CONCLUSION TO THE FIRST PART

THE preceding pages contain the execution of the first part of our plan; being an attempt to establish general principles respecting the systems of rays produced by the ordinary reflexion of light, at any mirror or combination of mirrors, shaped and placed in any manner whatsoever; and to shew that the mathematical properties of such a system may all be deduced by analytic methods from the form of ONE CHARACTERISTIC FUNCTION: as, in the application of analysis to geometry, the properties of a plane curve, or of a curve surface, may all be deduced by uniform methods from the form of the function which characterises its equation. It remains to extend these principles to other optical systems; to shew that in every such system, whether the rays be straight or curved, whether ordinary or extraordinary, there exists a Characteristic Function analogous to that which we have already pointed out for the case of the systems produced by the ordinary reflexion of light; to simplify and generalise the methods that we have given, for calculating from the form of this function all the other properties of the system; to integrate various equations which present themselves in the determination of mirrors, lenses, and crystals satisfying assigned conditions; to establish some more general principles in the theory of SYSTEMS OF RAYS, and to terminate with a brief review of our own results, and of the discoveries of former writers. But we have trespassed too long at present on the indulgence of mathematicians, and of the Academy, and must defer to another occasion the completion of this extensive design.

W. R. HAMILTON.

Observatory,
April 1828.

*PART SECOND**

ON ORDINARY SYSTEMS OF REFRACTED RAYS

XIV. *Analytic expressions of the Law of Ordinary Refraction.*

77. When a ray of light is refracted, in passing from one unchrystallised medium into another, we know by experience that the angles of incidence and refraction are in the same plane, and that the sine of the former has to the sine of the latter, a constant ratio (m), depending on the nature of the mediums, and on the colour of the ray. If then two forces, the one equal to unity, the other equal to (m), were to act at the point of incidence, in the directions of the two rays,† their resultant would be perpendicular to the refracting surface, and would be equal to

$$\cos pn + m \, . \cos \rho' n,$$

* [This part is now printed for the first time from the manuscript (Note Book 8 in the Library of Trinity College, Dublin). The manuscript is undated. The fact that the work follows so closely the headings published in the Table of Contents prefixed to Part First would lead us to suppose that it was written in 1827. But Graves (*Life of Sir W. R. Hamilton*, Vol. I, p. 366) states that as late as 1830 Hamilton was engaged in preparatory work for the Second Part. From an erased title to the manuscript it appears that Hamilton at one time intended to call the whole communication "Application of Analysis to Optics" (on the analogy of Monge's "Application de l'analyse à la géométrie"), instead of "Theory of Systems of Rays."]

† [The "direction" of the incident ray is the reverse of the direction of propagation.]

ρn, $\rho' n$, being the angles which the incident and the refracted rays make respectively with this resultant or normal (n); and if we represent by ρl, $\rho' l$, nl, the angles which the three lines ρ, ρ', n, make respectively with any assumed line (l), we shall have the following formula

$$\cos \rho l + m \,.\, \cos \rho' l = (\cos \rho n + m \,.\, \cos \rho' n) \,.\, \cos nl, \tag{A}'$$

which is the analytic expression of the known law of ordinary refraction, and includes the whole Theory of Dioptrics.

78. This formula (A)′ is susceptible of transformations analogous to those of the corresponding formula (A), in the Ist. Section of this Essay. Thus, by making the assumed line (l) coincide successively with three rectangular axes, we can find the angles which the refracted ray makes with those axes, knowing the corresponding angles for the incident ray, and the tangent plane to the refracting surface. In this manner we find, by elimination of $(\cos \rho n + m \,.\, \cos \rho' n)$, the following equations which are of very extensive application;

$$\frac{\alpha + m\alpha'}{\gamma + m\gamma'} = -p, \quad \frac{\beta + m\beta'}{\gamma + m\gamma'} = -q, \tag{B}'$$

$\alpha\ \beta\ \gamma$, $\alpha' \beta' \gamma'$, being the angles which the incident and the refracted rays* make with the axes of coordinates, and p, q, the partial differentials first order of the refracting surface. And if we represent by $\delta\rho$, $\delta\rho'$, δn, the variations of distance from any three assumed points, on the directions of the three lines ρ, ρ', n, (obtained by passing from the point of incidence to an infinitely near point in any assumed direction), these variations will be proportional to the cosines of the angles which the three lines make with that direction, and therefore the formula (A)′ may be thus written

$$\delta\rho + m\delta\rho' = (\cos \rho n + m \,.\, \cos \rho' n) \,.\, \delta n, \tag{C}'$$

which when we put $\delta n = 0$, that is when we suppose the infinitely near point taken on the refracting surface, reduces itself to the known *Principle of Least Action*,

$$\delta\rho + m\delta\rho' = 0, \tag{D}'$$

the distances ρ, ρ', being positive when they are measured on the rays themselves, negative when on the rays produced.

79. It follows immediately from (D)′, that if a surface be such as to refract rays accurately from one point to another, it must satisfy the equation

$$\rho + m\rho' = \text{const.}$$

This class of surfaces was first discovered by Descartes,† on which account we shall call them the *Cartesian Surfaces*; in the next Section we shall consider the more general question, to find a surface which shall refract to a given point the rays of a given system; in the mean time we may observe, that if we take one point upon any incident ray, and another on the refracted ray corresponding, the refracting surface will be touched by the Cartesian surface, constructed with those two points as foci.

XV. *On Focal Refractors, and on the Surfaces of Constant Action.*

80. The question, to find a surface which shall refract rays of a given refrangibility, from one given point to another, has been completely resolved by Descartes;† and Newton has given it a place in the First Book of the *Principia*,‡ in which he has also solved this more general problem,

* [The incident ray is still reversed.]

† [*Œuvres*, VI (Paris, 1902), pp. 424–431.]

‡ [Liber I, Prop. XCVIII.]

to find the second surface of a lens, which shall refract rays of one colour from one given point to another, the first surface being given, and both being surfaces of revolution round an axis passing through the two given points. I am going, in this Section, to treat the question in a more general manner, and to investigate by reasonings similar to those of the Second Section of this Essay, the equation of a surface, which shall refract to a given focus the rays of a given homogeneous* system. This question is analytically expressed by the following differential equation, which follows immediately from the formula (B)′,

$$\alpha dx + \beta dy + \gamma dz = -m(\alpha' dx + \beta' dy + \gamma' dz); \qquad \text{(E)}'$$

$\alpha\,\beta\,\gamma$ being given functions of $(x\,y\,z)$, namely the cosines of the angles which the incident ray passing through the point $(x\,y\,z)$ makes with the axes of coordinates; and $\alpha'\,\beta'\,\gamma'$ being other given functions of $(x\,y\,z)$, namely the cosines of the angles which the refracted ray, passing through that point $(x\,y\,z)$ and through the given focus $(X'\,Y'\,Z')$, makes with the axes of coordinates. The second member of this equation (E)′ being always an exact differential,

$$-m(\alpha' dx + \beta' dy + \gamma' dz) = m d\rho',$$

ρ' being the distance from the focus, it is necessary that the first member also should be an exact differential,† which produces the following condition

$$(\alpha + m\alpha')\left(\frac{d\beta}{dz} - \frac{d\gamma}{dy}\right) + (\beta + m\beta')\left(\frac{d\gamma}{dx} - \frac{d\alpha}{dz}\right) + (\gamma + m\gamma')\left(\frac{d\alpha}{dy} - \frac{d\beta}{dx}\right) = 0;$$

and since it may be proved, as in **8.**, that the three quantities

$$\frac{d\beta}{dz} - \frac{d\gamma}{dy}, \quad \frac{d\gamma}{dx} - \frac{d\alpha}{dz}, \quad \frac{d\alpha}{dy} - \frac{d\beta}{dx},$$

are proportional to α, β, γ, the equation (E)′ is not integrable unless these three quantities respectively vanish, that is unless the cosines of the angles which the incident ray makes with the axes are equal to the partial differential coefficients of a function of $(x\,y\,z)$; which requires that the incident rays may be cut perpendicularly by a series of surfaces.‡ And when this condition is satisfied, the integral of (E)′ is of the form

$$\rho + m\rho' = \text{const.}, \qquad \text{(F)}'$$

ρ, ρ' being the paths traversed by the light in going from any particular surface which cuts the incident rays perpendicularly, to the refractor, and from the refractor to the focus; from which it follows that the focal refractor is the enveloppe of a series of Cartesian surfaces, analogous to the ellipsoids, **10.**

81. Hence also it follows, that when homogeneous rays issuing from a given point have been refracted at a given surface, the rays of the refracted system are cut perpendicularly by that series of surfaces which are represented by the equation (F)′. In general, when rays of any one colour, issuing from a luminous point or from a perpendicular surface, have been any number of times reflected and refracted, it may be proved, by reasonings similar to those of **12.**, that the final rays are cut perpendicularly§ by that series of surfaces for which

$$\Sigma(m\rho) = \text{const.}, \qquad \text{(G)}'$$

* [That is, monochromatic.]

† [(E)′ is true only for displacements on the refracting surface: it is the fact that (E)′ is integrable, and not that its first member is an exact differential, which is to be used here.]

‡ [It has been shown by T. Levi-Civita that a given non-normal congruence may be transformed into another given non-normal congruence by two refractions (*Rend. Acc. Lincei*, (5) 9 (1900), pp. 185–189, 237–245).]

§ [See Appendix, Note 2, p. 463.]

$(m\rho)$ being the path traversed by the light in passing through any particular medium, multiplied by the refractive power of that medium; this series of surfaces I shall call the *Surfaces of Constant Action*, for the reasons mentioned in the IIId. Section of this Essay; and I shall dispense with expressing a number of remarks respecting them, which are analogous to the properties of the corresponding surfaces considered in that Section.

XVI. *On the Characteristic Function.*

82. We have seen, in the First Part of this Essay, that when rays issuing from a luminous point, or from a perpendicular surface, have been any number of times reflected, by any combination of mirrors, the reflected rays are cut perpendicularly by a series of surfaces; and that all the properties of the reflected system, which depend on the mutual position of the rays, may be deduced by analytic reasonings, from the form of a *Characteristic Function*, whose partial differential coefficients are equal to the cosines of the angles that the reflected ray makes with the axes of coordinates. We can now extend this result, to the systems produced by ordinary refraction, and we may lay it down, as a general Theorem of Optics, that when any system of homogeneous rays, issuing from a luminous point or from a perpendicular surface, has been any number of times modified by any combination of ordinarily reflecting and refracting surfaces, the final rays are cut perpendicularly by a series of surfaces, namely by the Surfaces of Constant Action, considered in the preceding Section; and the cosines of the angles which the ray passing through any assigned point of space, having for coordinates $(x\,y\,z)$, makes with the axes of coordinates, are equal to the partial differential coefficients of a function of those three quantities $(x\,y\,z)$, from the form of which function all the properties of the system may be deduced, and which for that reason I shall call the *Characteristic* of the system.*

XVII. *On the Principal Properties of a Refracted System.*

83. We have just shewn, that in any system of homogeneous rays, produced by ordinary reflection or refraction, the cosines $(\alpha\,\beta\,\gamma)$ of the angles which the ray passing through any given point $(x\,y\,z)$ makes with the axes of coordinates, are of the form

$$\alpha=\frac{dV}{dx}, \quad \beta=\frac{dV}{dy}, \quad \gamma=\frac{dV}{dz}, \qquad \text{(H)}'$$

V being the characteristic of the system. Hence it follows, that all those properties of reflected systems which were deduced in the preceding Part from the corresponding formulæ (M) (**23.**) belong also to homogeneous systems of refracted rays. Thus for example the two equations (**23.**), (**25.**)

$$\frac{dV}{dy}=f\left(\frac{dV}{dx}\right), \qquad \frac{\alpha}{\gamma}\cdot\frac{dz}{dx}+\frac{\beta}{\gamma}\cdot\frac{dz}{dy}=1,$$

the one containing an arbitrary function, the other in partial differentials of the first order, represent an infinite number of pencils, or surfaces, composed by the refracted rays; and the manner of employing them, for the solution of questions respecting the apparent form and magnitude of objects seen by lenses, and for problems of (painting and) perspective, is the same as in the case of reflected systems. And as I believe that the details into which I have entered respecting such systems, in the First Part of this Essay, are sufficient to shew the method that I

* [Hamilton here takes as characteristic $(\Sigma m\rho)/m_n$, where m_n is the refractive index of the final medium (cf. **106.**). In later papers he takes $\Sigma m\rho$ as characteristic.]

have pursued in my researches respecting the systems produced by ordinary refraction, I shall content myself with mentioning some of the principal results to which I have arrived, with regard to these latter systems, without stopping to enter into all the details of calculation.

84. The rays of a refracted system are in general tangents to two caustic surfaces, which are the locus of the centres of curvature of surfaces of constant action (**81.**); and to two series of caustic curves upon those surfaces, which are the *arêtes de rebroussement* of two corresponding series of developable pencils, which cut the refracting surface in two series of curves, that are called the *Lines of Refraction*. With respect to the manner of calculating these several circumstances of a refracted system, we may employ the formulæ (P) (Q) (R) of the First Part; deducing the partial differentials of the characteristic function, either immediately from that function itself, if it be given; or else from the equation of the refracting surface, and from the characteristic function of the incident system, by reasonings analogous to those of **29.** We may also employ the following formulæ, analogous to the equations (T) of the VIIth. Section,

$$\left.\begin{aligned}\rho\,\{(\gamma+m\gamma').\,dp+m\,(d\alpha'+p\,d\gamma')\}&=dx+p\,dz-(\alpha+\gamma p)\,(\alpha\,dx+\beta\,dy+\gamma\,dz),\\ \rho\,\{(\gamma+m\gamma').\,dq+m\,(d\beta'+q\,d\gamma')\}&=dy+q\,dz-(\beta+\gamma q)\,(\alpha\,dx+\beta\,dy+\gamma\,dz),\end{aligned}\right\}\qquad \text{(I)}'$$

in which $\alpha'\ \beta'\ \gamma'$ are the cosines of the angles that the incident ray makes with the axes, and (m) the sine of refraction divided by the sine of incidence.* The two directions on the refracting surface, and the two foci of the refracted ray, determined by these formulæ, are the directions of osculation, and the foci, of the greatest and least focal refractors, analogous to the focal mirrors considered in the VIIIth. Section; and the focal length of such a refractor, for any intermediate direction of osculation, has for expression

$$\rho=\frac{dx^2+dy^2+dz^2-d\rho^2}{(\gamma+m\gamma').\,d^2z+m.\,(d\alpha'dx+d\beta'dy+d\gamma'dz)},\qquad \text{(K)}'$$

which is analogous to the formula (A′) (**39.**) and may like it be transformed into the following

$$\frac{1}{\rho}=\frac{1}{\rho_1}.\cos^2\psi+\frac{1}{\rho_2}.\sin^2\psi,\qquad \text{(L)}'$$

ρ_1, ρ_2 being the two focal lengths of the given refracting surface, and ψ the angle, which the plane drawn through the refracted ray and through the direction of osculation, makes with the tangent plane to one of the developable pencils. We may also prove that the osculating focus, determined by this formula, is the point in which the given refracted ray is crossed by the projection of an infinitely near one, on the plane which passes through the given ray, and through the direction of osculation, from a consecutive point upon which direction the infinitely near ray is refracted; and all the formulæ and reasonings of the IXth. Section, respecting undevelopable and thin pencils in reflected systems, are true of refracted systems also.†

85. The intersection of the two caustic surfaces of a refracted system reduces itself in general to a finite number of *Principal Foci*, which are the centres of spheres that have contact of the second order with the surfaces of constant action; and the foci of osculating focal refractors, that have contact of the second order with the given refracting surface. These principal foci, together

* [The notation differs from that of **78.** Now α, β, γ refer to the refracted ray, α', β', γ' to the incident ray reversed, and m is the reciprocal of the refractive index. But (B)′ still retains the same form.]

† [This is obvious, since such properties are possessed by any normal congruence; it is only when connections are required between the incident and final systems that the theories for reflection and refraction require separate developments.]

with their corresponding rays, or *Axes of the Refracted System*, may be calculated by the formulæ (S′) (T′) (U′) of the Xth. Section, if we know the characteristic of the refracted system; they may also be deduced from the characteristic of the incident system, and from the equation of the refracting surface, by means of the formulæ (I)′ of the present Section, by considering the differentials in those formulæ as independent. Thus, for example, if the incident rays be parallel, the axes and the principal foci of the refracted system are determined by the following equations,

$$\left.\begin{aligned} \rho(\gamma+m\gamma').r &= 1+p^2-(\alpha+\gamma p)^2,\\ \rho(\gamma+m\gamma').s &= pq-(\alpha+\gamma p)(\beta+\gamma q),\\ \rho(\gamma+m\gamma').t &= 1+q^2-(\beta+\gamma q)^2, \end{aligned}\right\} \qquad (\mathrm{M})'$$

by which it may be proved that the axis of the refracted system is situated in the plane of the greatest osculating circle to the refractor, making with the normal an angle the square of whose cosine is equal to the ratio of the radii of curvature; and that a line drawn through the focus, parallel to the incident rays, will pass through the centre of the least osculating circle, and touch the locus of the centres of curvature of the refractor; properties which also follow from the theorem already established, that the principal focus of a refracted system is the focus of an osculating focal refractor which has contact of the second order with the given refracting surface; for when the incident rays are parallel, the focal refractor becomes an ellipsoid of revolution having its axis parallel to the incident rays, and its excentricity equal to the sine of refraction divided by the sine of incidence. In a similar manner we may find the principal foci and the axes when the incident rays diverge from a luminous point; either by the consideration of the osculating Cartesian surfaces; or by the following formulæ, into which the equations (I)′ in this case resolve themselves,

$$(\gamma+m\gamma').r=(1+p^2)\left(\frac{1}{\rho}+\frac{m}{\rho'}\right)-(\alpha+\gamma p)^2\left(\frac{1}{\rho}+\frac{1}{m\rho'}\right),$$

$$(\gamma+m\gamma').s=pq\left(\frac{1}{\rho}+\frac{m}{\rho'}\right)-(\alpha+\gamma p)(\beta+\gamma q)\left(\frac{1}{\rho}+\frac{1}{m\rho'}\right),$$

$$(\gamma+m\gamma').t=(1+q^2)\left(\frac{1}{\rho}+\frac{m}{\rho'}\right)-(\beta+\gamma q)^2\left(\frac{1}{\rho}+\frac{1}{m\rho'}\right),$$

and which may also be thus written

$$\left.\begin{aligned} (\gamma+m\gamma').r &= \frac{1+p^2-(\alpha+\gamma p)^2}{\rho}+m.\frac{1+p^2-(\alpha'+\gamma' p)^2}{\rho'},\\ (\gamma+m\gamma').s &= \frac{pq-(\alpha+\gamma p)(\beta+\gamma q)}{\rho}+m.\frac{pq-(\alpha'+\gamma' p)(\beta'+\gamma' q)}{\rho'},\\ (\gamma+m\gamma').t &= \frac{1+q^2-(\beta+\gamma q)^2}{\rho}+m.\frac{1+q^2-(\beta'+\gamma' q)^2}{\rho'}. \end{aligned}\right\} \qquad (\mathrm{N})'$$

And to find the principal foci, when the incident system has been produced by any given combination of reflecting or refracting surfaces, and when we do not know its characteristic function, we are to calculate by reasonings similar to those of **50.** the variations that $\alpha'\ \beta'\ \gamma'$, (the cosines of the angles that the incident ray makes with the axes of coordinates,) receive in passing from any given point of incidence to an infinitely near point upon the last refracting surface; these variations will give expressions of the form

$$d\alpha'+p\,d\gamma' = A\,dx+B\,dy,$$
$$d\beta'+q\,d\gamma' = B\,dx+C\,dy,$$

and substituting these expressions in the formulæ (I)′ we obtain the following equations

$$\left.\begin{aligned}\rho\{(\gamma+m\gamma')r+mA\}&=1+p^2-(\alpha+\gamma p)^2,\\ \rho\{(\gamma+m\gamma')s+mB\}&=pq-(\alpha+\gamma p)(\beta+\gamma q),\\ \rho\{(\gamma+m\gamma')t+mC\}&=1+q^2-(\beta+\gamma q)^2,\end{aligned}\right\}\qquad(\mathrm{O})'$$

which determine the axes and the principal foci of the last refracted system. In this manner we can find the image of a given luminous point formed by the rays of any one colour, after any number of reflections and refractions, and so calculate the image of any curve or surface, for example of a planet's disk, according to the principles explained in the XIth. Section.

86. The aberrations of homogeneous rays, in a refracted system, may be calculated from the form of the characteristic function, by the reasonings and formulæ of the XIIth. Section; the only difference being in the manner of computing from the incident system the partial differential coefficients of this characteristic function. The reasonings also, and the formulæ, for the density in a reflected system, contained in the XIIIth. Section, apply also to homogeneous sytems of refracted rays; observing only that in deducing the density at the refracting surface, immediately after refraction, from the density immediately before, we cannot, as in reflexion, suppose these two densities equal; but must consider the density just after refraction as being to the density just before, in the proportion of the cosine of incidence to the cosine of refraction. We may also remark that the sign of the quantity which distinguishes between the two principal cases considered in the XIIth. and XIIIth. Sections, of aberration and density at a principal focus, depends in refracted systems on the number of directions in which the given refractor is cut by the osculating focal refracting surface, as in reflected systems we have seen that it depends on the number of directions, in which the given reflector is cut by the osculating focal mirror.

XVIII. *On the determination of Reflecting and Refracting Surfaces, by their lines of Reflexion or Refraction.*

87. Having shewn how to calculate the lines of reflexion and refraction on any given reflector or refractor, the incident system being given, we might now reverse the question, and propose a number of problems similar to those which Monge has resolved, respecting the determination of surfaces by properties of their lines of curvature, in his *Application of Analysis to Geometry.* But as the complete discussion of these questions would be inconsistent with the limits of this Essay, I shall confine myself to the mention of a few, which have appeared to me interesting, either for their novelty or importance.

88. Malus, who first discovered the lines of reflexion and refraction, has remarked in the *Traité D'Optique* that when rays issuing from a luminous point fall upon a mirror, the two conic surfaces which are composed by the incident rays that pass through any given pair of the lines of reflexion, are perpendicular to one another; but that this is not the case for the conic surfaces which pass through the lines of refraction, when the rays issuing from a luminous point fall upon a refracting surface, unless the surface satisfy a certain partial differential equation of the second order, of which he has given the integral, namely the result of elimination of an arbitrary quantity (α) between the following system of equations, the luminous point being made the origin;

$$\left.\begin{aligned}x\,.\,\alpha+y\,.\,\phi+z&=\sqrt{(1+\alpha^2+\phi^2)}\,.\,\psi\,(x^2+y^2+z^2),\\ x+y\,.\,\phi'&=\frac{\alpha+\phi\,.\,\phi'}{\sqrt{(1+\alpha^2+\phi^2)}}\,.\,\psi,\end{aligned}\right\}\qquad(\mathrm{P})'$$

ϕ, ψ representing arbitrary functions, the former of α, the latter of $x^2+y^2+z^2$. This system is the same that Monge has obtained in the XXIVth. Section of the *Application of Analysis to Geometry*, for all surfaces whose normals touch a cone having its centre at the origin of coordinates; and accordingly Malus has concluded, that the normals of every refracting surface, possessing the required property, are tangents to a conic surface, whose centre is at the luminous point; and that therefore the refracted rays are tangents to the same cone, which consequently is one of the caustic surfaces. Malus has also remarked, that upon such a refractor, the lines of refraction are confounded with the lines of reflexion and with the lines of curvature; and he has shewn how the preceding results are modified, when the incident rays are parallel. But as the partial differential equation which Malus has given, for the analytic expression of the refractors that possess the required property, is one of excessive complication, I think that it may be useful to shew briefly, how the same question can be resolved, by the principles and formulæ of the present Essay.

89. For this purpose I resume the formulæ (I)′ of the preceding Section; these formulæ give, by elimination of (ρ), the following general equation for the lines of refraction,

$$\begin{aligned}&\{(\gamma+m\gamma')\,dp+m\,(d\alpha'+p\,d\gamma')\}\,\{dy+q\,dz-(\beta+\gamma q)\,(\alpha\,dx+\beta\,dy+\gamma\,dz)\}\\&\quad=\{(\gamma+m\gamma')\,dq+m\,(d\beta'+q\,d\gamma')\}\,\{dx+p\,dz-(\alpha+\gamma p)\,(\alpha\,dx+\beta\,dy+\gamma\,dz)\},\end{aligned}\qquad (\mathrm{Q})'$$

which may be put under the form

$$A\,dy^2+B\,dx\,dy+C\,dx^2=0;$$

and if we represent by τ, τ' the two values of $\left(\dfrac{dy}{dx}\right)$ furnished by this equation, we shall have the following equations for the tangent planes to the conic surfaces, the luminous point being origin,

$$\text{Ist.}\quad x\,\{(\beta'q-\gamma')\,\tau+\beta'p\}-y\,\{\alpha'p-\gamma'+\alpha'q\tau\}+z\,(\alpha'\tau-\beta')=0,$$

$$\text{IInd.}\quad x\,\{(\beta'q-\gamma')\,\tau'+\beta'p\}-y\,\{\alpha'p-\gamma'+\alpha'q\tau'\}+z\,(\alpha'\tau'-\beta')=0,$$

because these planes pass through the incident ray, and through the tangents to the lines of refraction; and the condition for their being perpendicular to one another, is

$$\begin{aligned}&\{(\beta'q-\gamma')\,\tau+\beta'p\}\,\{(\beta'q-\gamma')\,\tau'+\beta'p\}+\{\alpha'p-\gamma'+\alpha'q\tau\}\,\{\alpha'p-\gamma'+\alpha'q\tau'\}\\&\qquad\qquad+(\alpha'\tau-\beta')\,(\alpha'\tau'-\beta')=0,\end{aligned}$$

that is

$$A\,\{1+p^2-(\alpha'+\gamma'p)^2\}-B\,\{pq-(\alpha'+\gamma'p)\,(\beta'+\gamma'q)\}+C\,\{1+q^2-(\beta'+\gamma'q)^2\}=0,\qquad (\mathrm{R})'$$

because

$$C=A\,.\,\tau\tau',\qquad B=-A\,(\tau+\tau'),\qquad \alpha'^2+\beta'^2+\gamma'^2=1.$$

Now if we separate the equation (Q)′ into two parts, the one containing dp, dq, the other independent of them, and put for abridgment

$$\begin{aligned}&dp\,(dy+q\,dz)-dq\,(dx+p\,dz)+\{(\alpha+\gamma p)\,dq-(\beta+\gamma q)\,dp\}\,(\alpha\,dx+\beta\,dy+\gamma\,dz)\\&\qquad\qquad=A_1dy^2+B_1dy\,dx+C_1dx^2,\end{aligned}$$

$$\begin{aligned}&(d\alpha'+p\,d\gamma')\,\{dy+q\,dz-(\beta+\gamma q)\,(\alpha\,dx+\beta\,dy+\gamma\,dz)\}\\&\quad-(d\beta'+q\,d\gamma')\,\{dx+p\,dz-(\alpha+\gamma p)\,(\alpha\,dx+\beta\,dy+\gamma\,dz)\}=A_2dy^2+B_2dx\,dy+C_2dx^2,\end{aligned}$$

we shall have

$$A=(\gamma+m\gamma')\,A_1+mA_2,$$

$$B=(\gamma+m\gamma')\,B_1+mB_2,$$

$$C=(\gamma+m\gamma')\,C_1+mC_2,$$

and the condition (R)′ becomes

$$(\gamma + m\gamma')[R_1] + m[R_2] = 0,$$

$[R_1]$, $[R_2]$ representing what the first member of (R)′ becomes when we change $A\,B\,C$ to $A_1B_1C_1$, $A_2B_2C_2$. Besides from the relations

$$\rho'(d\alpha' + p\,d\gamma') = (\alpha' + p\gamma')(\alpha' dx + \beta' dy + \gamma' dz) - (dx + p\,dz),$$
$$\rho'(d\beta' + q\,d\gamma') = (\beta' + q\gamma')(\alpha' dx + \beta' dy + \gamma' dz) - (dy + q\,dz),$$
$$\alpha + p\gamma = -m(\alpha' + p\gamma'), \qquad \beta + q\gamma = -m(\beta' + q\gamma'),$$

ρ' being the distance of the luminous point from the point of incidence, it follows that the part $m[R_2]$ is identically nothing; which may also be proved by observing that every plane refractor possesses the property required; the condition therefore reduces itself to $[R_1] = 0$, that is

$$\begin{aligned} &r.\{pq(\beta' + q\gamma')^2 - (1 + q^2)(\alpha' + p\gamma')(\beta' + q\gamma')\} \\ &+ s.\{(1 + q^2)(\alpha' + p\gamma')^2 - (1 + p^2)(\beta' + q\gamma')^2\} \\ &+ t.\{(1 + p^2)(\alpha' + p\gamma')(\beta' + q\gamma') - pq(\alpha' + p\gamma')^2\} = 0; \end{aligned}$$

which, when we substitute for the cosines $\alpha'\,\beta'\,\gamma'$ the coordinates $x\,y\,z$ to which they are proportional, becomes

$$\{(y + qz)r - (x + pz)s\}\left(q - \frac{x + pz}{py - qx}\right) = \{(y + qz)s - (x + pz)t\}\left(p + \frac{y + qz}{py - qx}\right), \qquad \text{(S)}'$$

a partial differential equation, linear and of the second order, which has been integrated by Monge in the Section before referred to, and upon which therefore it would be superfluous here to delay. I may however remark, that we arrive at the same equation, when we investigate on similar principles the form of a reflecting or refracting surface, on which the lines of reflexion or refraction are perpendicular to one another, the incident rays being supposed to proceed from a luminous point placed at the origin; from which it follows that the property, observed by Malus to belong to that class of surfaces, whose normals are tangents to a cone having its centre at the luminous point, namely that on such a surface the lines of reflexion and refraction are confounded with the lines of curvature, is peculiar to the surfaces of that class, belonging to them only.

90. Many other questions may be proposed, respecting the determination of reflecting and refracting surfaces, by given properties of their lines of reflexion or refraction. A question of this sort conducts in general to a partial differential equation of the second order; and when we can succeed in finding the form of the integral, the determination of the arbitrary functions will still require, in general, the assistance of the Calculus of Finite Differences. As an example of this, it may be observed, that when rays parallel to the axis of (z) fall upon a reflecting surface, the lines of reflexion are given by the differential equation

$$dp\,dy - dq\,dx = 0 = s(dy^2 - dx^2) + (r - t)\,dx\,dy; \qquad \text{(T)}'$$

if then it were required to determine the mirror by the condition that one set of these lines should be in planes parallel to the plane of (xz), we should have to integrate the partial differential equation

$$s = 0;$$

and if, by a different choice of coordinates, we made these planes parallel to one bisecting the angle between the planes of (xz) (yz), the equation would then become

$$r - t = 0,$$

which is a particular case of the celebrated Equation of Vibrating Chords.

91. There is one problem of this kind which deserves to be considered separately; it is the question, to find a reflecting or refracting surface, on which there shall be only one set of lines of reflection or refraction, the incident rays being supposed to come from a given point, or to belong to a given rectangular system. This question is analytically expressed by equalling to nothing the radical which enters into the value of $\left(\frac{dy}{dx}\right)$, deduced from the equation of the lines of reflexion or refraction; which furnishes a partial differential equation of the second order, analogous to that which Monge has given in the XIXth. Section of the *Analyse*, for that class of surfaces whose two sets of lines of curvature are confounded with one another. Monge, in integrating that equation by very ingenious reasonings, has found that instead of two arbitrary functions, the integral does not contain even one, but represents spheric surfaces only. This result appeared extraordinary to Monge, because he did not perceive that the partial differential equation to be integrated, resolves itself into two distinct equations; a decomposition which was first observed by Meusnier,* and to which I also had been conducted, in considering the same equation in an optical point of view, when I had found, contrary to the opinion of Malus, that the rays of a reflected or refracted system are always cut perpendicularly by a series of surfaces, namely by the surfaces of constant action. For it followed from that theorem, that if the lines of reflexion or refraction coincide on the reflecting or refracting surface, the lines of curvature must coincide upon the surfaces which cut the rays perpendicularly; and therefore that these latter surfaces must satisfy the partial differential equation,

$$4(rt-s^2)(1+p^2+q^2)-[(1+q^2)r-2pqs+(1+p^2)t]^2=0,$$

which is the one integrated by Monge in the Section before referred to; and this I found to resolve itself into the two following,†

$$\frac{r}{1+p^2}=\frac{s}{pq}=\frac{t}{1+q^2},$$

which Monge has shewn to be peculiar to spheric surfaces. It follows from this decomposition, or from that indicated at the beginning of the Xth. Section of this Essay, that the partial differential equation expressing the condition for the lines of reflexion or refraction coinciding, resolves itself in like manner into two distinct equations; and that the integral does not contain any arbitrary functions, but represents that class of focal surfaces, considered in the IInd. and XVth. Sections of this Essay.

XIX. *On the determination of Reflecting and Refracting Surfaces, by means of their Caustic Surfaces.*

92. The question, to find a surface which shall reflect or refract the rays of a given system, so as to make them tangents to a given caustic surface, is interesting in an analytic point of view, and may perhaps admit of useful applications; I am going therefore in this Section to offer some remarks upon it, and on the theory that Monge has given, for the solution of the corresponding questions in the *Applications of Analysis to Geometry.*

* [*Mémoires présentés à l'Académie Royale des Sciences*, 10 (1785), p. 501.]

† [In another manuscript (Note Book 19) we find this effected by reducing the equation to the form

$$(1+p^2+q^2)(A-B)^2+p^2q^2(A+B)^2=0,$$

where

$$A=\frac{r}{1+p^2}-\frac{s}{pq},\quad B=\frac{t}{1+q^2}-\frac{s}{pq}.]$$

93. In that profound and beautiful work, Monge has considered a number of problems, respecting the determination of curve surfaces,* by means of their centres of curvature; he has shewn that the question, to find a surface whose normals are tangents to a given surface, conducts in general to a partial differential equation of the first order, or to an integral equation involving an arbitrary function; and that if the surface which the normals are to touch, be not entirely given, but only belong to a given class, for instance if we only know that it is a conic or developable surface, then new arbitrary functions come in, and the partial differential equation rises to an order higher than unity.

These questions are treated in the XXIIId. and two following Sections of the *Analyse*; and the plan which Monge has adopted in those Sections, is first to obtain the functional and differential equations of the surfaces sought, by geometrical reasonings respecting the generation of those surfaces; then to deduce the former equations from the latter, by methods of integration founded on the theory that he has given, respecting the characteristics of surfaces, belonging to a given class, or represented by a given partial differential equation. In this manner he has found, for example, that if the normals to a curve surface be tangents to a sphere, they are also tangents to a conic surface having its summit at the centre of that sphere; and that the curve surface sought, can be generated by making the plane containing the involute of a circle, whose radius is equal to the radius of the sphere, roll round the conic surface, in such manner that the circle, which rolls along with its involute, may have its centre constantly at the summit of the cone. I am going to shew that these and similar results, together with the methods that lead to them, may be applied to the determination of reflectors and refractors by means of their caustic surfaces; and then to point out, and to integrate, another class of partial differential equations, of the second and higher orders, which the same problems conduct to, but which have not been considered by Monge.

94. For this purpose I observe that it has been already proved, that the rays of a reflected or refracted system are cut perpendicularly by a series of surfaces, namely by the surfaces of constant action; and that if we know the equation of one of these surfaces, together with a corresponding surface for the incident system, we can deduce the equation of the reflecting or refracting surface, by the general formula

$$\Sigma\,(m\rho) = \text{const.},$$

without the necessity of any integration. If then it be required, to find a reflecting or refracting surface, which shall make the rays of a given system tangents to a given caustic surface, it is sufficient to find, by the methods of Monge, the form of the surfaces whose normals touch that given caustic surface; these will be the surfaces of constant action, of the reflected or refracted system, and the difficulty of the problem will be reduced to elimination alone.

95. With respect to that new class of partial differential equations, alluded to in **93.**, I observe that since the coordinates of a focus of a reflecting or refracting surface, are functions of the coordinates and partial differentials of that surface as high as the second order; if we establish any given relation between these coordinates, in order to express that the focus is contained upon a given caustic surface, we get a partial differential equation of the second order, which is more general than the one furnished by the preceding methods; and which therefore would seem to indicate, that besides the surfaces discovered by those methods, there exist an infinite number of other surfaces,

* [The correct but unusual expression "curve surface" appears consistently.]

possessing the property required. Thus for example, the two foci of a mirror, if the incident rays be parallel to the axis of (z), are determined by the formulæ

$$X = x - \frac{2p\rho}{1+p^2+q^2}, \quad Y = y - \frac{2q\rho}{1+p^2+q^2}, \quad Z = z + \left(\frac{1-p^2-q^2}{1+p^2+q^2}\right).\rho, \tag{U$'$}$$

ρ being the focal length, determined by the quadratic equation

$$4\rho^2(rt-s^2) - 2\rho(1+p^2+q^2)(r+t) + (1+p^2+q^2)^2 = 0; \tag{V$'$}$$

and if we establish any given relation between $X\,Y\,Z$; for instance if we put

$$X^2 + Y^2 + Z^2 = 1,$$

to express that one of the two foci is always on a sphere, having its centre at the origin, and its radius equal to unity; the mirror will in this case be represented by the following partial differential equation of the second order,

$$1 = x^2 + y^2 + z^2 + \rho^2 + \frac{2\rho}{1+p^2+q^2}\{(1-p^2-q^2)\,z - 2(px+qy)\}, \tag{W$'$}$$

in which ρ is to be considered as a given function of $p\,q\,r\,s\,t$, deduced from the quadratic equation (V)$'$ already mentioned; or by that equation (V)$'$, ρ being considered as a given function of $x\,y\,z\,p\,q$, deduced from (W)$'$. To integrate this partial differential equation (V)$'$, let us investigate, according to the method of Monge, the *characteristic curve*; that is, the curve in which the surface represented by the equation (V)$'$ is touched by a consecutive surface of the same kind. For this purpose Monge has shewn that we are to employ the equation

$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0, \tag{X$'$}$$

$R\,S\,T$ representing, for abridgment, the coefficients of dr, ds, dt, obtained by differentiating the proposed equation of the second order; in the present case we have by (V)$'$,

$$R = 4\rho^2 t - 2\rho(1+p^2+q^2),$$
$$S = -8\rho^2 s,$$
$$T = 4\rho^2 r - 2\rho(1+p^2+q^2),$$

and since by the same equation (V)$'$, these coefficients are connected by the relation

$$4RT - S^2 = 0, \tag{Y$'$}$$

the quadratic equation (X)$'$ has its roots equal, and furnishes the two following equations, both belonging to the characteristic curve,

$$2R\,dy - S\,dx = 0, \quad 2T\,dx - S\,dy = 0, \tag{Z$'$}$$

that is

$$2\rho\,.\,dq = (1+p^2+q^2)\,dy, \quad 2\rho\,.\,dp = (1+p^2+q^2)\,dx. \tag{A$'$$'$}$$

These equations express, that the characteristic curve is one of the lines of reflexion upon the unknown mirror; namely that line for which the focal distance is equal to ρ; and if we combine them with the formulæ (U)$'$, which determine the coordinates of the focus, we find the following expressions for the differentials of those coordinates,

$$\left.\begin{aligned} dX &= -2p\,.\,d\left(\frac{\rho}{1+p^2+q^2}\right),\\ dY &= -2q\,.\,d\left(\frac{\rho}{1+p^2+q^2}\right),\\ dZ &= (1-p^2-q^2)\,.\,d\left(\frac{\rho}{1+p^2+q^2}\right).\end{aligned}\right\} \tag{B$'$$'$}$$

Substituting these values in the differential of (W)′, which may be thus written

$$X.dX+Y.dY+Z.dZ=0, \qquad \text{(C}'\text{)}'$$

we find this other equation for the characteristic curve,

$$\{-2(pX+qY)+(1-p^2-q^2)Z\}.d\left(\frac{\rho}{1+p^2+q^2}\right)=0; \qquad \text{(D}'\text{)}'$$

unless therefore the factor in the brackets vanishes, that is unless we have

$$-2(px+qy)+(1-p^2-q^2)z+(1+p^2+q^2)\rho=0, \qquad \text{(E}'\text{)}'$$

a particular case which we shall consider presently; we must have the other factor nothing, that is

$$d\left(\frac{\rho}{1+p^2+q^2}\right)=0,$$

and therefore, by (B′)′,

$$dX=0, \quad dY=0, \quad dZ=0. \qquad \text{(F}'\text{)}'$$

We see then, that in general, the rays reflected from any one characteristic curve, upon the unknown mirror represented by the partial differential equation (V)′, all converge to some one focus $X\,Y\,Z$ upon the given sphere,

$$X^2+Y^2+Z^2=1;$$

and therefore that the mirror is touched in the whole extent of this curve, by a paraboloid of revolution, having its axis parallel to the incident rays, and its focus at the point $X\,Y\,Z$; from which it follows that all the reflected rays pass through some one curve upon the given sphere, and that the mirror is the enveloppe of a series of paraboloids, whose axes are all vertical, and whose foci are all ranged upon this arbitrary spheric curve. The complete integral of (V)′ is therefore represented by the following system of equations,

$$\left.\begin{aligned} \sqrt{\{(x-X)^2+(y-Y)^2+(z-Z)^2\}} &= P+z-Z, \\ -\frac{(x-X)\,dX+(y-Y)\,dY+(z-Z)\,dZ}{\sqrt{\{(x-X)^2+(y-Y)^2+(z-Z)^2\}}} &= dP-dZ, \end{aligned}\right\} \qquad \text{(G}'\text{)}'$$

P being the semi-parameter of the generating paraboloid, and $X\,Y\,Z$ being considered as functions of P, two of which are arbitrary, and the third deduced from those two by the given equation

$$X^2+Y^2+Z^2=1.$$

96. But although the class of mirrors, to which we have been conducted by the integration of the preceding paragraph, satisfies the partial differential equation (V)′, and possesses the property, expressed by that equation, of having one focus always contained on the given sphere; it is evident that these mirrors do not possess that other property which was originally required, of making the reflected rays tangents to the given caustic sphere; and that with respect to them, this sphere is in no other sense a caustic, than as containing a curve through which the rays all pass. On the other hand, it is clear that the mirrors of that other class, which reflect the rays so as to touch the sphere, and which are obtained by the methods explained in the beginning of this Section, must also be included in the partial differential equation (V)′; since they also have one set of foci contained upon the given sphere. This difficulty is removed by the Theory of Singular Primitives; and by observing that although the factor (E′)′, which enters into the equation (D′)′ of the characteristic of the mirror, must be suppressed when we wish to give to the integral all the generality of which it is susceptible, there yet exists a series of mirrors for which that factor vanishes; less general indeed, than the class represented by the system (G′)′, which contains two arbitrary functions; but, like that class, satisfying the proposed equation (V)′, and represented by the singular primitive of that equation, namely by the formula (E′)′, considered as a partial differential equation of the first order; and it is this less general series of mirrors which satisfies the original condition, and of which the integral equation, with one arbitrary function, is to be obtained by the methods of Monge, and by the properties of the surfaces of constant action.

97. It is easy to generalise the preceding reasonings, and to shew that the partial differential equation of the second order which represents all the reflectors or refractors that have one set of their foci contained upon a given caustic surface, the incident system being given, is susceptible of two distinct integrations, and conducts to two classes of surfaces entirely different from one another. The one class, represented by a system of equations with two arbitrary functions, includes all those reflecting or refracting surfaces which make the rays pass through any assumed curve upon the given caustic surface; and this class may be considered as the enveloppe of a series of focal reflectors or refractors which have their foci ranged on that assumed curve. The other class contains those surfaces which make the rays touch the given caustic surface; and these are represented by a partial differential equation of the first order, which is a singular primitive of the proposed. Similar reasonings may be applied to the corresponding questions in the *Application of Analysis to Geometry*; and we may observe, that if instead of being given the equation of the locus, of foci in the one case, or centres in the other, we are only given a class of surfaces to which that locus belongs; as, for example, if it were required to find a mirror which should reflect rays issuing from a given luminous point, in such a manner that one set of foci should be upon a conic surface having the luminous point for its centre; we should then be conducted to a partial differential equation, of an order higher by unity, than that which expresses that the rays are tangents to a locus belonging to the class proposed; and the former equation would have the latter for a singular primitive, while the complete integral would belong to a more general series of reflecting or refracting surfaces, the rays from which, instead of touching the locus, pass through some curve upon it.

98. In all the preceding questions, we have supposed one only of the two caustic surfaces given; because these two surfaces are connected with one another, by a relation arising from the property which I have proved that they possess, of being always the two surfaces of centres of curvature of the surfaces of constant action; from which it follows, by what Monge has proved respecting centres of curvature in general, that from whatever point we look at the two caustic surfaces of a reflected or refracted system, the profiles of those two surfaces will cut one another perpendicularly. It is not possible, therefore, to find a reflector or refractor which shall make the rays of a given system tangents to two given surfaces, unless those surfaces satisfy the geometrical condition just mentioned, or the analytic condition corresponding, investigated by Monge in the *Analyse*; and we should recognise this, by being conducted to an equation in ordinary differentials, not satisfying the condition of integrability. In general, if it be required to determine a reflector or refractor, by properties of its caustic curves or surfaces; and if the number and nature of the given conditions be such as to conduct to an equation in ordinary differentials, this equation will not be integrable, that is, it will not represent a series of surfaces, if the conditions contain anything inconsistent, either with one another, or with the general properties of reflected and refracted systems; for example, if it were required that the rays should be tangents to a given set of small circles on a sphere, which would be inconsistent with the general property, that every caustic curve is the shortest between two points upon the surface to which it belongs. (Rays touching a small circle on a sphere could never be collected in one point by any number of ordinary reflections or refractions.)

99. The only other remark that I shall make upon questions of this nature, is that if it were required to find a reflector or refractor, whose two sets of foci coincide with one another, we should obtain the equation of the surface, in partial differentials of the second order, by equalling to nothing the radical which enters into the general expression for the focal length; but that in consequence of the decompositions mentioned in **91.** this equation would resolve itself into two others, which can only be satisfied by those focal reflectors or refractors, whose two caustic surfaces not only coincide with one another, but reduce themselves to one single point. However, although the two equations thus obtained, cannot be satisfied by all the points of any other surface; and though they determine in general upon any given reflector or refractor, a finite number of points, which I have called the *vertices* of the surface; yet, there are some particular surfaces, for which those two equations acquire a common factor, and are satisfied by all the points of a certain curve, which is analogous to the line of spheric curvature that exists upon some curve surfaces, and which from that analogy, may be called the *line of focal curvature.* We may remark also that when a line of this kind exists, there exists a corresponding line of foci, the intersection of the two caustic surfaces.

XX. *On the Caustics of a given Reflecting or Refracting Curve.*

100. We have seen that when the rays of a given system are reflected or refracted at a given surface, there are only two particular sets of curves upon that surface, (namely the lines of reflexion or refraction,) which possess the property, that the rays from any one of them are tangents to a caustic curve. We have shewn how to find these curves, together with the caustics corresponding; and we have proved (**31.**) that those caustics possess the distinguishing property, that their arc, comprised between any two given limits, is equal to the increment received by the characteristic function of the system in passing from one end of the arc to the other; so that in reflected systems, this arc is equal to the difference of the two polygon paths traversed by the light in arriving at the two extremities; and, in refracted systems, to the difference of the two sums obtained by multiplying every side of each polygon path by the refractive power of the medium in which it lies, the refractive power of the last medium being unity. If then it were required to find a reflecting or refracting curve which should have a given caustic curve, it would be sufficient to make a focal mirror in the one case, or a focal refracting surface in the other, (that is, a surface which would reflect or refract the rays of the given incident system to some one point XYZ, and which may therefore be represented by the equation

$$\rho + \rho' = \text{const.}, \quad \text{or}, \quad \rho + m\rho' = \text{const.},$$

ρ', ρ being the paths traversed by the light, from a given surface cutting the incident rays perpendicularly, to the reflector or refractor, and from this to the focus) move in such a manner that its focus may be always at some point on the given caustic curve, and that the constant of its equation may be equal to the arc of that curve counted from any assumed point; the enveloppe of the series of focal surfaces thus obtained, will touch the unknown reflector or refractor in the whole extent of the reflecting or refracting curve; and therefore the intersection of this enveloppe, with the developable locus of tangents to the given caustic curve, will be one of the reflecting or refracting curves to which that caustic corresponds.

101. In this manner we can find the curves which have a given caustic; let us now reverse the question, and investigate the caustics of a given curve. Those who have hitherto written upon Optics have considered only one such caustic, namely that which is touched by the rays reflected or refracted from a plane curve, in the plane of that curve, the incident rays also being supposed to lie in the same plane. Malus indeed, has shewn that the rays reflected or refracted from a curve of double curvature are tangents to a caustic curve, if that curve be a line of reflection or refraction on the surface to which it belongs; but I do not know that any one has remarked, that a given curve may have an infinite number of caustics, corresponding to the infinite number of different surfaces on which it may be a line of reflexion or refraction. I am going to point out the means of determining these different caustic curves, and to shew that they have for their locus a surface, possessing some remarkable properties.

102. For this purpose I observe that when a ray of light falls upon a given reflecting or refracting curve, we cannot entirely determine the direction of the reflected or refracted ray, unless we know the tangent plane to the reflecting or refracting surface; but the principle of least action

$$\delta\rho + \delta\rho' = 0, \quad \delta\rho + m\,\delta\rho' = 0,$$

proves that the projection of the element of the arc, on the reflected or refracted ray, bears a given ratio to the projection of the same element on the incident ray, and therefore that the reflected [or refracted] ray must be one of the sides of a given cone of revolution, whose centre is at the point of incidence, and whose axis touches the reflecting or refracting curve. If then the rays of a given incident system fall upon a given curve, the reflected or refracted rays will all be contained on the series of cones, constructed in the manner just mentioned; they will therefore be tangents to the enveloppe of that series; and this enveloppe will be the locus of the caustic curves. Besides, if we observe that the osculating plane of any particular caustic curve, contains two consecutive rays, and therefore passes through two consecutive points upon the reflecting or refracting curve, we shall see that this plane contains the axis of the generating cone, and is therefore perpendicular to the surface of that cone, and consequently to the enveloppe; from which it follows that the caustic curves are the shortest between two points upon that enveloppe, and may therefore be found by stretching a thread in the direction of any particular ray, and winding it about the enveloppe. With respect to the equations of the generating cone, the enveloping surface, and the caustic curves, these may easily be deduced from the principles and formulæ that have been already laid down; I shall only observe, therefore, that the enveloppe of the cones is also the enveloppe of all the caustic surfaces, corresponding to the infinite number of reflectors or refractors upon which the given curve is a line of reflexion or refraction; and that if we looked at this enveloppe, from any point on that given curve, its profile would always be circular. It belongs therefore to a general class of surfaces, which we may call *Surfaces of Circular Profile*; and this class includes, as particular cases, not only spheres, surfaces of revolution, and developable surfaces, but also the class which Lancret* considered, generated by a circular cone of constant angle, moving in such a manner that its axis is constantly tangent to a given curve, while its centre is at the point of contact.

* [*Mémoires présentés à l'Institut des Sciences par divers savans*, 2 (1811), p. 19.]

XXI. *On the conditions of Achromatism.*

103. It has been shewn, that when rays of any one colour issuing from a luminous point have been any number of times reflected or refracted, there are in general a finite number of isolated foci, at which the density of light is greatest, and at which the image of the luminous point is formed. Denoting the coordinates of such a focus by $X\,Y\,Z$, they may be considered as functions of the colour of the rays, or of the quantity (m) which measures the degree of refrangibility at the last refracting surface; and if we suppose this quantity (m) to receive a small increment (δm), that is if we pass from the rays of a given colour to those of another colour, the focal coordinates will receive corresponding increments, and will become

$$\left.\begin{aligned} X+\delta X &= X+\frac{dX}{dm}\,.\,\delta m+\frac{1}{2}\,\frac{d^2X}{dm^2}\,.\,\delta m^2+\ldots,\\ Y+\delta Y &= Y+\frac{dY}{dm}\,.\,\delta m+\frac{1}{2}\,\frac{d^2Y}{dm^2}\,.\,\delta m^2+\ldots,\\ Z+\delta Z &= Z+\frac{dZ}{dm}\,.\,\delta m+\frac{1}{2}\,\frac{d^2Z}{dm^2}\,.\,\delta m^2+\ldots,\end{aligned}\right\} \qquad \text{(H}'\text{)}'$$

formulæ by which we can determine the chromatic aberrations, when we have calculated the partial differential coefficients $\left(\dfrac{dX}{dm}\right)$ &c. And if we confine ourselves to the first power of (δm), the conditions of achromatism will be

$$\frac{dX}{dm}=0, \qquad \frac{dY}{dm}=0, \qquad \frac{dZ}{dm}=0. \qquad \text{(I}'\text{)}'$$

104. With respect to the calculation of the partial differential coefficients which enter into these formulæ, and their use in the construction of achromatic lenses, the discussion of these points would lead us too far from the plan and subject of this Essay, which is intended only to contain the general principles of Optics; but I hope to finish at some future time, a Memoir of a more practical nature, in which the principles and formulæ, here laid down, shall be applied to the construction of reflecting and refracting telescopes.*

XXII. *On Systems of curved Rays.*

105. We have hitherto supposed that the actions of the reflecting or refracting surfaces take place at finite intervals, and therefore that the course of the light is a polygon of a finite number of sides; but if we suppose the light to pass through a medium of continually varying density, such as the atmosphere of the Earth, then the direction of the ray will alter continually, and the polygon will become a curve. To find the equations of this curve, we are to investigate, by the Calculus of Variations, the increment which the definite integral, $\int v\,d\rho$, receives, when we pass from one curved ray to another infinitely near it, (v) being the refractive power of the medium, and $(d\rho)$ the element of the path; and then to express, according to the principle of least action, that this increment would be nothing, if the limits of the integral were fixed. In this manner we find, first for the variation of the integral,

$$\begin{aligned}\delta\int v\,d\rho=\int\delta\,(v\,d\rho)&=v\,(\alpha\,\delta x+\beta\,\delta y+\gamma\,\delta z)-v'\,(\alpha'\delta x'+\beta'\delta y'+\gamma'\delta z')\\ &+\int\left\{\frac{dv}{dx}\,.\,d\rho-d\,.\,v\,\frac{dx}{d\rho}\right\}.\,\delta x+\int\left\{\frac{dv}{dy}\,.\,d\rho-d\,.\,v\,\frac{dy}{d\rho}\right\}.\,\delta y+\int\left\{\frac{dv}{dz}\,.\,d\rho-d\,.\,v\,\frac{dz}{d\rho}\right\}.\,\delta z, \qquad \text{(K}'\text{)}'\end{aligned}$$

* [See footnote to p. 165.]

v, α, β, γ, being the refractive power of the medium and the cosines of the angles which the direction of the ray makes with the axes, at the last limit, and $v', \alpha', \beta', \gamma'$, the corresponding quantities at the first limit; and secondly, for the equations of the ray,

$$\frac{dv}{dx}.d\rho = d.v\frac{dx}{d\rho}, \quad \frac{dv}{dy}.d\rho = d.v\frac{dy}{d\rho}, \quad \frac{dv}{dz}.d\rho = d.v\frac{dz}{d\rho}, \qquad \text{(L}')'$$

three formulæ which are equivalent to but two distinct ones, as appears by adding them, multiplied respectively by (dx, dy, dz), and attending to the following identical relations,

$$dx^2 + dy^2 + dz^2 = d\rho^2, \qquad dx.d\frac{dx}{d\rho} + dy.d\frac{dy}{d\rho} + dz.d\frac{dz}{d\rho} = 0.$$

106. It may be shewn by means of these equations, that the osculating planes of the curved ray are always perpendicular to the strata of uniform refractive power, which have for equation

$$v = \text{const.}; \qquad \text{(M}')'$$

a result to which Malus also has arrived, by different reasonings. But it is more important to observe that the rays themselves are perpendicular to the surfaces of constant action, that is to the surfaces for which the definite integral, $\int v d\rho$, is equal to any given quantity; the rays being supposed to issue originally from a luminous point, or from a perpendicular surface, and this point or surface being taken for the first limit of the integral; in such a manner that the part

$$(\alpha'\delta x' + \beta'\delta y' + \gamma'\delta z')$$

vanishes, in the expression (K′)′ for the variation of that definite integral. In fact, this variation then becomes, in consequence of the equations (L′)′,

$$\delta\int(v d\rho) = v.(\alpha\delta x + \beta\delta y + \gamma\delta z), \qquad \text{(N}')'$$

and if we put it equal to nothing, that is if we suppose the definite integral equal to any constant quantity, we shall have

$$\alpha\delta x + \beta\delta y + \gamma\delta z = 0, \qquad \text{(O}')'$$

which proves, as has been already stated, that the curved rays are cut perpendicularly by the surfaces of constant action; in the same manner as it was before proved, that the systems of straight rays, produced by a finite number of reflections or refractions, are cut perpendicularly by their corresponding surfaces. And as in those systems, the direction of the ray passing through a given point of space was determined by the equations

$$\alpha = \frac{dV}{dx}, \quad \beta = \frac{dV}{dy}, \quad \gamma = \frac{dV}{dz},$$

$(\alpha\ \beta\ \gamma)$ being the cosines of the angles which the ray makes with the axes, and (V) the sum of the several paths traversed by the light, in arriving at the given point $(x\ y\ z)$, multiplied each by the refractive power of the medium in which it lies, and divided by the power of the last; so, in the systems of curved rays which we are now considering, the cosines $(\alpha\ \beta\ \gamma)$ are determined by the equations

$$\alpha = \frac{1}{v}.\frac{dI}{dx}, \quad \beta = \frac{1}{v}.\frac{dI}{dy}, \quad \gamma = \frac{1}{v}.\frac{dI}{dz}, \qquad \text{(P}')'$$

(I) being the definite integral, $\int v\,d\rho$, taken to the point $(x\ y\ z)$, and (v) the refractive power of the medium at that point. Finally, as in the systems of straight rays we have called the function (V) the characteristic function, and have shewn that all the properties of the system may be deduced from it; so, in the systems of curved rays, we may call the function (I) the *characteristic function*,* and may deduce all the properties of the system from it, provided that we know also the form of the function (v), that is, the Law of the Refractive Power of the medium; we can even deduce the latter function from the former, by means of the following relation,

$$v^2 = \left(\frac{dI}{dx}\right)^2 + \left(\frac{dI}{dy}\right)^2 + \left(\frac{dI}{dz}\right)^2. \qquad (Q')'$$

* [It is the function here denoted by I that is afterwards adopted generally as characteristic function, and denoted by V.]