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Modeling of turbulence by ensembles of vortices with inviscid interaction(*)

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IN THE PAPER the accuracy of some vortex models of the turbulence theory are discussed. It is shown that the numerical modeling of turbulent spectra based on the dynamics of point (discrete) vortex systems requires the introduction of the intrinsic structure of vortices. For the statistically equilibrium blobs model of coherent structures in turbulent shear layer, the correctness with respect to the principle of informational entropy nondecreasing was established. The linear stability of model vortex blobs was proved. By generalization of this model, rotational MHD flows with coaxial turbulent shear layers were studied. The results are in a good agreement with the experimental data.

W pracy przedyskutowano dokładność pewnych modeli wirowych w teorii turbulencji. Pokazano, że modelowanie numeryczne widm turbulencji oparte na dynamice punktowych (dyskretnych) układów wirów wymaga wprowadzenia struktury wewnętrznej tych wirów. W przypadku statystycznie równowagowych modeli pęcherzykowych struktur w burzliwej warstwie ścinanej stwierdzono poprawność względem zasady niemalejącej entropii. Wykazano liniową stateczność modelowych pęcherzyków wirowych. Uogólnienie tej metody pozwoliło na przeanalizowanie wirowych przepływów MHD z współosiowymi warstwami przepływów ścinających. Otrzymane wyniki wykazują dobrą zgodność z danymi doświadczalnymi.

В работе исследуются вопросы корректности некоторых вихревых моделей теории турбулентности. Показано, что при численном моделировании спектров турбулентности на основе динамики систем точечных (дискретных) вихрей необходимо вводить внутреннюю структуру вихрей. Для модели когерентных структур в турбулентном слое сдвига, использующей статистически равновесные [вихревые] „таблетки", установлена ее коррекность в отношении принципа неубывания информационной энтропии. Доказана линейная устойчивость модельных вихревых стр ктур. На основе обобщения этой модели исследованы вращающиеся МГД-течения с коаксиальными т рбулентными сдвиговыми слоями. Полученные при этом результаты хорошо согласуются с экспериментальными данными.

1. Introduction

UP-TO-DATE IDEAS about the mechanisms of developed hydrodynamic turbulence show that many processes in turbulent flows can be considered within the framework of a conception of inviscid interaction of regions with concentrated vorticity.

Thus conditions are created for modeling the effects of a developed turbulence by means of point (discrete) vortex ensembles in an ideal fluid. For the first time the possibility of this approach was indicated in the works [1, 2]. At present a large number of works is known where the vortex ensembles with Hamiltonian dynamics are used to construct statistical turbulence models. Some of these results are given in a review [3]. Numerous

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papers are devoted to applications of discrete vortex systems for numerical modeling of turbulence and vortex flows at large Reynolds numbers [4]. It can be noted that vortex models possess a certain universality and have a wide range of uses: from small scale (subgrid) turbulence simulation up to the studies of large vortex structures. Their indubitable merit is that they are immediately directed at describing vortex distributions and evolution which, in modern theories and experiments on turbulence, acquire a growing significance. Also it is essential that being relatively simple, these models truly reflect the principle of nonlinear effects of vortex dynamics.

The present work deals with the problems connected with certain applications of vortex models to the turbulence theory problems. The peculiarity of the application of discrete vortices dynamic systems to the numerical modeling of isotropic turbulence spectra is dealt with in Sect. 2. The properties of the model of plane coherent vortex structures using statistically equilibrium distributions of vortices are investigated in Sect. 3. The results of the modeling of coherent structures in annular shear layers and in rotating MHD flows obtained by this model generalization are presented in Sect. 4.

To model two-dimensional turbulence effects a system of straightline vortex filaments with the same circulation $\varkappa$ was used as an original one. The dynamics of the system in the ideal noncompressible fluid without the boundaries and the external velocity field is described [5] in a canonical form:

$$\frac{d\mathbf{r}_i}{dt} = -\varkappa \mathbf{e}\times \nabla_{\mathbf{r}_i} H_N, \qquad H_N = -\sum_{i<j}^{N} \ln |\mathbf{r}_i - \mathbf{r}_j|, \qquad i,j = 1, \ldots, N. \tag{1.1}$$

The system (1.1) has 4 integrals of motion:

$$H_N = \text{const}, \qquad \mathbf{R} = N^{-1}\sum_{i=1}^{N} \mathbf{r}_i = \text{const}, \qquad L^2 = N^{-1}\sum_{i=1}^{N} (\mathbf{r}_i - \mathbf{R})^2 = \text{const}, \tag{1.2}$$

connected with the conservation of energy, momentum and angular momentum, respectively.

In a three-dimensional case the point vortices model with dipole-dipole interaction [6] was considered. The dynamics of the system is described in terms of Lamb momenta of vortices $\{\mathbf{p}_1, \ldots, \mathbf{p}_N\}$ and their radius-vectors $\{\mathbf{r}_1, \ldots, \mathbf{r}_N\}$ canonically conjugated with respect to a Hamiltonian:

$$\begin{gathered} H_N = \sum_{i=1}^{N} T(|\mathbf{p}_i|) + \sum_{i<j}^{N} \Phi_{ij} \\ T(|\mathbf{p}|) = A|\mathbf{p}|^{1/k}, \qquad \Phi_{ij} = -(4\pi)^{-1}\left[\frac{(\mathbf{p}_i\cdot\mathbf{p}_j)}{|\mathbf{r}_{ij}|^3} - 3\,\frac{(\mathbf{p}_i\mathbf{r}_{ij})(\mathbf{p}_j\mathbf{r}_{ij})}{|\mathbf{r}_{ij}|^5}\right]. \end{gathered} \tag{1.3}$$

Here the integrals of motion are the Hamiltonian H_N as well as a summarized momentum and a summarized angular momentum:

$$\mathbf{P}_0 = \sum_{i=1}^{N} \mathbf{p}_i, \qquad \mathbf{M}_0 = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i. \tag{1.4}$$

2. Modeling of isotropic turbulent spectra by dynamics of point vortices systems

The possibility to reproduce energy cascade processes is one of the principle requirements to any universal model of a developed turbulence. In particular, it would look natural that point vortices systems that are the limit at $r \to 0$ of small eddy formations, must contain corresponding effects in the region of high wavenumbers.

When modeling an energy spectrum, the spectral density function of turbulent energy $E(k)$ is to be calculated according to the results of the numerical integration of the system (1.1) or (1.3). In the spatial case according to the definition

$$\int_0^\infty dkE(k) = \frac{1}{2}\left\langle \int_0^\infty dk\left[\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta |\mathbf{\Omega}(\mathbf{k}, t)|^2\right]\right\rangle.$$

Here

$$\mathbf{\Omega}(\mathbf{k}, t) = (2\pi)^{-3} \int d\mathbf{r} e^{-i\mathbf{kr}} \boldsymbol{\omega}(\mathbf{r}, t)$$

is a Fourier transformation of the vorticity field of a point vortices system.

$$\boldsymbol{\omega}(r, t) = -\sum_{n=1}^N \mathbf{p}_n \times \nabla_r \boldsymbol{\delta}(\mathbf{r}-\mathbf{r}_n).$$

The angular brackets mean averaging with regards to angular variables and time when integrating Eq. (1.3). By angular averaging one can derive the following expression for energy spectrum modeling:

$$E(k) = \left\langle \underline{\frac{k^2}{48\pi^5}}\left\{\sum_{n=1}^N p_n^2 + \sum_{n<m}^N 3(\mathbf{p}_n \cdot \mathbf{p}_m)\left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} - \frac{\sin\alpha}{\alpha^3}\right] - \frac{|\mathbf{p}_n||\mathbf{p}_m|\cos\theta_n\cos\theta_m}{|\mathbf{r}_n - \mathbf{r}_m|^2}\left[\frac{3\cos\alpha}{\alpha^2} + \left(1 - \frac{3}{\alpha^2}\right)\frac{\sin\alpha}{\alpha}\right]\right\}\right\rangle_t, \tag{2.1}$$

where $\alpha = k|\mathbf{r}_n - \mathbf{r}_m|$, θ_n, θ_m are angles between momentums and the radius-vector $\mathbf{r}_n - \mathbf{r}_m$ and the angular brackets mean now only the temporal averaging.

In the case of a rectilinear vortices filaments system (1.1) an analogous expression was obtained earlier in [7]. It looks as follows:

$$E(k) = \left\langle \underline{\frac{\varkappa^2}{k}}\left[N + 2\sum_{j<n}^N J_0(k|\mathbf{r}_j - \mathbf{r}_n|)\right]\right\rangle_t, \tag{2.2}$$

$J_0(x)$ is the Bessel function.

The characteristic feature of Eqs. (2.1) and (2.2) is the presence of underlined terms which determine the asymptotics at $k \to \infty$ and are connected with the self-influence of vortices (with their infinite self-energy). It is quite obvious that it is impossible to obtain the necessary asymptotics of the spectral energy density on the basis of these relations ($E(k) \sim k^{-5/3}$). Moreover, it is seen from Eq. (2.2) that neglecting the first term because of its nonphysical nature doesn't give the necessary result since $J_0(x) \sim O(x^{-1/2})$ at $x \to \infty$. If we turn to the enstrophy cascade modeling where the asymptotics looks as follow

$$E(k) \sim k^{-3}, \tag{2.3}$$

the deviation will be still greater. The same thing takes place in the case of the formula (2.1). The observed shortcoming of point vortex models can be explained in the following way. On the basis of the representation theorem [8] for introducing the stochastic fields by random measure integrals, the random vorticity field in the second case can be presented [9] as follows:

$$\omega(\mathbf{r}) = \int \omega_0(\mathbf{r}-\mathbf{r}_1)\varrho(d\mathbf{r}_1), \tag{2.4}$$

$\omega_0(\mathbf{r})$ is a sure (nonrandom) function, $\varrho(d\mathbf{r})$ is a random measure.

The meaning of this representation is that a random vortex field can be represented statistically indistinguishable (in the sense of coincidence of any average value) by a sum of translates of a fixed vortex multiplied by random coefficients. From Eq. (2.4) it follows:

$$\omega(\mathbf{r}) \cong \sum_{i=1}^{N} \omega_0(\mathbf{r}-\mathbf{r}_i)\varrho(d\mathbf{r}_i).$$

It is obvious that Eq. (2.4) gives a theoretical probabilistic basis for modeling the random vortex distribution by means of discrete vortices of a fixed intrinsic structure.

If $\varrho(d\mathbf{r})$ is a Wiener measure, then from Eq. (2.4) and the Parseval equality

$$\langle\omega(\mathbf{x})\omega(\mathbf{x}+\mathbf{r})\rangle = \int d\mathbf{k}e^{-i\mathbf{k}\mathbf{r}}|\Omega_0(\mathbf{k})|^2.$$

In particular, for an isotropic function $\omega_0(r) = \omega_0(|r|)$ the enstrophy is represented as follows:

$$\langle|\omega|^2\rangle = \int dkk|\Omega_0(\mathbf{k})|^2 = \int dkk^2E(k),$$

from this the corresponding spectrum is

$$E(k) \sim \frac{1}{k}|\Omega_0(\mathbf{k})|^2. \tag{2.5}$$

Thus the energy spectrum is completely defined by a spectrum of a fixed sure function by the translates of which the random vortex field is represented. An analogous conclusion can be arrived at in a spatial case as well, while considering the componentwise representations of a random vortex field similar to Eq. (2.4). In purely point models the translates of the δ-function are dealt with, which, according to what was mentioned above, leads to a distortion of spectra. From this it follows that discrete models with a definite intrinsic vortices structure are to be used for an adequate numerical modeling of spectra. In particular, to obtain a correct asymptotics of Eq. (2.3) circular vortices of $\omega_0(|\mathbf{r}|) \sim |\mathbf{r}|^{-1}$ structure used in [10] can be taken.

Since numerical modeling is necessarily of an approximated character, there exist many ways to determine the vortices structure for which satisfactory results can be obtained. In this connection it can be shown that the use of algorithms of a "VORTEX" type [11] enables one to obtain acceptable results directly for point models [12]. This does not contradict the conclusion reached above. In these algorithms, particularly in a plane case, on each time step a grid vortex function is evaluated instead of a direct integration of systems (I.II). Then, by discrete Fourier transformation (DFT), the stream function is restored, and according to this function the spectrum is calculated. DFT operates with the functions of the limited spectrum $k < k_{max} \sim 1/h$, h is a step of a spatial grid. This limitation implies, as a matter of fact, an implicit transition to circular vortices systems with an effective radius $r = h$. At that

$$\omega_0(|\mathbf{r}|) = \begin{cases} c, & |\mathbf{r}| \leqslant h, \\ 0, & |\mathbf{r}| > h, \end{cases} \quad \Omega_0(k) \sim \int d\mathbf{r}\omega_0(|\mathbf{r}|)J_0(k|\mathbf{r}|) = \frac{h}{k}J_1(kh).$$

Since $hk \sim O(1)$, then from Eq. (2.5) $E(k) \sim k^{-3}$. This is what was required.

3. On one model of coherent vortex structures in turbulent shear layers

The systems of large vortex formations which are called now coherent structures (CS) were discovered for the first time in experiments [13] in turbulent shear layers. The estimates transacted have shown that these regions of relatively ordered vortex motion mainly contribute to turbulent characteristics and, in particular, to Reynolds stresses. This gave an impact for intensive investigation of CS in free turbulent flows and first of all, in plane shear layers, wakes and jets. It was discovered that CSs appear as a result of the Kelvin–Helmholtz instability development on the mean velocity profile with an inflection point. Their evolution moves mainly by pairwise mergings of small eddies and leads to vorticity distributions in CS, universal for the given flow. They possess a considerable symmetry and do not depend on a Reynolds number. However, there are practically no data on the intrinsic structure of CSs. This is why different hypotheses have to be introduced in theoretical and numerical CS models.

As a simplified CS model in a shear layer, an inviscid evolution of a rectilinear vortex blobs row was considered in [14]. In the model the structures were supposed to be formed in the limit of an infinite number of pairwise mergings: when vortex pairing, the conservation laws of vorticity energy and angular momentum are satisfied. For the following mergence, the conservation laws are put down as

$$N_m\Gamma_m = N_{m+1}\Gamma_{m+1}, \tag{3.1}$$

$$N_m\bar{\bar{E}}_m + U_m = N_{m+1}\bar{\bar{E}}_{m+1} + U_{m+1}, \tag{3.2}$$

$$M_m = M_{m+1}. \tag{3.3}$$

Here N_m is a number of CS in a row on m step, Γ_m is a velocity circulation of a model vortex, M_m is a full angular momentum, $\hat{E}_m$ is the internal energy of a separate vortex, U_m is an interaction energy of N_m vortices. At pairwise merging $N_m = 2N_{m+1}$, $\Gamma_{m+1} = 2\Gamma_m$.

For the closure of a model it was supposed that after the next merging in each newly formed intermediate structure a statistically equilibrium vortex distribution from a one--parameter λ-family described in [15] is established. In this case a dimensionless distribution (d.f.) of the vortex satisfies a nonlinear equation which looks as

$$F(\boldsymbol{\eta}_1) = c\exp\left[-(1+\lambda)\eta_1^2 + 4\lambda\int d\boldsymbol{\eta}_2 \ln|\eta_{12}|F(\boldsymbol{\eta}_2)\right], \tag{3.4}$$

where the following scaling of variables is used (see Eq. (1.2)

$$\boldsymbol{\eta} = (r-\mathbf{R})/L, \quad F(\boldsymbol{\eta}) = \bar{F}(\boldsymbol{\eta})L^2, \quad \eta = |\boldsymbol{\eta}|,$$

$\bar{F}(\eta)$ is a d.f. of vorticity, c is a constant providing the normalizing condition of a vortex circulation

$$\int d\eta F(\eta) = 1. \tag{3.5}$$

The conservation laws that correspond to Eq. (1.2) in these variables can be presented as follows:

$$E(\lambda) = -(4\pi)^{-1} \iint d\eta_1 d\eta_2 \ln \eta_{12} F(\eta_1) F(\eta_2) = \text{const}, \tag{3.6}$$
$$\int d\eta \eta F(\eta) = 0, \quad \int d\eta \eta^2 F(\eta) = 1.$$

The λ parameter in Eq. (3.4) is a dimensionless reciprocal temperature of distribution defined as:

$$8\pi\lambda = \left(\frac{\partial \hat{S}}{\partial E}\right)_{L^2} = \left(\frac{\partial S}{\partial E}\right)_{L^2}, \tag{3.7}$$

where

$$\hat{S} = -\int d\mathbf{r} \bar{F}(\mathbf{r}) \ln \bar{F}(\mathbf{r}) = S(\lambda) + \ln L^2, \tag{3.8}$$
$$S(\lambda) = -\int d\eta F(\eta) \ln F(\eta).$$

The range of means of the λ parameters is $(-1, \infty)$. For the distribution (3.4) there take place [15] the following character dependences of the reciprocal temperature λ and entropy $S(\lambda)$ on the dimensionless energy $E(\lambda)$. In the limit $\lambda \to \infty$ distribution corresponds to the Rankine's vortex, at $\lambda = 0$ the distribution (3.4) is a Gaussian function. At $\lambda \to -1$ the energy of distributions (see Fig. 1) sharply increases and the latter become more and more peaked.

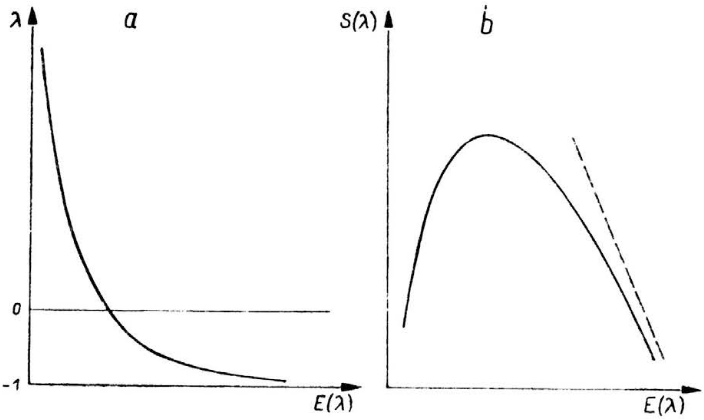

FIG. 1.

If we denote a root-mean square CS radius as L_m, a distance between CS neighbouring centrums as D_m, and put down an integral energy as

$$\bar{E}_m = \Gamma_m^2 [E_m(\lambda_m) - (4\pi)^{-1} \ln L_m],$$

then from the conservation laws (3.1) there follows [14, 16]

$$2E(\lambda_{m+1}) = E(\lambda_m)+(4\pi)^{-1}\ln\frac{\pi}{2}\frac{L_{m+1}^2}{L_m D_m}, \tag{3.9}$$

$$L_{m+1}^2 = L_m^2+\frac{1}{4}D_m^2. \tag{3.10}$$

Considering these relations as recurrent, from Eqs. (3.9) and (3.10) at $m \to \infty$ one can derive the expressions for the maximum value of the intermittency coefficient $\gamma_m = D_m/2L_m$. In particular, Eq. (3.9) yields [16]

$$\gamma_\infty = \pi \exp[-4\pi E(\lambda_\infty)], \tag{3.11}$$

whereas from Eq. (3.10)

$$\gamma_\infty = \sqrt{3}. \tag{3.12}$$

The formulae (3.11) and (3.12) yield a satisfactory agreement with experimental data [17] in the limits of their errors. The schematic model [14, 16] reflects correctly the physical effects observed in the experiments: the evolution by pairwise merging with energy transfer from small eddies to larger ones; the supposed inviscid character of the interaction mechanism, the universal nature and symmetry of CS. All this stimulated our investigations connected with the physical noncontradictority of the given model and its application to other flows.

3.1

It is easy to show that statistically equilibrium distributions (3.4) provide the conditional extremum (maximum) of the entropy (3.8) at the given integrals (3.5) and (3.6).

When the external field is absent, a row of blobs with $F(\lambda)$ distributions can be considered as a closed statistical system, the informational entropy of which must increase (not decrease). But there exist facts which make it nonobvious. In the process of transition to the self-similar regime, the energy of separate structures increases monotonously. According to the estimates [14, 16], at a certain step the vorticity distributions in eddies pass over to the negative temperature domain where the growth of energy is accompanied by the decreasing of entropy $S(\lambda)$ of separate structures (see Fig. 1, b). The merging process itself outwardly looks as a certain regularization. It gives an idea that during such a process the entropy as a measure of "chaos" must decrease.

The most critical situation with respect to the sign of the entropy changing is the case of a strong coupled row where, when merging, the vortices practically move without any deviation from the median. At that, apart from the energy conservation law (3.9), the angular momentum conservation law (3.10) is also satisfied. Because of the universal character of the formula (3.8), the entropy of a vortex blobs row on the m step is represented as

$$\hat{S}_m^{\text{III}} = -\int d\eta f^{(m)}(\eta)\ln f^{(m)}(\eta)+\ln L^2, \tag{3.13}$$

where $f^{(m)}(\mathbf{n})$ is a dimensionless function of a vortex distribution in a system of blobs on the whole, L^2 characterizes its full angular momentum. From definition of the model it follows that

$$\bar{f}^{(m)}(\mathbf{r}) = \frac{1}{2N_m}\sum_{i=1}^{2N_m} \bar{F}^{(m)}(\mathbf{r})\chi(\mathbf{r}_i),$$

$F^{(m)}(\mathbf{r})$ is an equilibrium d.f. on the m step in a separate vortex, $\chi(\mathbf{r}_i)$ is a characteristic function of the domain occupied by the i-vortex. Taking this into account and changing the dimensionless variables in Eq. (3.13), one can obtain

$$\hat{S}_m = -\int d\eta F^{(m)}(\eta)\ln F^{(m)}(\eta)+\ln 2N_m L_m^2 = S(\lambda_m)+\ln 2N_m L_m^2.$$

From this

$$\Delta\hat{S}_m = \hat{S}_{m+1}-\hat{S}_m = S(\lambda_{m+1})-S(\lambda_m)+\ln\frac{1}{2}\frac{L_{m+1}^2}{L_m^2}. \tag{3.14}$$

While merging, the intermittency parameter monotonously increases and at $m\to\infty$ $\gamma_m\to\gamma_\infty$ (see Eq. (3.12)). In this limit the structures pass over to large negative temperature domain [14, 16] $|\lambda_m|\to|\lambda_\infty|\simeq 1$, and their energy tends to a limit value of $E(\lambda_\infty)$. In this region the asymptotical relation for the entropy

$$S(\lambda_m) = 1+\ln\pi-8\pi E(\lambda_m) \tag{3.15}$$

is valid [15] (see the dotted line in Fig. 1b). Equations (3.14), (3.15) and (3.10) yield

$$\lim_{m\to\infty}\Delta\hat{S}_m = \lim_{m\to\infty}\ln\frac{1}{2}\left(1+\frac{1}{4}\gamma_m^2\right) = \ln 2,$$

that is, asymptotically the evolution in the row proceeds along with the growth of the entropy, and the given model is correct with respect to the nondecreasing information entropy law in closed statistical systems.

3.2(1)

An essential property of CS is their stability with respect to a small scale turbulence background and their collective interaction.

In this connection the vorticity distributions used for the modeling of CS and, in particular, the distributions (3.4) must possess a certain reserve of a hydrodynamic stability. Therefore the stability of statistically equilibrium vortex distributions with respect to two-dimensional inviscid disturbances was studied.

The stream function of disturbances was considered in the form

$$\psi(\eta,\alpha,t) = \mathrm{Re}\,\psi(\eta)\exp[i(m\alpha-\omega t)].$$

The linearization of Euler equations of an incompressible fluid with respect to a plane nondisturbed flow with vortex distribution (3.4)–(3.6) yields the following equation for the amplitude disturbance function:

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\psi}{d\eta}\right)-\left(\frac{m^2}{\eta^2}+\frac{F'_\eta}{\eta(V-\xi)}\right)\psi = 0. \tag{3.16}$$

(1) These results have been obtained jointly with A. G. Melnikov.

Here $\xi = \omega/m$,

$$V(\eta) = \frac{1}{\eta^2}\int\limits_0^\eta ds sF(s).$$

As boundary conditions we had

$$\psi(0) = 0, \quad \frac{d\psi}{d\eta}\underset{\eta\to\infty}{\to} 0. \tag{3.17}$$

The flow will be stable in a linear approximation at $\mathrm{Im}\,\xi \leqslant 0$. The problem (3.16), (3.17) is analogous to the Rayleigh spectral problem [18] under the axial symmetry of the flow.

From Eq. (3.16), using ordinary transformations for spectral problems, one can obtain

$$\frac{dW}{d\eta} = \mathrm{Im}\,\xi \frac{F'_\eta}{|V-\xi)|^2}|\psi|^2, \tag{3.18}$$

where $W(\eta)$ is the real function in the form of

$$W(\eta) = \frac{i}{2}\eta\left(\frac{d\psi}{d\eta}\psi^* - \frac{d\psi^*}{d\eta}\psi\right).$$

The asterisk denotes the complexly conjugated value. From the boundary conditions (3.17) it follows:

$$W(0) = 0, \quad W(\eta)\underset{\eta\to\infty}{\to} 0.$$

Thereat the integration of Eq. (3.18) yields

$$\mathrm{Im}\,\xi\int\limits_0^\infty d\eta \frac{F'_\eta}{|V-\xi|^2}|\psi|^2 = 0.$$

The latter will be true if the derivative of the vortex profile changes the sign in the flow region. Otherwise $\mathrm{Im}\,\xi = 0$ and the flow is neutrally stable. The differentiation of Eq. (3.4) and the transformation of the right-hand side of the integral yields

$$F'_\eta(\eta) = F(\eta)\left[-2\eta(1+\lambda)+\frac{8\pi\lambda}{\eta}\int\limits_0^\eta ds sF(s)\right].$$

Since the function $F(\eta)$ is nonnegative, then for $-1 < \lambda < 0$ $F'_\eta \leqslant 0$. Thus the negative temperature vortex distributions used in the given model are linearly stable with respect to two-dimensional inviscid disturbances. For positive λ the stability with respect to such disturbances was obtained only for asymptotics as $\lambda \to 0$ and $\lambda \to \infty$.

4. The vortex model of coherent structures in free rotating annular shear layers and MHD-flows

In this section the model [14] is generalized for application to the study of CS in circular shear layers and MHD-flows with coaxial streamlines and axial magnetic field. In such a flow after developing the Kelvin–Helmholtz instability the coaxial CS rows are formed.

Thereat the number of CS depends on the regime parameters. CS behaviour in axial symmetrical shear layer is of interest in problems of zonal atmosphere circulation and their laboratory model. MHD-flows of such a type (the Lenhert flows [19–21]) have also certain applications.

4.1

Peculiarities of scheme [14] application to the flows of this type can be considered on the basis of a circular vortex blobs row.

As an initial flow where Kelvin–Helmholtz instability is developed, one can consider an axial symmetrical discontinuity jump of the azimuthal velocity on the radius R_0 corresponding to the vortex distribution

$$\omega(\mathbf{r}) = \Gamma_0 \delta(\mathbf{r}-R_0),$$

Γ_0 is a circulation. Thereat a total flow energy in the initial state can be represented as

$$\varepsilon_0 = -\frac{1}{4\pi}\iint d\mathbf{r}_1 d\mathbf{r}_2 \omega(\mathbf{r}_1)\omega(\mathbf{r}_2)\ln|\mathbf{r}_1-\mathbf{r}_2| = -\frac{\Gamma_0^2}{4\pi}\ln R_0. \tag{4.1}$$

As an intermediate state of number m entering the conservation laws (3.1)–(3.3) N_m vortex blobs of equal circulation Γ_m located uniformly on the circumference of radius R_m is considered. The energy of blobs interaction in a circular row can be put down as

$$U_m = -\frac{\Gamma_m^2}{4\pi}N_m \ln \prod_{k=1}^{N_m-1} 2R_m \sin\left(\frac{k\pi}{N_m}\right) = -\frac{\Gamma_m^2 N_m^2}{4\pi}\ln R_m + \frac{\Gamma_m^2}{4\pi}N_m \ln\frac{R_m}{N_m}. \tag{4.2}$$

It was assumed that the blob appearing after a pairwise merging is set out in the centre of the chord tightening the centres of merging blobs. In this case the row radius of the intermediate state is expressed as

$$R_m = R_{m-1}\cos\frac{\pi}{N_{m-1}} = R_0 \sin\frac{\pi}{N_m}\Big/\left[2^m \sin\frac{\pi}{2^m N_m}\right]. \tag{4.3}$$

Taking Eq. (4.2) into account there follow from Eqs. (3.1)–(3.3) the relations corresponding to Eqs. (3.9)–(3.10):

$$2E(\lambda_{m+1}) = E(\lambda_m) + \frac{1}{4\pi}N_{m-1}\ln\frac{R_{m-1}}{R_m} + \frac{1}{4\pi}\ln\left[\frac{R_{m-1}}{L_{m-1}N_{m-1}}\left(\frac{L_m N_m}{R_m}\right)^2\right], \tag{4.4}$$

$$R_{m+1}^2 + L_{m+1}^2 = R_m^2 + L_m^2. \tag{4.5}$$

From this, considering Eq. (4.3), at $m \to \infty$ there follows the expression for the intermittency coefficient in the annular CS row:

$$\gamma_\infty = \frac{\pi R_\infty}{N_\infty L_\infty} = \pi \exp\left[-4\pi E(\lambda_\infty)\right]\left(\frac{\sin \pi/N_\infty}{\pi/N_\infty}\right)^{N_\infty}, \tag{4.6}$$

$$\gamma_\infty = \sqrt{3}\,\frac{\sin \pi/N_\infty}{\pi/N_\infty}, \tag{4.7}$$

It is obvious from Eqs. (4.6) and (4.7) that in this case the axial geometry influence is considerable at comparatively small $N_\infty < 10$. At $N_\infty \to \infty$ these formulae pass over to Eqs. (3.11) and (3.12). The asymptotical limit is practically achieved at $N_\infty \sim 15$.

For the Lenhert type flows two coaxial rows of CS were considered as a zero approximation. As an initial flow two concentric azimuthal velocity discontinuity jumps were considered. But the formulae obtained lacked a link with the magnetic field, and on their basis it has been impossible to obtain correlating relations for the experimental data [20, 21].

4.2

It is rather easy to achieve necessary improvement of the model if we confine ourselves to a strong magnetic field approximation.

To be more concrete, the experiments [20, 21] are considered below. Under these conditions [21] the following picture of the flow is to be seen in the regime before the development of instability (see Fig. 2). In a rotating layer above the electrodes with mean

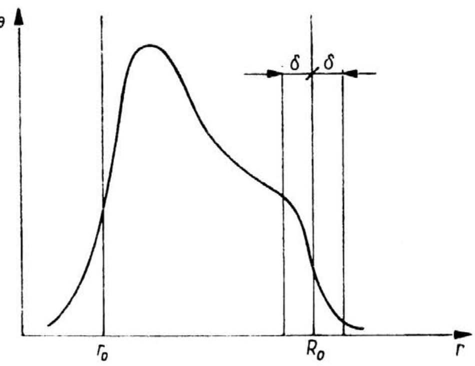

FIG. 2.

radii r_0, R_0, a plane potential flow is formed limited by shear layers centered with respect to the radii r_0, R_0. At that the azimuthal velocity profile can be represented, like [22], as follows:

$$(4.8)\qquad V_\theta = \begin{cases} K/2r_0\left(1+\Phi(x/\delta)\right), & x = r-r_0, \\ K/r, & r_0+\delta \gtrsim r \gtrsim R_0-\delta, \\ K/2r_0(\left(1-\Phi(x/\delta)\right) & x = r-R_0. \end{cases}$$

Here $\delta = 2h/\sqrt{\mathrm{Ha}_h}$ is an effective semiwidth of a shear layer, determined by the layer height h in the axial direction and by the Hartman number $\mathrm{Ha}_h = Bh\sqrt{\frac{\mu}{\sigma}}$, K is a certain dimensional constant, $\Phi(x)$ is a probability integral, B is a magnetic induction, σ is an electrical conductivity, μ is a dynamic viscosity coefficient.

Generally speaking, one needs not consider intermediate states of a row as it was done above. It was supposed that conservation laws of energy and vorticity of each sign are satisfied for the whole sequence of stationary (bifurcational) states dependent on the value B, including the regime before the loss of stability. In addition it was assumed that the instability having been developed, the total vorticity of the given sign is uniformly distributed among the vortices of the corresponding row.

For steady regimes at $\mathrm{Ha}_h > 10^3$, when the number of blobs in rows is relatively large [20], the ideas of symmetry permit to apply the conservation laws for the vortex component of each sign separately. Below the relations obtained for the outer shear layer are given. The vorticity energy of the outer shear layer (vortex row) is the sum of the internal energy and the energy in the velocity field induced by the inner shear layer row. The conservation law looks as follows:

$$\varepsilon_0 + V_0 = \varepsilon_\infty + V_\infty. \tag{4.9}$$

To obtain the expressions in explicit form, the left hand side integrals (4.9) were calculated with an accuracy up to the terms of the first order with respect to the parameter δ/R_0. The self energy of the outer layer ε_0 in the approximation required equals to

$$\varepsilon_0 = -\frac{(N_\infty \Gamma_\infty)^2}{4\pi}\left(\ln R_0 + \frac{1}{\sqrt{2\pi}}\frac{\delta}{R_0}\right). \tag{4.10}$$

The term V_0 equals

$$\nabla_0 = (N_\infty \Gamma_\infty)^2 1/2\pi \ln R_0. \tag{4.11}$$

In the formulae (4.10) and (4.11) the dimension factor $N_\infty \Gamma_\infty$ allowing for the conservation of a total vorticity in the layer was separated. The terms proportional to $\ln R_0$ correspond to the passage from the shear layer of the finite thickness to the tangential discontinuity jump of the azimuthal velocity (see Eq. (4.1)). In the final state of the row consisting of N_∞ vortices equally located on the circumference of the radius R_∞ it can be put down as:

$$\varepsilon_\infty = N_\infty \bar{E}_\infty + U_\infty, \tag{4.12}$$

where, as formerly, $\bar{E}_\infty$ is the energy of a separate vortex, U_∞ is the total energy of their interaction. The latter equals to (see Eq. (4.2)).

$$U_\infty = -\frac{(\Gamma_\infty N_\infty)^2}{4\pi}\ln R_\infty + \frac{N_\infty \Gamma_\infty^2}{4\pi}\ln\frac{R_\infty}{N_\infty}. \tag{4.13}$$

In this case the energy of interaction between inner and outer vortex rows is

$$V_\infty = (2\pi)^{-1}(N_\infty \Gamma_\infty)^2 \ln R_\infty. \tag{4.14}$$

Substitution of the expressions (4.10)–(4.14) into Eq. (4.9) yields

$$\exp\left(\sqrt{2\pi}/\lambda\right)\lambda(R_\infty/R_0)^{N_\infty+1} = 2\pi\exp(-4\pi E_\infty). \tag{4.15}$$

Here the value $\tilde{\delta} = |u_1 - u_2|/|\partial u/\partial x|_{\max}$ is taken as a scaling length; u_1, u_2 are asymptotical velocity values on the boundaries of the shear layer. For the chosen profile (4.8) $\tilde{\delta} = \delta\sqrt{\pi}.$ The value $\lambda = 2\pi R_0/(N_\infty \tilde{\delta})$ has the meaning of the nondimensional wavelength of the Kelvin–Helmholtz mode (compare with the definition in Eq. (4.6)), the development of which leads to a stationary state with N_∞ vortices in the outer row. The dimensionless energy of a separate vortex E_∞ is determined by the expression

$$\bar{E}_\infty = \Gamma_\infty^2\left(E_\infty - (4\pi)^{-1}\ln\tilde{\delta}\right).$$

On the basis of the MHD-flow analysis in a separate CS, it was shown that E_∞ is independent of the regime parameters, in particular, of the magnetic field value B, and is universal;

this corresponds to the ideas about the nature of CS in a plane case [17]. Proving this exceeds the frameworks of the present paper.

Using Eq. (4.15), one can evaluate the energy of large scale turbulent fluctuations, the intermittency coefficient, the number of CS in the row depending on the magnetic field value, etc. In particular, if we assume that in accordance with the data [21] $R_0 \simeq R_\infty$, because of the universal character of E_∞, Eq. (4.15) yields

$$N_\infty/\sqrt{B} = \text{const.} \tag{4.16}$$

Table 1

B, T	0.37	0.555	0.74	1.11	1.43	1.51
f, Hertz	0.45	0.58	0.73	0.85	0.95	1.00
f/f_{max}	0.45	0.58	0.73	0.85	0.95	1.00
$\sqrt{B/B_{max}}$	0.495	0.60	0.705	0.855	0.94	1.00

The Table 1 presents the comparison of this conclusion and the experimental data [21]. Here, instead of the relation of the vortices numbers, the frequency relations proportional to it directly measured in experiments are taken. The greatest deviation does not exceed 10% and lies in the region of comparatively small values of B where the asymptotical approximation used grows worse.

Comparing the results of Sects. 4.1 and 4.2, one can see that the account of the finite thickness of a shear layer can lead to qualitatively different results than when modeling it by a tangential velocity discontinuity jump. In particular, such an approach to CSs in an ordinary shear layer will probably require another class of statistically equilibrium distributions [23, 24] constructed on the Fermi type statistics.

When selecting the results, the authors aimed at emphasizing the important role of the intrinsic structures of vortices while modeling different aspects of turbulence.

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