XV.

ON CAUSTICS*

PART FIRST

[1824]

To the Reverend Archdeacon Brinkley, Professor of Astronomy in the University of Dublin and President of the Royal Irish Academy, this first paper on Caustics is respectfully inscribed.

The problems of Optics, considered mathematically, relate for the most part to the intersections of the rays of light, proceeding from known surfaces, according to known laws.

In the present paper it is proposed to investigate some general properties common to all such Systems of Rays, and independent of the particular surface, or particular law. It is intended, in another paper, to point out the applications of these mathematical principles to the actual laws of Nature.

A fortnight ago, I believed that no writer had ever treated of Optics on a similar plan. But within that period, my Tutor, the Reverend Mr. Boyton, to whom I had communicated some of my results, has shewn me in the College Library a beautiful Memoir of Malus on the subject, entitled "Traité d'Optique," and presented to the Institute in 1807.†

Those who may take the trouble to compare his Memoir with mine, will perceive a difference in method and extent. With respect to those results which are common to both, it is proper to state that I had arrived at them in my own researches, before I was aware of the existence of his.

Dublin, 7 *S. Cumberland Street*,
December 6, 1824.

General Principles.

I.

Let us conceive that from every point of a given surface

$$F(x', y', z') = 0$$

proceeds a ray of light, represented by its equations

$$\frac{x-x'}{z-z'} = \mu, \quad \frac{y-y'}{z-z'} = \nu, \qquad \text{(R)}$$

in which μ, ν are known functions of x', y', z' and their partial differentials $\frac{dz'}{dx'}$, $\frac{dz'}{dy'}$, &c., which we will denote by p, q, &c.

* [See Appendix, Note 1, p. 462. This paper deals with the properties of a general rectilinear congruence, and with its singularities; for a recent treatment of the singularities, see G. Julia, *Éléments de Géométrie infinitésimale* (Paris, 1927), pp. 204–217.]

† [*Mémoires présentés à l'Institut par divers savans*, 2 (1811), pp. 214–302.]

Such a system of rays has several properties, independent of the form of the functions F, μ, ν: these properties we propose to investigate. When the equation of the surface is given, we can deduce from it the partial differentials p, q, &c., and therefore μ, ν, as functions of x', y', z'. This is a consequence of our fundamental hypothesis that from each point of the surface proceeds but one ray of the system.

If it be required to find the ray which passes through a given external point x'', y'', z'', we have these three equations

$$\frac{x''-x'}{z''-z'}=\mu, \quad \frac{y''-y'}{z''-z'}=\nu, \quad F(x', y', z')=0,$$

to determine x', y', z', the point on the surface from which the ray proceeds.

But since these equations are in general of a degree higher than unity, let the exponents of their respective degrees be

$$M, N, \Phi,$$

and the final equation will be of the degree marked by the product of these exponents $MN\Phi$, which we will denote by n. This is then the number of possible solutions, and n different rays may intersect in one point x'', y'', z''.

If n be an even number, all the roots may be imaginary. Then no ray passes through the assumed external point x'', y'', z''. But if n be an odd number, which requires that the equation of the surface should not be of an even degree, then one root at least is always real, and every point of space has at least one ray passing through it.

If the three equations in x', y', z' reduce themselves to two, they express not a finite number of points but a curve on the given surface, all the rays from which intersect in the point x'', y'', z'', and compose a cone, having that point for its vertex.

Finally if the three equations reduce themselves to one, all the rays intersect in the point x'', y'', z'', which is then the focus of the system.

If we trace any curve, plane or of double curvature, on the given surface, the rays proceeding from that curve will compose a surface of rays. We will obtain its equation by eliminating x', y', z', between the equations of the ray (R) and those of the curve

$$F(x', y', z')=0, \quad \Psi(x', y', z')=0.$$

This curve I call the *base* of the surface of rays. There are an infinite number of such surfaces, corresponding to the infinite number of possible bases: but they have all one common property, and are represented by one common equation in partial differentials of the first order,

$$\mu\frac{dz}{dx}+\nu\frac{dz}{dy}-1=0, \tag{P}$$

in which we are to write, instead of μ, ν, their values as functions of x, y, z, deduced from the equations of the ray and of the given surface.

(P) has been obtained by elimination of the arbitrary function Ψ. It expresses that the tangent plane to the surface of rays entirely contains the ray passing through its point of contact.

This equation (P) furnishes a criterion whereby to ascertain whether a proposed surface $V=0$ is composed of rays: and if not, to determine the curve of contact, in the whole extent of which it is enveloped by a system of rays. For if $V=0$ represent a surface of rays, the equation

$$\mu\cdot\frac{dV}{dx}+\nu\cdot\frac{dV}{dy}+\frac{dV}{dz}=0=v$$

is identically satisfied: and if not, $V=0$, $v=0$ are the equations of the curve of contact. This curve separates, on the proposed opaque surface $V=0$, between the part illumined, and the part obscure.

There are two particular surfaces, or rather two naps of one surface, which though not surfaces of rays, yet satisfy the equation of condition (P). Their equations

$$\Sigma = 0, \quad \Sigma' = 0,$$

without containing an arbitrary function, or even an arbitrary constant, are nevertheless integrals of that equation in partial differences of the first order.

In fact they are singular primitives of that equation: and they belong to the two caustic surfaces, touched by all the rays. This subject will be resumed in the IIId Article of this paper.

To find the surface of rays passing through a given curve $V=0$, $v=0$, we will first find its base, $\Psi(x', y', z')=0$, by eliminating x, y, z between the equations of the given curve, and those of the ray. But it is necessary to observe, that since through each point of the given curve may pass n rays, the problem has in general n solutions: or rather the surface of rays has in general n branches, which all intersect in that curve. When $V=0$, $v=0$ are the equations of a given ray, instead of a curve, the surface of rays has only $n-1$ branches: for one evidently in this case coincides with that given ray.

To find the surface of rays, enveloping a given surface $V=0$, we are first to find the curve of contact, and so reduce this problem to the last.

These two problems contain the complete solution of the problem of shadows. The caustic surfaces have no determined shadow. Every other curve or surface may in general have n shadows.

Sometimes all the rays of the system intersect the curve $V=0$, $v=0$. In this case, that curve takes the place of a caustic surface: and the equation

$$\Psi(x', y', z')=0,$$

found by elimination of x, y, z between the equations of the curve and those of the ray, is identically satisfied by that of the given surface $F(x', y', z')=0$.

The subject of caustic surfaces will be more fully treated in Article III.

II.

In the general equations of the ray, μ, ν are known functions of x', y', z' and their partial differentials. But these latter are themselves functions of x', y', z', derived from the equation of the given surface

$$F(x', y', z')=0:$$

so that, all reductions made, we may denote the differentials of z', μ, ν as follows,

$$dz' = p\,dx' + q\,dy', \quad d\mu = \zeta\,dx' + \eta\,dy', \quad d\nu = \zeta'\,dx' + \eta'\,dy',$$

p, q, ζ, η, ζ', η' being all functions of x', y', z'.

This being laid down, suppose that from a given ray we wish to pass to a consecutive ray which intersects it, we must of course differentiate its equations, making x', y' vary, and z', μ, ν, as functions of these, but supposing $dx=0$, $dy=0$, $dz=0$. That is, we must combine with the equations of the ray, namely

$$\frac{x-x'}{z-z'}=\mu, \quad \frac{y-y'}{z-z'}=\nu, \tag{R}$$

their derived equations

$$\frac{\mu . dz' - dx'}{z - z'} = d\mu, \quad \frac{\nu . dz' - dy'}{z - z'} = d\nu. \tag{1}$$

These four equations determine not only the coordinates of intersection, but the direction by which we must pass on the given surface, to obtain a consecutive ray intersecting the given ray. There are two such directions: and their properties will form the subject of the present Article.

Eliminating $z - z'$ between the two last equations we obtain the following,

$$(\mu dz' - dx') d\nu = (\nu dz' - dy') d\mu, \tag{2}$$

which when we substitute for dz', $d\mu$, $d\nu$ their values, becomes quadratic in $\frac{dy'}{dx'}$, and determines the directions. Before making this substitution, I will observe that when the rays are supposed perpendicular to the surface, on which hypothesis

$$\mu = -p, \quad \nu = -q,$$

this last equation becomes

$$(p dz' + dx') dq = (q dz' + dy') dp,$$

a known form for the equation of the curves of greatest and least curvature.

By a reasoning precisely analogous to that which is usually employed with respect to these curves of curvature, we infer that equation (2) is the differential equation belonging to a class of curves, two of which pass through every point on the given surface, and the rays proceeding from which intersect consecutively. These curves we will denote by (s), (s′).

Let us now develope (2), which determines their directions. It takes the form

$$A dy'^2 + B dy' dx' - C dx'^2 = 0, \tag{3}$$

in which

$$A = \eta' \mu q + \eta (1 - \nu q),$$
$$B = \zeta' \mu q + \zeta (1 - \nu q) - \eta' (1 - \mu p) - \eta \nu p,$$
$$C = \zeta' (1 - \mu p) + \zeta \nu p.$$

Suppose the roots of (3) are $\frac{dy'}{dx'} = \epsilon$ and $\frac{dy'}{dx'} = \epsilon'$: which gives us

$$\epsilon + \epsilon' = -\frac{B}{A}, \quad \epsilon\epsilon' = -\frac{C}{A}.$$

Then the tangent to the curve of double curvature (s) has for its equations

$$\frac{y - y'}{x - x'} = \frac{dy'}{dx'} = \epsilon, \quad \frac{z - z'}{x - x'} = p + q\frac{dy'}{dx'} = p + q\epsilon, \tag{t}$$

and in the same manner the tangent to the other curve (s′) is given by the equations

$$\frac{y - y'}{x - x'} = \epsilon', \quad \frac{z - z'}{x - x'} = p + q\epsilon'. \tag{t'}$$

Representing by θ the angle under which the curves (s), (s′), or their tangents (t), (t′), cut each other, we easily deduce

$$(1 + p^2 + q^2)(\epsilon - \epsilon')^2 = N^2 \tan^2 \theta,$$

N being

$$1 + \epsilon\epsilon' + (p + q\epsilon)(p + q\epsilon') = N.$$

If we observe that these formulæ contain only symmetric functions of ϵ, ϵ', the roots of (3), we will find no difficulty in calculating them in terms of A, B, C, the coefficients of that equation. Thus we obtain

$$(1+p^2+q^2)(B^2+4AC)=(AN)^2\tan^2\theta, \tag{4}$$
$$AN=(1+p^2)A-pqB-(1+q^2)C.$$

By means of this equation (4), if the law and the surface be given, we can find the angle θ, made by the two directions at any given point on the surface: or that angle being given, we can find a curve on the surface, the locus of the points at which the two directions cut under that given angle.

If the surface be given but not the law, (4) is the equation of condition which that law must satisfy, in order that at every point of the given surface, the two directions should cut under the required angle θ.

Finally if the law and the angle are given, but not the surface, (4) is the equation in partial differences of the latter.

Two particular cases deserve notice: 1st. $\theta=90°$, 2nd. $\theta=0$. When $\theta=90°$, that is, when the two directions cut at right angles,

$$\tan\theta=\infty, \quad \text{or} \quad (1+p^2)A-pqB-(1+q^2)C=0.$$

This condition becomes, when we substitute for A, B, C their values,

$$\zeta'(1+q^2-\mu p)+\zeta p(\nu+q)=\eta' q(\mu+p)+\eta(1+p^2-\nu q),$$

and it is satisfied of itself when the rays are perpendicular to the surface: for then

$$\nu+q=0, \quad \mu+p=0, \quad 1+q^2-\mu p=1+p^2-\nu q, \quad \zeta'=\eta=-\frac{d^2z'}{dx'\,dy'}.$$

When $\theta=0$, that is, when the two directions coincide,

$$\tan\theta=0, \quad \text{or} \quad B^2+4AC=0.$$

The curve on the given surface which has for its equations the two following

$$F(x',y',z')=0, \quad B^2+4AC=0, \tag{Λ}$$

is a limit of a remarkable nature. Every ray proceeding from a point on one side of that limit, is intersected by two consecutive rays: every ray from a point upon it is intersected by but one consecutive ray: and finally every ray from a point on the other side of it, is not intersected by any consecutive ray.

For if $B^2+4AC>0$, the roots of $A\,dy'^2+B\,dy'\,dx'-C\,dx'^2=0$ are real, and $\tan\theta$ is real:
if $B^2+4AC=0$, ... equal, and $\tan\theta=0$:
if $B^2+4AC<0$, ... imaginary, and $\tan\theta$ is imaginary.

This limit (Λ) is in general touched by all the curves (s), (s′), and its equations form a singular primitive of their differential equation (3).*

* [This is not necessarily true. (Λ) is a locus of cusps of (s), (s′), but is not, in general, touched by these curves; (Λ) does not, in general, satisfy (3); cf. E. Picard, *Traité d'Analyse*, III (1908), chap. III; E. Goursat, *Mathematical Analysis* (Hedrick-Dunkel), II (2), p. 198 *et seq.* The error in Hamilton's reasoning is explained in the Appendix, Note 22, p. 504.]

But we must observe that if any one of these hypotheses respecting the values of B^2+4AC be for any particular surface or law impossible, the limit (Λ) in that case does not exist at all, or at least not in the sense we have described.

If, for example, B^2+4AC be essentially positive, or essentially negative, in the one case every ray is intersected by two consecutive rays, in the other case no two consecutive rays intersect.

Again if $B^2+4AC=0$ is possible, but $B^2+4AC<0$ impossible, the two directions coincide at the limit, but they do not become imaginary at either side of it. In this case the limit (Λ) is not touched by the curves (s), (s′), but cuts them obliquely. I will mention two cases of this, which are both of them found in Nature.

1st Case. When B^2+4AC is an exact square: in this case the differential equation resolves itself into two rational factors.

2nd Case. When $B^2+4AC=0$ can only be satisfied by putting separately*

$$A=0, \quad B=0, \quad C=0:$$

and when these suppositions are possible.

In the VIth Article of this Essay † will be found investigated a class of laws, including normals, reflected, and refracted rays, which come under this Second Case. The consideration of the point to which, in this case, the limit (Λ) generally reduces itself, is important in the Theory of Optics: and in the Second Part of this Essay † it will be introduced with all necessary development.

Let us conceive the system of equations

$$F(x', y', z')=0, \quad A\,dy'^2+B\,dy'dx'-C\,dx'^2=0$$

actually integrated. From their nature, the integral may be put under the form‡

$$F(x', y', z')=0, \quad (c-\phi)^2=\psi,$$

the first being as before the equation of the given surface, and the second determining upon it the curves (s), (s′). ϕ, ψ represent rational functions of x', y', z', and c is the arbitrary constant.

To complete the integral, that is, to determine the constant by the condition of the curves (s), (s′) passing through a given point of the surface, we have

$$c'=[\phi]+[\psi]^{\frac{1}{2}}, \quad c''=[\phi]-[\psi]^{\frac{1}{2}},$$

$[\phi]$, $[\psi]$ representing what ϕ, ψ become at the given point, and c', c'' being the corresponding values of c: so that the equations of the two curves passing through that point are

$$(c'-\phi)^2=\psi \quad \text{(s)}, \qquad \text{and} \qquad (c''-\phi)^2=\psi \quad \text{(s}'\text{)}.$$

These curves are distinct, coincident, or imaginary, according as

$$[\psi]>0, \quad =0, \quad \text{or} \quad <0,$$

conditions which correspond to those before given, namely $B^2+4AC>0$, $=0$, or <0.

They are altogether independent in their nature, if the function ψ is an exact square, in which case B^2+4AC is an exact square also. There exists then no one curve touched by them all, but all of one kind may intersect in one point.

* [This happens when the rays form a normal congruence.]

† [Not included in the MS.]

‡ [See Appendix, Note 22, p. 504.]

We shall get an idea of these circumstances, if we recollect that the curves of greatest and least curvature on a surface of revolution are the meridian and the parallel: the latter is always a circle, but the former may be any plane curve; there exists no one curve touched by all of either species, but all the meridians intersect in the vertex of the surface.

In general, it may happen that through some particular point of the given surface pass more than two curves (s), (s′): in this case the quadratic equation

$$Ady'^2 + Bdy'dx' - Cdx'^2 = 0$$

is not proper to determine their directions: analysis shews this by the coefficients vanishing.

If there be three directions, they are given by the equation of the third degree

$$dA \,.\, dy'^2 + dB \,.\, dy' \,.\, dx' - dC \,.\, dx'^2 = 0,$$

and in general if there be a finite number of directions $= n$, they are given by

$$d^{n-2} A \,.\, dy'^2 + d^{n-2} B \,.\, dy' \,.\, dx' - d^{n-2} C \,.\, dx'^2 = 0,$$

which becomes, at this point, an equation of the first order, and nth degree.

The surfaces of rays proceeding from the curves (s), (s′) are all developable, because composed of lines which intersect consecutively. Their equations are deduced from the equations of the curves (s), (s′), by the method of the Ist Article.

In the next Article, we will treat of their aretes:* in the present we will confine ourselves to their tangent planes.

There are in general two such developable surfaces passing through every ray: their tangent planes contain the elements of the two curves (s), (s′), that is to say, their tangents (t), (t′). Hence the equations of those planes are

Ist. $\quad (x - x')\{(1 - \nu q)\,\mathcal{E} - \nu p\} + (y - y')\{\mu q \mathcal{E} - (1 - \mu p)\} + (z - z')\{\nu - \mu \mathcal{E}\} = 0, \quad$ (T)

IInd. $\quad (x - x')\{(1 - \nu q)\,\mathcal{E}' - \nu p\} + (y - y')\{\mu q \mathcal{E}' - (1 - \mu p)\} + (z - z')\{\nu - \mu \mathcal{E}'\} = 0; \quad$ (T′)

$\mathcal{E}$, $\mathcal{E}'$ are the roots of the equation $A\mathcal{E}^2 + B\mathcal{E} - C = 0$: so that $\mathcal{E} + \mathcal{E}' = -\dfrac{B}{A}$, $\mathcal{E}\mathcal{E}' = -\dfrac{C}{A}$. The trigonometric tangent of the angle ω under which these two developable surfaces, or their tangent planes, cut each other, is given by the equation

$$(1 - \mu p - \nu q)^2 (1 + \mu^2 + \nu^2)(B^2 + 4AC) = N'^2 \tan^2 \omega,$$

N' being $AA' + BB' - CC'$, and

$$A' = (1 - \mu p)^2 + \nu^2 (1 + p^2),$$
$$B' = (\mu + p)(\nu + q) - pq(1 + \mu^2 + \nu^2),$$
$$C' = (1 - \nu q)^2 + \mu^2 (1 + q^2).$$

This formula is susceptible of an important reduction. In fact, if we develope the quantity here denoted by N' we find it is of the form

$$N' = (1 - \mu p - \nu q) N'',$$

N'' being

$$\mu(\nu + q)\zeta + (1 - \mu p + \nu^2)\eta - (1 - \nu q + \mu^2)\zeta' - \nu(\mu + p)\eta':$$

so that, setting aside the particular case in which $1 - \mu p - \nu q = 0$, our angular formula becomes

$$(1 + \mu^2 + \nu^2)(B^2 + 4AC) = N''^2 \tan^2 \omega. \qquad (5)$$

* [Throughout the MS. this word is spelt without the circumflex accent.]

It is easy to apply to this equation (5) the remarks that were made in page 349, on the uses of equation (4). The same particular cases, here also, deserve notice, namely $\omega = 0$ and $\omega = 90°$.

When $\omega = 0$, $\tan\omega = 0$, $B^2 + 4AC = 0$: that is to say, the tangent planes to the developable surfaces coincide at the limit (Λ).

When $\omega = 90°$, that is, when the developable surfaces cut at right angles, we have $\tan\omega = \infty$, $N'' = 0$,

$$\mu(\nu + q)\zeta + (1 - \mu p + \nu^2)\eta = (1 - \nu q + \mu^2)\zeta' + \nu(\mu + p)\eta'. \qquad (6)$$

This condition is always satisfied when the rays are perpendicular to the surface: for then

$$\nu + q = 0, \quad \mu + p = 0, \quad 1 - \mu p + \nu^2 = 1 - \nu q + \mu^2, \quad \eta = -\frac{d^2z'}{dx'\,dy'} = \zeta'.$$

It is also satisfied by reflected and refracted rays.

I will conclude this Article by shewing how some of the preceding formulæ may be simplified, 1st, by a choice of the axes: 2nd, of the independent variables.

1st. When our object is not to examine the general properties of the system, but circumstances connected with some particular ray, we may assume that ray for the axis of z, which gives $x' = 0, y' = 0, \mu = 0, \nu = 0$. On this hypothesis the equations of the ray become (R)......$x = 0, y = 0$ their derived equations (1) become

$$-\frac{dx'}{z - z'} = d\mu, \quad -\frac{dy'}{z - z'} = d\nu;$$

the condition of intersection (2) becomes

$$d\nu \,.\, dx' = d\mu \,.\, dy';$$

its developed equation (3) becomes

$$\eta\, dy'^2 + (\zeta - \eta')\, dy'\, dx' - \zeta'\, dx'^2 = 0,$$

the coefficients being

$$A = \eta, \quad B = \zeta - \eta', \quad C = \zeta';$$

the tangent planes become

Ist. $y - \mathfrak{S}x = 0$ (T), IInd. $y - \mathfrak{S}'x = 0$ (T');

the formula for their mutual inclination becomes

$$B^2 + 4AC = (A - C)^2 \tan^2\omega,$$

or

$$(\zeta - \eta')^2 + 4\eta\zeta' = (\eta - \zeta')^2 \tan^2\omega,$$

and the condition for their rectangularity, $A = C$ or $\eta = \zeta'$.

In this position of the axes, $\dfrac{dy'}{dx'}$ is the trigonometric tangent of the angle, which a plane containing the ray and cutting the surface in a direction corresponding to $\dfrac{dy'}{dx'}$, makes with the vertical plane zx, which latter also contains the ray. We will denote this angle by ϖ, and we have, for the two tangent planes

$$\eta \tan^2\varpi + (\zeta - \eta')\tan\varpi - \zeta' = 0,$$

which may be thus written

$$\eta - \zeta' + (\zeta - \eta')\sin 2\varpi = (\eta + \zeta')\cos 2\varpi,$$

under which form we shall have occasion to employ it in the IVth Article.*

* [Not included in the MS.]

Finally if we denote $-\dfrac{1}{z-z'}$, the reciprocal of the focal distance taken negatively, by Δ, we have from equation (1)

$$\Delta = \zeta + \eta \mathcal{E} = \zeta + \eta \tan \varpi.$$

2nd. The other simplification, which however we shall not make use of, consists in supposing the independent variables to be $x' - \mu z'$, $y' - \nu z'$, the coordinates of the intersection of the ray with the horizontal plane. Denoting these by a, b, the equations (R) of the ray become

$$x - a = \mu z, \quad y - b = \nu z:$$

the condition of two consecutive rays being in the same plane is

$$d\mu \, . \, db = d\nu \, . \, da:$$

and the condition of the two planes being at right angles is

$$\mu\nu \, . \, \frac{d\mu}{da} + (1+\nu^2)\frac{d\mu}{db} = (1+\mu^2)\frac{d\nu}{da} + \mu\nu \, . \, \frac{d\nu}{db}.$$

III.

The general equations of the ray being

$$\frac{x-x'}{z-z'} = \mu, \quad \frac{y-y'}{z-z'} = \nu, \tag{R}$$

we shewed that to obtain its intersection with a consecutive ray of the same kind, we must combine with (R) the derived equations

$$\frac{\mu dz' - dx'}{z-z'} = \zeta dx' + \eta dy' = d\mu, \quad \frac{\nu dz' - dy'}{z-z'} = \zeta' \, dx' + \eta' \, dy' = d\nu:$$

equations which we will now write thus

$$x = x' + \mu\Gamma, \quad y = y' + \nu\Gamma, \quad z = z' + \Gamma, \tag{R}$$

$$\Gamma = \frac{\mu dz' - dx'}{d\mu} = \frac{\nu dz' - dy'}{d\nu} \quad (1), \qquad \Gamma \text{ being } z - z'.$$

The equation of condition

$$(\mu \, dz' - dx') \, d\nu = (\nu dz' - dy') \, d\mu \tag{2}$$

was shewn to indicate two directions by one or other of which we must pass on the given surface, in order that two consecutive rays should intersect. Being developed it was put under the form

$$A \, dy'^2 + B dy' dx' - C dx'^2 = 0, \tag{3}$$

and shewn to be the differential equation of a class of curves (s), (s′), analogous to those of greatest and least curvature, two of which pass through every point of the given surface, (except those points which are at or beyond the limit (Λ),) and of which we represented the integral equation by

$$(\phi - c)^2 = \psi.$$

We calculated the tangents to these curves, and the angle under which they cut each other: we shewed that the rays proceeding from them intersect consecutively, and so compose developable surfaces (D), (D′): we calculated the tangent planes to these surfaces, and the angle ω under which they cut each other.

These were the principal subjects of the last Article. In the present we propose to consider the aretes of those developable surfaces; the coordinates, loci, and other circumstances connected with the intersections of consecutive rays.

To find the coordinates x, y, z of the point in which the given ray from the point x', y', z' is intersected by a consecutive ray from the point

$$x' + dx', \quad y' + \varepsilon dx', \quad z' + (p + q\varepsilon)\, dx',$$

or from the point

$$x' + dx', \quad y' + \varepsilon' dx', \quad z' + (p + q\varepsilon')\, dx',$$

ε, ε' being roots of the equation $A\varepsilon^2 + B\varepsilon - C = 0$, we have the following equations:

$$x = x' + \mu\Gamma, \quad y = y' + \nu\Gamma, \quad z = z' + \Gamma, \tag{R}$$

$$1 - \mu p + \zeta\Gamma + (\eta\Gamma - \mu q)\,\varepsilon = 0 \quad \text{(i)}, \qquad 1 - \mu p + \zeta\Gamma + (\eta\Gamma - \mu q)\,\varepsilon' = 0 \quad \text{(i}'\text{)}.$$

The three equations marked (R) are common to both points of intersection: with them we must combine (i) for the first point, and (i′) for the second.

The focal distance of the surface in each direction, that is, the distance of the point of intersection from the surface, measured on the ray, has for its expression

$$\pm \{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{\frac{1}{2}} = \pm\, \Gamma \,.\, \sqrt{1 + \mu^2 + \nu^2} = \frac{1}{\Delta}\,.$$

To find the arete (a) of the developable surface (D), that is, the locus of the intersections of the rays from the curve (s), eliminate x', y', z', Γ between the equations of that curve (s), and the four equations marked (R), (i): in the same manner to find the arete (a′) of the developable surface (D′), eliminate x', y', z', Γ between the six equations (s′), (R), (i′).

These aretes (a), (a′) are the *caustics* of the curves on the given surface (s), (s′): for they are touched by all the rays from those curves (s), (s′). These latter are the only curves on the given surface which have caustics.

The tangent planes (T), (T′) of which we found the equations in page 351, are the osculating planes of the caustics (a), (a′): for the tangent plane to a developable surface is always the osculating plane of its arete.

Every ray of the system, within the limit (Λ), is the intersection of two developable surfaces, and the common tangent of two caustics.

The locus of the caustics (a) is a curve surface (C), touched by all the rays, and containing upon it all their intersections of the first kind: we will call it the first caustic surface. Its equation is obtained by eliminating x', y', z', Γ between the equations (R), (i) and that of the given surface, $F(x', y', z') = 0$.

In the same manner, the locus of the caustics (a′) is a curve surface (C′), touched also by all the rays, and containing upon it all the intersections of the second kind. We will call it the second caustic surface, and obtain its equation by eliminating x', y', z', Γ between (R), (i′) and $F(x', y', z') = 0$.

These two caustic surfaces are only, in general, two different branches of one and the same great surface: for if we represent their equations by $\Sigma = 0$, $\Sigma' = 0$, we find that these are in general *irrational* factors of the equation $\Sigma\Sigma' = 0$.

This collective equation is immediately obtained by eliminating x', y', z', $\frac{dy'}{dx'}$ between the equations of the surface and ray, and their derived equations: or, in our present notation, by eliminating x', y, z', Γ between the equation of the given surface, and these four other equations

$$x = x' + \mu\Gamma, \quad y = y' + \nu\Gamma, \quad z = z' + \Gamma, \tag{R}$$

$$(\eta'\Gamma + 1 - \nu q)(\zeta\Gamma + 1 - \mu p) = (\eta\Gamma - \mu q)(\zeta'\Gamma - \nu p). \tag{I}$$

The last equation may be thus written

$$A''\Gamma^2 + B''\Gamma + C'' = 0,$$

in which

$$A'' = \eta'\zeta - \eta\zeta',$$
$$B'' = (1 - \mu p - \nu q)(\zeta + \eta') + \nu(p\eta + q\eta') + \mu(p\zeta + q\zeta'),$$
$$C'' = 1 - \mu p - \nu q.$$

We cannot give a complete discussion of this equation, until we have shewn how to calculate the partial differentials of the caustic surface. However some obvious remarks may be inserted here.

If $\frac{C''}{A''} > 0$, the roots, if real, are of the same sign, and the two foci are at the same side of the given surface. If $\frac{C''}{A''} < 0$, the roots are necessarily real, and of opposite signs: then the two foci are at opposite sides of the given surface.

If $B'' = 0$, the roots, if real, are equal in quantity, but different in sign: then the two foci are at opposite sides, and at equal distances.

If $B''^2 - 4A''C'' = 0$, the roots are equal, and the foci coincide. This condition being developed is found to be the same as $B^2 + 4AC = 0$.

If $A'' = 0$, one of the roots is infinite, and two consecutive rays parallel. In this case $d\mu = 0$, $d\nu = 0$: that is

$$\zeta dx' + \eta dy' = 0, \quad \zeta' dx' + \eta' dy' = 0:$$

either of these equations determines the direction on the surface in which one of its focal distances is infinite.

Every ray from the curve on the given surface, determined by the equation

$$A'' = \eta'\zeta - \eta\zeta' = 0,$$

is asymptot to two branches of a caustic situated on different sides of the given surface: and the surface of rays from the same curve is asymptotic to two branches of a caustic surface, situated likewise at different sides of the given surface. If $A'' = 0$ be identically satisfied, one of the developable surfaces is cylindrical.

Let us now consider the problem "To find the partial differentials of the caustic surface," its equation in x, y, z being given as the result of elimination of x', y', z', Γ between the five equations, $F(x', y', z') = 0$, (R) and (I).

Instead of stopping at this, I will propose at once the more general problem "Given a system of m equations between $m + n$ variables, to find the partial differential coefficient of any one of them, relative to the variation of any other."

Let us represent the coefficient sought by $\frac{dz}{dx}$: and conceive $m - 1$ variables eliminated between the m equations, so as to leave one equation between $n + 1$ variables, including x and z.

Of these we will consider z as the dependent variable, and the n others as independent. If now we differentiate, suppressing all the differentials except those of x and z, the resulting equation in dx, dz gives the coefficient.

But the elimination which we have here supposed, may be impossible, and is never necessary. In fact if we attend to the principle of the method, we find that it consists in supposing constant $n-1$ out of the n independent variables: on which hypothesis the ratio $dz:dx$ is the coefficient sought.

If then we differentiate the m given equations, supposing $n-1$ of the variables constant, the m resulting equations, being linear equations between m partial differentials, contain an easy solution of the problem.

Perhaps I may have not succeeded in expressing myself with clearness, on a subject so very general: but I hope that the preceding remarks will be understood by observing their application to the case before us.

In this case, we have a system of five equations between seven variables, namely $x, y, z, x', y', z', \Gamma$: consequently two are independent; for these we will take x, y. Now differentiating, and supposing y constant, we obtain five linear equations between $\frac{dz}{dx}, \frac{dx'}{dx}, \frac{dy'}{dx}, \frac{dz'}{dx}, \frac{d\Gamma}{dx}$, five partial differentials of which we only want the first. But since these are all linear equations there is no difficulty, except the length of the calculation, (which in this case is abridged by some reductions that present themselves,) in finding each separately. In a similar manner I have calculated $\frac{dz}{dy}$: and these are the results, after all reductions:*

$$\frac{\nu Q-1}{\mu Q}=\frac{\zeta'\Gamma-\nu p}{\zeta\Gamma+1-\mu p}=\frac{\eta'\Gamma+1-\nu q}{\eta\Gamma-\mu q},$$

$$\frac{\mu P-1}{\nu P}=\frac{\zeta\Gamma+1-\mu p}{\zeta'\Gamma-\nu p}=\frac{\eta\Gamma-\mu q}{\eta'\Gamma+1-\nu q}:$$

P, Q here denote the partial differentials $\frac{dz}{dx}, \frac{dz}{dy}$, of the caustic surface.

The two expressions thus found for $\frac{\nu Q-1}{\mu Q}$ are equal among themselves, in consequence of equation (I): and it is easy to see that they are the reciprocals of those for $\frac{\mu P-1}{\nu P}$.

* [These results may easily be obtained from the equations

$$x=x'+\mu\Gamma, \quad y=y'+\nu\Gamma, \quad z=z'+\Gamma.$$

Taking an arbitrary displacement dx', dy', dz' on the surface $F(x', y', z')=0$, we find

$$\frac{dx-\mu dz}{dy-\nu dz}=\frac{(1-\mu P)\,dx-\mu Q\,dy}{-\nu P\,dx+(1-\nu Q)\,dy}=\frac{(1+\zeta\Gamma-\mu p)\,dx'+(\eta\Gamma-\mu q)\,dy'}{(\zeta'\Gamma-\nu p)\,dx'+(1+\eta'\Gamma-\nu q)\,dy'}.$$

But, since Γ satisfies (I) (p. 355), this fraction is independent of dy'/dx'. Therefore

$$\frac{1-\mu P}{-\nu P}=\frac{-\mu Q}{1-\nu Q}=\frac{1+\zeta\Gamma-\mu p}{\zeta'\Gamma-\nu p}=\frac{\eta\Gamma-\mu q}{1+\eta'\Gamma-\nu q}.]$$

Hence we deduce

$$\frac{\nu Q-1}{\mu Q}\cdot\frac{\mu P-1}{\nu P}=1:$$

that is,

$$1-\mu P-\nu Q=0.$$

Now this equation in partial differences of the first order, belonging to the caustic surfaces, is precisely the same as that which we found in the first Article, to belong to all surfaces of rays, by elimination of an arbitrary function.

And it is on the analysis here described, that I founded the assertion in that Article, that the equations of the two caustic surfaces, which we will still consider as distinct, and represent by $\Sigma=0$, $\Sigma'=0$, are singular primitives of the equation in partial differences

$$1-\mu\frac{dz}{dx}-\nu\frac{dz}{dy}=0.$$

We have already considered these surfaces as formed by two different generations, and given two corresponding methods to find their equations.

In the first method we obtained each separately, considering it as the locus of the caustic curves (a) or (a′) of its own kind. In the second method we obtained their equations collectively, considering the great surface of which they are in general different branches, as the one locus of all the intersections of consecutive rays.

In each method, their equations, whether separately or combined, are presented as the result of elimination.

Now it is a received principle in Analysis, that whatever is represented by a system of equations, may be represented by an infinite number of systems equivalent: in the same manner as, in Mechanics, forces which have no resultant, may be replaced by an infinite number of equivalent systems.

Conformably to this general principle, I am going to shew that the caustic surfaces may have an infinite number of different generations, among which we are at liberty to choose, in any particular case, that which may offer the greatest facility.

Conceive any curve whatever, with an arbitrary parameter, traced on the given luminous surface $F(x', y', z')=0$: the rays proceeding from that curve will compose a surface of rays with an arbitrary parameter, which we will represent by

$$\Phi(x, y, z, \epsilon)=0=u.$$

If we consider a consecutive surface of the same kind,

$$u+\frac{du}{d\epsilon}.d\epsilon=0,$$

we find that it intersects the former in three different ways.

1st. In the ray proceeding from the intersection of their consecutive bases.

2nd. Every curve of the form (s) cuts the two bases in two points: the consecutive rays, one on each surface, proceeding from these two points, intersect each other: the locus of such intersections is a curve common to the two consecutive surfaces of rays, and this curve is on the first caustic surface.

3rd. By a similar reasoning, the two consecutive surfaces of rays have another curve of intersection on the second caustic surface.

The ray and the two curves have for their common expression

$$u = 0, \quad \frac{du}{d\epsilon} = 0,$$

and if we eliminate ϵ between these, we obtain in x, y, z an equation

$$E . \Sigma . \Sigma' = 0:$$

$\Sigma = 0$, $\Sigma' = 0$ are the caustic surfaces; $E = 0$ is a surface of rays enveloping all those of the assumed form $\Phi(x, y, z, \epsilon) = 0$.

In general every surface of rays envelopes both caustic surfaces. But there exists one remarkable exception which it is necessary to notice. The developable surfaces of the first kind (D) do not touch the first caustic surface (C): and in general the developable surfaces do not touch the loci of their own aretes: a circumstance sufficiently evident, because their tangent planes osculate to those aretes: but I believe it may be useful to shew how the preceding reasoning fails in this case.

In fact it is evident that a curve of the form (s) does not cut the bases of two consecutive developable surfaces (D), though it may coincide with one of them: so that if we represent by $D = 0$, $D' = 0$, the general equations of the two developable surfaces, with the arbitrary constant, and eliminate c between $D = 0$, $\frac{dD}{dc} = 0$, we obtain an equation $L\Sigma' = 0$, which does not contain the first caustic surface. In the same manner, eliminating c between $D' = 0$, $\frac{dD'}{dc} = 0$, we obtain $L\Sigma = 0$.

The common factor $L = 0$ belongs to a remarkable surface, the principal properties of which I will rapidly enumerate. 1st. It envelopes all the developable surfaces of both kinds (D), (D'); 2nd. it is a limiting surface, analogous to the limiting curve (Λ), since through every ray at one side of it pass two developable surfaces, through every ray upon it passes but one, and through every ray beyond it passes none; 3rd. it has for base the limiting curve (Λ); 4th. it envelopes both the caustic surfaces in one and the same curve of contact; 5th. it is itself developable and this curve of contact is its own arete; 6th. that arete is a curve of rebroussement on the caustic surfaces, in which the two branches touch each other and at which they terminate; 7th. the same arete is itself a caustic curve, and the one in which the density of light is greatest; 8th. it is touched by all the caustic curves of both kinds, and it is the caustic of the limit (Λ).*

This is a proper place to point out some developments, and some demonstrations, which may assist in conceiving the connexion between the curves (s), (s') and the limit (Λ), and at the same time between the surfaces of rays proceeding from them, that is, the developable surfaces (D), (D') and the limiting surface (L).

We have represented the general form of the equations of the curves (s), (s') by

$$F(x', y', z') = 0, \quad (\phi - c)^2 = \psi,$$

* [Some of these properties of (L) (the surface of rays through (Λ)) are derived from the assumption (see footnote to p. 349) that (Λ) is touched by the curves (s), (s'). In general (L) is not touched by the developables of the system, nor is it itself developable, but it does touch both the caustic surfaces in a single curve.]

of which we need only attend to the latter, because the former belongs to the given surface. If we eliminate the arbitrary constant by differentiation we find the following,

$$4\psi \cdot d\phi^2 = d\psi^2, \tag{d}$$

which must be identical with the original differential equation of the curves,

$$A\,dy'^2 + B\,dy'dx' - C\,dx'^2 = 0\,;$$

expanding and comparing, we find

$$\lambda A = \left(\frac{d\psi}{dy'}\right)^2 - 4\psi \cdot \left(\frac{d\phi}{dy'}\right)^2,$$

$$\lambda B = 2\left\{\frac{d\psi}{dy'} \cdot \frac{d\psi}{dx'} - 4\psi \cdot \frac{d\phi}{dy'} \cdot \frac{d\phi}{dx'}\right\},$$

$$-\lambda C = \left(\frac{d\psi}{dx'}\right)^2 - 4\psi \cdot \left(\frac{d\phi}{dx'}\right)^2,$$

$$\lambda^2(B^2 + 4AC) = 16\psi \cdot \left\{\frac{d\psi}{dx'} \cdot \frac{d\phi}{dy'} - \frac{d\psi}{dy'} \cdot \frac{d\phi}{dx'}\right\}^2,$$

λ being an indeterminate coefficient, which proves that when $\psi > 0, = 0, < 0$, or an exact square, we have at the same time $B^2 + 4AC > 0, = 0, < 0$, or an exact square.

It is evident that the differential equation (d) has a singular solution, $\psi = 0$, $d\psi = 0$. These hypotheses belong to the limit (Λ), and shew that this latter curve is touched by all of the form (s), (s′).* It has however only a contact of the first order with each: for if we differentiate (d), and put $\psi = 0$, $d\psi = 0$, we find

$$d^2\psi \cdot (d^2\psi - 2d\phi^2) = 0:$$

the first factor belongs to the limit, the second to the curve: so that they have at their common point a common tangent, but not a common osculating circle.

Since at the limit $d\psi = 0$, $d^2\psi = 2d\phi^2$, or $d^2\psi > 0$, ψ is a minimum. Only one curve of the form (s), (s′) passes through each point of the limit. In fact these two curves are in general of the same nature, and no otherwise differ than by the value of the constant c: which at the limit has but one value.

Perhaps a simple geometrical illustration may be useful, to make these circumstances easy of conception. Imagine a series of circles of given radius, all touching externally a given circle, in its own plane: through every external point there pass in general two such circles; these we may conceive to represent the curves (s), (s′). Through every point upon the given circle passes but one, namely the circle of the given radius, which touches it externally at that point. In this case the given circle is the limit (Λ), and no curve of the kind we are considering passes through any point within it. It is necessary however to observe that in this case there is *another limit*, also a circle, touched by all of the given form, through every point of which passes but one, and through every point beyond which passes none: this is a peculiarity of the instance we have selected, and is not found in the general case.

But when the function ψ is an exact square which we will represent by Ω^2, the differential equation

$$4\psi \cdot d\phi^2 - d\psi^2 = 0 \tag{d}$$

* [This is not true in general. See Appendix, Note 22, p. 504.]

resolves itself into two factors: one of them is $\Omega^2 = 0$, the other $d\phi^2 - d\Omega^2 = 0$: the first belongs to the limit, which is still a particular solution, and the second belongs to the curves which on this hypothesis become

$$\phi + \Omega = c \quad \text{(s)}, \qquad \phi - \Omega = c \quad \text{(s')}:$$

in this, as in all cases, at the limit the two values of the constant become equal, but the curves themselves do not touch either the limit or each other.* In this case then, (Λ) is no longer a limit, properly speaking, nor is (L) a limiting surface.

In this case also the curves (s), (s') and the surfaces (D), (D') are altogether independent. Nor does there then exist any one surface touched by all the developable surfaces of both kinds: but as we observed in page 350, that in this case the curves of one kind may all pass through one common point, so we will now observe, that all the developable surfaces of one kind may touch one common surface, or pass through one common line: which surface or line is then to be considered as one of the caustic surfaces, in this case independent.

It is proper to take our illustrations from the properties of normals to a surface, because these being treated of by all elementary writers, are familiar to the conception, and because Monge has taken them for the principal subject of his profound and beautiful "Analyse." Suppose then the rays all normal to a surface of revolution: one of the developable surfaces is the meridian plane, the other is a right cone, having its vertex in the axis: their natures are different; all those of the first kind pass through a common line, namely the axis: all those of the other kind touch one common surface, namely the one formed by the evolute of the meridian revolving round the axis.

The angle under which the developable surfaces cut the loci of their own aretes, is the same as that under which they cut each other, namely ω, calculated in the IInd Article. This is also the angle under which the osculating planes of the caustic curves (a), (a') cut their own caustic surfaces (C), (C'). Hence it follows that when $\omega = 90°$, the caustic curves are the shortest between two points on their caustic surfaces. This is the case for normals, for reflected and for refracted rays. In the Vth Article† we will prove that, in the first of these three cases, the caustic curves are the shortest on another kind of locus, which consequently touches the caustic surface in the whole extent of the caustic curve.

Whenever the condition $\omega = 90°$ is satisfied, a thread drawn in the direction of the ray, and, at the point where it meets either of the caustic surfaces, freely wound round that surface, would trace upon it the caustic curve corresponding: and conversely such a thread freely unwound would trace out on the given surface the curve (s) or (s'): so that the caustics are analogous to evolutes.

In fact evolutes are a particular case of caustics, when the latter, as in this paper, are considered in all their generality. In the case of evolutes, the rays are perpendicular to the surface: the curves (s), (s') are the curves of greatest and least curvature on the latter: the developable surfaces (D), (D') are the normal surfaces, with which Monge occupied himself so much, and which cut at right angles: the caustic surfaces (C), (C') are the loci of the centres of curvature, which answer to the foci, and finally, the caustic curves (a), (a') are the evolutes of

* [There is some confusion in the argument, since (unless $A = B = C = 0$ on (Λ)) the equation

$$A\,dy'^2 + B\,dy'\,dx' - C\,dx'^2 = 0$$

(which defines (s), (s')) is satisfied by a unique direction when $B^2 + 4AC = 0$. The curves (s), (s') therefore touch each other at the points where they meet (Λ).]

† [Not included in the MS.]

the curves of greatest and least curvature, which Monge proved in a similar manner to be the shortest lines between two points on their own loci.

Monge also proved that if from any point of space we could see the two surfaces which are the loci of the centres of curvature, their profiles would appear to cut at right angles. I will only stop to observe that if ω be any constant angle, the profiles of the two caustic surfaces would appear to cut under that angle.

We may now resume the discussion of the equation of focal distance in page 355,

$$A''\Gamma^2 + B''\Gamma + C'' = 0,$$

which was there left incomplete. We there discussed the cases $A'' = 0$ and $B'' = 0$. The hypothesis

$$C'' = 0 = 1 - \mu p - \nu q$$

belongs to a curve on the given surface, in the whole extent of which, one focal distance being $= 0$, one caustic surface intersects the given surface.

This is a remarkable curve: let us consider some of its properties.

1st. It is, as we have seen, the intersection of the given surface with a caustic surface.

2nd. But it is at the same time a curve of contact between the given surface and that caustic surface: for if we make

$$1 - \mu p - \nu q = 0, \quad \Gamma = 0$$

in the expression calculated in page 356 for the partial differentials of the caustic surface, P, Q, we find $P = p$, $Q = q$.

3rd. It is a singular curve on the caustic surface, for if we calculate the partial differentials, second order, of that surface, we find them, at that curve, in general infinite.*

4th. The equation which in general determines $\tan\omega$, disappears when

$$1 - \mu p - \nu q = 0.$$

5th. On the same hypothesis the equation which determines the directions of the curves (s), (s′), namely

$$A\,dy'^2 + B\,dy'dx' - C\,dx'^2 = 0,$$

resolves itself into two rational factors, of which one being $\mu dy' - \nu dx' = 0$, that is, $\nu - \mu\mathfrak{G} = 0$, changes the equations in page 348, for the tangent (t) to the curve (s), into these,

$$\frac{y - y'}{x - x'} = \frac{\nu}{\mu}, \quad \frac{z - z'}{x - x'} = \frac{\mu p + \nu q}{\mu} = \frac{1}{\mu}, \tag{t}$$

because in the present case $1 - \mu p - \nu q = 0$: and these equations (t) shew that the tangent to one of the curves (s), (s′) is in this case the ray itself.

6th. Finally the curve

$$F(x', y', z') = 0, \quad 1 - \mu p - \nu q = 0,$$

is that in which the given surface is itself enveloped by a surface of rays.

It remains to consider the case $B''^2 - 4A''C'' = 0$.

* [Hamilton must have been led to this conclusion by an error of calculation. The partial derivatives of the second order for the caustic surface are infinite only if

$$2\mu(\zeta p + \zeta' q) + 2\nu(\eta p + \eta' q) + \mu^2 r + 2\mu\nu s + \nu^2 t = 0.]$$

In general when we seek the intersection of the two caustic surfaces, or of the two branches of the great caustic surface, which is composed of them both, we suppose the two foci to coincide, and therefore the equation in Γ to take equal roots.*

This hypothesis furnishes the condition $B''^2 - 4A''C'' = 0$: which being developed is found to be the same as $B^2 + 4AC = 0$. Hence we conclude

1st. the curve of intersection is also a curve of contact: for $B^2 + 4AC = 0$ gives $\tan \omega = 0$;

2nd. this curve is therefore one of rebroussement on the great caustic surface;

3rd. the surface of rays passing through it has for base the limit (Λ), [and]

4th. is therefore the developable limiting surface (L);†

5th. the curve itself is the arete of that surface (L) and has all the properties mentioned in page 358: that is to say

6th. it is the caustic of the limit (Λ), and touched by all the caustics of both kinds;

7th. for this caustic, the density of light is a maximum: we will call it the *principal caustic.*

It does not suit the plan of this paper to enter into a detail of exceptions: we will however mention the most extensive and important.

Ist. When $B^2 + 4AC$ is an exact square.

IInd. When $B^2 + 4AC = 0$ can only be satisfied by putting separately $A = 0$, $B = 0$, $C = 0$, and when these suppositions are possible.

These are the same exceptions that we mentioned in page 350, respecting the nature of the limit (Λ). We are going to consider them separately.

In the Ist Case, the caustic surfaces are not branches of one and the same surface, but are absolutely distinct and independent. They do not touch at their intersection, nor do they terminate there. That intersection is not a caustic; nor is the surface of rays (L) passing through it, developable; nor a limiting surface: nor the base (Λ) a limiting curve.

However the intersection of the two caustic surfaces is still an important curve, since for every point of it two foci coincide, and the density of light is greatest.

In the IInd Case, the limit (Λ) generally reduces itself to a point: and there exists a class of laws which render not only the equation itself,

$$A\,dy'^2 + B\,dy'dx' - C\,dx'^2 = 0,$$

but all its derived equations identical at that point: so that the ray from that point is intersected by an infinite number of consecutive rays.

Moreover they all intersect it in one and the same point, for the equation

$$A''\Gamma^2 + B''\Gamma + C'' = 0$$

takes equal roots: to this point the *principal caustic* in this case reduces itself: we will call it the *principal focus.* The point on the given surface to which in this case the limit (Λ) reduces itself, and from the portion of the surface near to which the light converges accurately to the principal focus, I will call the *vertex* of the surface.

* [Hamilton leaves out of consideration an intersection for which a common point is a focus on two different rays.]

† [This, and the properties which follow, are based on the assumption that (s), (s′) touch (Λ).]

These two points exist in Nature, and it is obvious how important the consideration of them must be. It will of course be introduced in the second part of this paper,* whenever I have leisure to complete it.

To find the coordinates of the vertex of the given surface, we may employ any three of the four equations

$$F(x', y', z') = 0, \quad \eta' \mu q + \eta (1 - \nu q) = 0, \quad \zeta \nu p + \zeta' (1 - \mu p) = 0, \quad \mu q \zeta' = \eta \nu p:$$

and to find the coordinates of the principal focus, x, y, z, we have the following,

$$\Gamma = \frac{\nu p}{\zeta'} = \frac{\mu q}{\eta}, \quad x = x' + \mu \Gamma, \quad y = y' + \nu \Gamma, \quad z = z' + \Gamma.$$

Γ is the projection of the focal distance on the axis of z; that distance is

$$\Gamma \sqrt{1 + \mu^2 + \nu^2} = \frac{1}{\Delta}.$$

I had prepared an Article on the intersections of rays at finite distances from each other, which was to have been inserted in this place: but the comparative unimportance of the subject, and the press of other matter, have induced me to relinquish that intention.

* [Not included in the MS.]