta_,+(\mathrm{c}-A)\alpha_0\alpha_,\beta_,.$$

In transforming these equations I made use of the following suppositions and changes of notation: I put

$$\mathrm{e}-A=2E; \quad 3\mathrm{a}-2A=2F; \quad \mathrm{b}=2G; \quad \mathrm{c}-A=2H;$$

$$K=\tfrac{1}{4}E(\alpha_,^2+\beta_,^2)^2+\tfrac{1}{2}\alpha_0^2(F\alpha_,^2+G\beta_,^2)+H\alpha_0\alpha_,(\alpha_,^2+\beta_,^2);$$

and therefore

$$x''=\alpha_,(z''+A)\left(1+\frac{\alpha_,^2+\beta_,^2}{2}\right)+\frac{dK}{d\alpha_,}; \quad y''=\beta_,(z''+A)\left(1+\frac{\alpha_,^2+\beta_,^2}{2}\right)+\frac{dK}{d\beta_,}.$$

Next I took a new origin and a new unit, making

$$-E=1;$$

$$x''+\tfrac{1}{2}H\alpha_0^3(G-F)=x'''; \quad y''=y'''; \quad z''+A+2\alpha_0^2\left(H^2+\frac{G+F}{4}\right)=z''';$$

* [Hamilton here refers to formulæ established on the Third Sheet, second page, of the MS. These formulæ, in which a is written instead of α_0, are as follows:]

$$\Delta'' a=0; \quad \Delta''\frac{1}{A}=(\mu-1)\left(\frac{1}{r'}-\frac{1}{r}\right)=\text{power of lens, taken negatively};$$

$$\Delta''\frac{\mathrm{a}}{A^2}=-\frac{1}{3\mu}\Delta''\frac{1}{A}; \quad \Delta''\frac{\mathrm{b}}{A^2}=-\frac{1}{\mu}\Delta''\frac{1}{A}; \quad \Delta''\frac{\mathrm{c}}{A^3}=-\frac{1}{\mu}\left(\frac{1}{A}+\frac{1}{r}+\frac{\mu}{r'}\right)\Delta''\frac{1}{A};$$

$$\Delta''\frac{\mathrm{e}}{A^4}=-\frac{1}{\mu}\left\{\frac{2}{A^2}+\frac{4}{A}\left(\frac{1}{r}+\frac{\mu}{r'}\right)+\frac{2-\mu}{r^2}+\frac{\mu(3-2\mu)}{rr'}+\frac{\mu(2\mu-1)}{r'^2}\right\}\Delta''\frac{1}{A}.$$

These formulæ relate to the change in the coefficients of the Characteristic Function, produced by passage through an infinitely thin spheric lens *in vacuo*, of refractive index μ, and of radii r, r' (positive when convexities are turned towards incident rays). [These formulæ are deduced without difficulty from the general form of V given above (writing $\frac{V}{\mu}$ instead of V inside the lens), on putting

$$z_,=\frac{(x_,^2+y_,^2)}{2r}+\frac{(x_,^2+y_,^2)^2}{8r^3}, \quad z_,=\frac{(x_,^2+y_,^2)}{2r'}+\frac{(x_,^2+y_,^2)^2}{8r'^3},$$

at the first and second surfaces respectively, and using the condition of the continuity of V $(\Delta V=0)$.]

and making also

$$\alpha = \alpha_{,} - H\alpha_0, \quad \beta_{,} = \beta, \quad H^2 + \tfrac{1}{2}(F - G) = I, \quad I\alpha_0^2 = k,$$

I found*

$$x''' = \alpha(z''' - \alpha^2 - \beta^2 + k) + H\alpha_0 z''', \quad y''' = \beta(z''' - \beta^2 - \alpha^2 - k).$$

Finally considering z''' as small of the order k, or α^2, and changing the direction of the coordinates, so as to take for new vertical axis the ray for which $\alpha = 0$, $\beta = 0$, and therefore $\alpha_{,} = H\alpha_0$, and representing the new rectangular coordinates by x, y, z, I found the equations

$$x = \alpha(z + k - \alpha^2 - \beta^2); \quad y = \beta(z - k - \alpha^2 - \beta^2).$$

These two equations seem very fit to be taken as fundamental in investigating the aberrations of a lens for ordinary rays which emerge slightly inclined to its axis, and in examining the shape and position of the caustic surfaces near their cusps and principal foci. x, y, z are rectangular coordinates of any point near these foci, the axis of z being a certain ray which we may call the *central ray*† of the system; α, β are the cosines of the angles which the ray passing through the point x, y, z makes with the axes of x, y; k is a small constant which vanishes when the system is of revolution, but which for other systems we may suppose to be > 0, since if it presented itself at first as < 0, we should only have to interchange the coordinates x, y, and then it would become > 0. The unit of length is the coefficient of longitudinal aberration, or the longitudinal divided by the square of the sine of the angular, in either of the two coordinate diametral planes of xz, yz, passing through the central ray. In the plane of xz, the ordinate of the central focus is $= -k$; in the plane of yz, the corresponding ordinate, or ordinate of the second focus of the central ray, is $= +k$; thus the origin of coordinates which we have chosen bisects the interval between the two foci of the central ray. To distinguish these two foci more completely we may remark that the point $x = 0$, $y = 0$, $z = -k$, has no ray passing through it, except the central ray, because the equations

$$0 = -\alpha(\alpha^2 + \beta^2), \quad 0 = \beta(2k + \alpha^2 + \beta^2), \quad k > 0,$$

can only be satisfied by making $\beta = 0$, $\alpha = 0$; but that the point $x = 0$, $y = 0$, $z = k$, has not only the central ray passing through it, but two others also, in the plane of xz, for which $\beta = 0$, $\alpha^2 = 2k$. Knowing therefore that there exists a central ray and that one of the two foci of this ray, or points of contact with the caustic surfaces, is a point through which passes no other near ray of the system, while the other focus has two other near rays passing through it, contained in one common plane; we may define the former to be the *First* and the latter the *Second* Focus of the central ray; and placing the origin of coordinates midway between them, may define the positive direction of the coordinate z to be the direction from the first towards the second; the plane of the two near rays which cross the central ray in the second focus we may take for the plane of xz, and the plane perpendicular to this, and passing through the central ray, will then be the plane of yz; finally with respect to the unit of length, it is the distance between the foci of the central ray, divided by the square of the sine of the inclination of that ray to either of the two other rays which cross it at its second focus. We shall also see that there are two principal rays or axes of the system which are contained in the plane of xz, and which have their angle bisected at the origin by the central ray; and we might define the origin, the ray with its positive direction or the positive axis of z, and the plane of xz with respect to these two principal rays.

* [In deriving these expressions, it is assumed that z''' is small.]

† [Cf. p. 300.]