Arch. Mech., 36, 2, pp. 207–218, Warszawa 1984

On Eulerian and Lagrangean objectivity in continuum mechanics

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In continuum mechanics the commonly-used definition of objectivity (or frame-indifference) of a tensor field does not distinguish between Eulerian, Lagrangean and two-point tensor fields. This paper highlights the distinction and provides a definition of objectivity which reflects the different transformation rules for Eulerian, Lagrangean and two-point tensor fields under an observer transformation. The notion of induced objectivity is introduced and its implications examined.

Powszechnie używane w mechanice ośrodka ciągłego pojęcie obiektywności (lub niezależności od układu odniesienia) pola tensorowego nie rozróżnia pól tensorowych eulerowskich, lagranżowskich oraz dwupunktowych. Praca niniejsza uwypukla te różnice i wprowadza taką definicję obiektywności, która odzwierciadla różnice praw transformacji w tych polach przy transformacji obserwatora. Wprowadzono pojęcie obiektywności wzbudzonej i przeanalizowano jego konsekwencje.

Повсеместно используемое в механике сплошной среды понятие объективности (или независимости от системы отсчета) тензорного поля не различает эйлеровых, лагранжевых и двухточечных тензорных полей. Настоящая работа отмечает эти разницы и вводит такое определение объективности, которое отражает разницы законов преобразований в этих полях при преобразовании наблюдателя. Введено понятие возбужденной объективности и проанализированы его следствия.

1. Introduction

With the motion of a deformable continuous body there are associated certain scalar quantities which can be regarded as intrinsic to the material constituting the body in the sense that all observers attach the same value to each such quantity. One example of such a scalar is the mass density of the material; another is the extension of an arbitrary line element of material. The speed of a material particle, on the other hand, depends on the choice of observer since different observers are in relative motion in general. These scalar quantities are measured in terms of scalar, vector and tensor fields which transform according to certain rules under a change of observer (or change of frame of reference (1)).

The purpose of this paper is to formalize the "observer indifference" of scalars described above by means of a definition which is reflected in the transformation rules of the associated vector and tensor fields under a change of observer. Fields satisfying this definition are said to be *objective* (1). An important consequence of the definition is that objective Eulerian and Lagrangean tensor fields have different transformation rules. This is emphasized because previous definitions of objectivity have not distinguished between Eulerian

(1) The terms "observer" and "frame of reference" are often used synonymously, as also are "objective" and "frame-indifferent".

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and Lagrangean tensor fields (see, for example, [4] and [5]). Before discussing the notion of objectivity in detail, we summarize some basic ideas relating to observers.

Events (or phenomena) which occur in the physical world are manifested in space and time through the perception of what is referred to loosely as an "observer". We suppose that the space in which events are recorded by an "observer" is the (three-dimensional) Euclidean point space E ([2]) and that time is measured on the real line R. We may regard an "observer" as being equipped to measure physical quantities and, in particular, to monitor the relative positions of points in E and the progress of time in R. Formally, an *observer*, 0 say, is defined as a mapping (in fact, a homeomorphism) which assigns a pair $(\mathbf{x}, t) \in E \times \mathsf{R}$ to an event in the physical world, where $\mathbf{x}$ is the *place* and t the *time* of the event *as perceived by* 0 and $E \times \mathsf{R}$ denotes the Cartesian product if E and R.

Let $\mathbf{x}$ and $\mathbf{x}_0$ be points in E. Then the point difference $\mathbf{x}-\mathbf{x}_0$ is an element of the vector space, denoted E, which is called the *translation space* of E. The distance between the points $\mathbf{x}$ and $\mathbf{x}_0$ is denoted $|\mathbf{x}-\mathbf{x}_0|$. Similarly, if t and t_0 are times in R, then $t-t_0$ is a time interval in R (which may be positive or negative). Thus the events recorded by 0 as $(\mathbf{x}, t)$ and $(\mathbf{x}_0, t_0)$ are separated by distance $|\mathbf{x}-\mathbf{x}_0|$ and time $t-t_0$.

In continuum mechanics it is stipulated that different observers should agree about (a) the distance between events, (b) time intervals between events and (c) the order in which events occur. This means that $|\mathbf{x}-\mathbf{x}_0|$ and $t-t_0$ are preserved under a mapping from $E \times \mathsf{R}$ to $E \times \mathsf{R}$ which corresponds to a change of observer. Thus, if the events recorded by 0 as $(\mathbf{x}, t)$ and $(\mathbf{x}_0, t_0)$ are recorded by a second observer, 0* say, as $(\mathbf{x}^*, t^*)$ and $(\mathbf{x}_0^*, t_0^*)$ the most general one-to-one mapping from $E \times \mathsf{R}$ to $E \times \mathsf{R}$ which satisfies these requirements is specified by the equations

$$\mathbf{x}^*-\mathbf{x}_0^* = \mathbf{Q}(t)(\mathbf{x}-\mathbf{x}_0), \tag{1.1}$$

$$t^* = t-a, \tag{1.2}$$

where $a \in \mathsf{R}$ is a constant and $\mathbf{Q}(t)$ is an orthogonal (second-order) tensor which can be regarded as a linear mapping from E to E ([3]). Note that no preferred choice of origin for E is involved in Eq. (1.1).

The mapping from $E \times \mathsf{R}$ to $E \times \mathsf{R}$ characterized by Eqs. (1.1) and (1.2) is called an *observer transformation* and corresponds to a change of observer from 0 to 0*. It is assumed that $\mathbf{Q}(t)$ is suitably smooth. Essentially an observer transformation merely changes the description in $E \times \mathsf{R}$ of an event.

For future reference we note that Eq. (1.1) may also be written as

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x}+\mathbf{c}(t), \tag{1.3}$$

where $\mathbf{c}(t)$ is an arbitrary vector in E with $\mathbf{x}$ and $\mathbf{x}^*$ now interpreted as the position vectors in E of the points $\mathbf{x}$ and $\mathbf{x}^*$ relative to an arbitrary choice of origins in E for 0 and 0*, respectively.

([2]) Here E is taken to be the same for all observers, but we note that more abstract work does not require such a restriction [6].

([3]) A more general treatment (see [6] and the comments in [4]) assumes that the Euclidean point space in which events are observed, E_t, is different for each distinct instant of time t. With each *instantaneous Euclidean point space* E_t is associated an *instantaneous translation space* E_t.

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2. Some fields associated with the deformation and motion

As is usual in continuum mechanics, we regard a body consisting of continuously distributed material as a smooth three-dimensional manifold, B say, whose points are called *material points*.

A *configuration* of B (as observed by 0) is a homeomorphism $\boldsymbol{\chi} : B \to E$ which takes material points to the places they occupy in E. We identify a generic material point in B by the label X so that

(2.1) $$\mathbf{x} = \boldsymbol{\chi}(X), \quad X \in B,$$

where $\mathbf{x}$ is the place occupied by X in the configuration $\boldsymbol{\chi}$. It is assumed that $\boldsymbol{\chi}$ and its inverse $\boldsymbol{\chi}^{-1}$ have sufficient regularity for our requirements.

We write $\mathscr{B} = \boldsymbol{\chi}(B) \equiv \{\boldsymbol{\chi}(X): X \in B\}$ for the region of E occupied by B in the configuration $\boldsymbol{\chi}$. Since no confusion should arise, we also refer to $\mathscr{B}$ as a configuration of B. As the body moves and deforms, the region it occupies in E changes continuously, and a *motion* of B is defined as a one-parameter family of configurations $\boldsymbol{\chi}_t : B \to E$, where the subscript t identifies the time as parameter.

Let $\mathscr{B}_t = \boldsymbol{\chi}_t(B)$ and write

(2.2) $$\mathbf{x} = \boldsymbol{\chi}_t(X) \equiv (X, t), \quad X \in B,$$

generalizing Eq. (2.1). (Of course, t may be restricted to some subset of R, but this need not be specified here.)

For reference purposes, it is convenient to identify a certain fixed (but arbitrarily chosen) configuration of B so that material points are labelled during motion by their places in that configuration. Let $\boldsymbol{\chi}_0$ denote such a fixed configuration and write

(2.3) $$\mathbf{X} = \boldsymbol{\chi}_0(X),$$

where $\mathbf{X}$ is the place of the material point X in the configuration $\boldsymbol{\chi}_0$. Also set $\boldsymbol{\chi}_0(B) = \mathscr{B}_0$. The subscript zero may, but need not, correspond to $t = 0$ in Eq. (2.2).

A fixed configuration $\mathscr{B}_0$ is called a *reference configuration* and $\mathscr{B}_t$ the *current configuration* of the body in the motion specified by Eq. (2.2).

On eliminating X between Eqs. (2.2) and (2.3), we obtain

(2.4) $$\mathbf{x} = \boldsymbol{\chi}\{\boldsymbol{\chi}_0^{-1}(\mathbf{X}), t\} \equiv \boldsymbol{\varkappa}(\mathbf{X}, t) \equiv \boldsymbol{\varkappa}_t(\mathbf{X}),$$

where the one-parameter mapping $\boldsymbol{\varkappa}_t : \mathscr{B}_0 \to \mathscr{B}_t$ thus defined specifies the *deformation* from the reference configuration $\mathscr{B}_0$ to the current configuration $\mathscr{B}_t$. The definition of $\boldsymbol{\varkappa}$, of course, depends on the choice of reference configuration.

Thus far we have related the motion to a single observer 0. Under an observer transformation whose spatial part is written as Eq. (1.3) the motion Eq. (2.2) itself changes according to

(2.5) $$\boldsymbol{\chi}^*(X, t^*) = \mathbf{Q}(t)\boldsymbol{\chi}(X, t) + \mathbf{c}(t),$$

where $\mathbf{x}^* = \boldsymbol{\chi}^*(X, t^*)$ is the description by 0* of the motion observed by 0 as Eq. (2.2). We remark that if $\mathbf{Q}(t)$ is restricted to being proper orthogonal, then Eq. (2.5) may be interpreted as a motion recorded by 0 consisting of a rigid-body motion superposed on the motion (2.2). We return to this point later.

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In the following we examine some specific scalar, vector and tensor fields associated with the motion in order to distinguish those which depend intrinsically on 0 and those which are essentially independent of 0.

First, the velocity $\dot{\boldsymbol{\chi}}(X, t) \equiv \partial\boldsymbol{\chi}(X, t)/\partial t$ of the material point X transforms according to

$$\dot{\boldsymbol{\chi}}^*(X, t^*) = \mathbf{Q}(t)\dot{\boldsymbol{\chi}}(X, t)+\dot{\mathbf{c}}(t)+\dot{\mathbf{Q}}(t)\boldsymbol{\chi}(X, t) \tag{2.6}$$

under an observer transformation. Clearly, the velocity is directly linked to the choice of observer through the relative motion of observers implicit in $\dot{\mathbf{c}}(t)$ and $\dot{\mathbf{Q}}(t)$. Similar remarks apply to the acceleration $\ddot{\boldsymbol{\chi}}(X, t)$.

In this paper it is assumed for simplicity that all observers select the same reference configuration so that the particle X is allocated the same reference point $\mathbf{X}$ in E by each observer. This assumption affects the details but not the principle of our subsequent argument ([4]).

From Eqs. (2.4) and (2.5) we see that the *deformation gradient* ([5])

$$\mathbf{A}(\mathbf{X}, t) \equiv \mathrm{Grad}\boldsymbol{\varkappa}(\mathbf{X}, t) \tag{2.7}$$

has the transformation rule

$$\mathbf{A}^*(\mathbf{X}, t^*) = \mathbf{Q}(t)\mathbf{A}(\mathbf{X}, t). \tag{2.8}$$

It follows that

$$\det\mathbf{A}^* = \pm\det\mathbf{A}, \tag{2.9}$$

the sign on the right-hand side of Eq. (2.9) being $+(-)$ if $\mathbf{Q}(t)$ is proper (improper) orthogonal. In Eq. (2.9) and for the remainder of this Section we omit the arguments from tensors (which may be regarded as fields over either $\mathscr{B}_0$ or $\mathscr{B}_t$ through Eq. (2.4)). If $\mathbf{Q}$ is proper orthogonal, then Eq. (1.1) represents a rotation of vectors in E.

It is usual to adopt the physically sensible convention that relative orientation of triads of material line elements is preserved under deformation. This means that

$$\det\mathbf{A} > 0 \tag{2.10}$$

and this convention is preserved under an observer transformation provided $\mathbf{Q}$ is proper orthogonal. We adopt this convention here for all observers and therefore rule out what may be regarded as physically unrealistic deformations in which a material becomes a mirror image of itself ([6]).

([4]) If 0 and 0* select different reference configurations so that the material point X is allocated points $\mathbf{X}$ and $\mathbf{X}^*$ in E by 0 and 0* respectively, then

$$\mathbf{X}^* = \mathbf{Q}_0\mathbf{X}+\mathbf{c}_0,$$

where $\mathbf{c}_0$ is a constant vector and $\mathbf{Q}_0$ is a constant orthogonal tensor.

([5]) Grad (with upper case G) and grad (with lower case g) denote the gradient operation relative to $\mathbf{X}$ and $\mathbf{x}$ respectively.

([6]) This viewpoint is not accepted universally and many authors admit improper orthogonal $\mathbf{Q}$ in Eq. (1.3). However, new light on the problem has recently been shed by MURDOCH [3] who regards the space E^* in which 0* records events as distinct from E. He assumes that the sign of det $\mathbf{A}$ is the same for all motions recorded by any given observer, allows different observers to disagree about orientation, and concludes that the implications for material response functions are independent of whether $\mathbf{Q}$ is proper or improper orthogonal, $\mathbf{Q}$ being a linear mapping from E to E^* (the translation space of E^*).

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From Eq. (2.8) it follows that the right and left Cauchy–Green deformation tensors $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ satisfy

(2.11) $$\mathbf{A}^*\mathbf{A}^{*T} = \mathbf{QAA}^T\mathbf{Q}^T,$$

(2.12) $$\mathbf{A}^{*T}\mathbf{A}^* = \mathbf{A}^T\mathbf{A},$$

respectively. Similarly for the respective inverses $\mathbf{BB}^T$ and $\mathbf{B}^T\mathbf{B}$ of $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$, where $\mathbf{B}$ is the inverse of $\mathbf{A}^T$.

Next, the polar decomposition

(2.13) $$\mathbf{A} = \mathbf{RU} = \mathbf{VR}$$

yields

(2.14) $$\mathbf{V}^* = \mathbf{QVQ}^T, \quad \mathbf{R}^* = \mathbf{QR}, \quad \mathbf{U}^* = \mathbf{U},$$

where $\mathbf{R}$ is proper orthogonal and $\mathbf{U}$ and $\mathbf{V}$ (the left and right stretch tensors respectively) are positive definite and symmetric.

The Green strain tensor

(2.15) $$\mathbf{E} = \frac{1}{2}(\mathbf{A}^T\mathbf{A}-\mathbf{I}) = \frac{1}{2}(\mathbf{U}^2-\mathbf{I})$$

and the Almansi strain tensor

(2.16) $$\mathbf{F} = \frac{1}{2}(\mathbf{I}-\mathbf{BB}^T) = \frac{1}{2}(\mathbf{I}-\mathbf{V}^{-2})$$

therefore transform according to

(2.17) $$\mathbf{E}^* = \mathbf{E}, \quad \mathbf{F}^* = \mathbf{QFQ}^T.$$

Under the deformation a material line element $\mathbf{dX}$ based on the point $\mathbf{X}$ in $\mathscr{B}_0$ maps onto a line element $\mathbf{dx}$ at $\mathbf{x}$ in $\mathscr{B}_t$ according to $\mathbf{dx} = \mathbf{AdX}$. We refer to $\mathbf{dX}$ and $\mathbf{dx}$, respectively, as *Lagrangean* and *Eulerian line elements*. From Eq. (1.3) we see that Eulerian line elements transform according to

(2.18) $$\mathbf{dx}^* = \mathbf{Qdx},$$

while a Lagrangean line element $\mathbf{dX}$ is unaffected by an observer transformation (in accordance with our assumption that all observers select the same reference configuration).

If $\mathbf{dX}'$ denotes a second Lagrangean line element at $\mathbf{X}$, then $\mathbf{dx}' = \mathbf{AdX}'$ is the corresponding Eulerian line element and the strain tensors (2.15) and (2.16) are connected through

(2.19) $$\mathbf{dX}'\cdot(\mathbf{E}^*\mathbf{dX}) = \mathbf{dX}'\cdot(\mathbf{EdX}) = \mathbf{dx}'\cdot(\mathbf{Fdx}) = \mathbf{dx}^{*\prime}\cdot(\mathbf{F}^*\mathbf{dx}^*),$$

use having been made of Eqs. (2.17) and (2.18). We shall comment on this shortly.

From Eqs. (2.7) and (2.8) we obtain $\dot{\mathbf{A}} = \mathbf{\Gamma A}$ and $\dot{\mathbf{A}}^* = \mathbf{Q}\dot{\mathbf{A}}+\dot{\mathbf{Q}}\mathbf{A}$, where $\mathbf{\Gamma}$ is the velocity gradient tensor grad $\mathbf{v}$ and $\mathbf{v}(\mathbf{x}, t) = \dot{\boldsymbol{\varkappa}}(\mathbf{X}, t)$ with $\mathbf{x} = \boldsymbol{\varkappa}(\mathbf{X}, t)$. It follows that the body spin $\mathbf{\Omega} = \frac{1}{2}(\mathbf{\Gamma}-\mathbf{\Gamma}^T)$ and Eulerian strain-rate $\mathbf{\Sigma} = \frac{1}{2}(\mathbf{\Gamma}+\mathbf{\Gamma}^T)$ have transformation rules

(2.20) $$\mathbf{\Omega}^* = \mathbf{Q\Omega Q}+\dot{\mathbf{Q}}\mathbf{Q}^T$$

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and

$$\mathbf{\Sigma}^* = \mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T \tag{2.21}$$

under an observer transformation.

Time differentiation of Eq. (2.19) (at fixed **X**) now yields

$$\mathbf{dX}' \cdot (\dot{\mathbf{E}}\mathbf{dX}) = \mathbf{dx}' \cdot (\mathbf{\Sigma}\mathbf{dx}) = \mathbf{dx}^{*\prime} \cdot (\mathbf{\Sigma}^*\mathbf{dx}^*). \tag{2.22}$$

The body spin, which is a measure of the instantaneous rigid rotation of a triad of (Eulerian) line elements, is clearly influenced by the rotation of observers through their relative spin $\dot{\mathbf{Q}}\mathbf{Q}^T$. Tensors such as $\mathbf{E}$, $\dot{\mathbf{E}}$ and $\mathbf{\Sigma}$ are not affected in this way. These are measures of extension and rate of extension of material line elements and also of the changing angles between pairs of line elements. The scalars attached to these tensors through Eqs. (2.19) and (2.22) are independent of the observer and this "observer indifference" is reflected in the transformation rules $(2.17)_1$ and (2.21).

It is important to notice the distinction between the transformation rules for *Eulerian tensors*, such as $\mathbf{A}\mathbf{A}^T$ and $\mathbf{\Sigma}$, for *Lagrangean tensors* such as $\mathbf{A}^T\mathbf{A}$ and $\mathbf{E}$ and for *two-point tensors* such as $\mathbf{A}$. We regard each of these tensors as *objective* in the sense that each may be associated with "observer-indifferent" scalars as described above although the tensors themselves satisfy transformation rules of the form $(2.17)_1$, $(2.17)_2$ or (2.8). (Note that $\mathbf{dx}' \cdot \mathbf{dx} = \mathbf{dx}'. (\mathbf{A}\mathbf{dX})$). We now justify and formalize this terminology which, at first sight, appears to conflict with the usual convention.

3. Eulerian and Lagrangean objectivity

In the literature tensors are often regarded as real multilinear mappings over the translation space E, either explicitly as in [4] or, more commonly, implicitly. This means that the distinctions between Eulerian, Lagrangean and two-point tensors cannot be made clear. The difficulty can be overcome in two ways. First, by associating the reference and current configurations with distinct translation spaces, E_0 and E_t, respectively, so that, for example, a (Lagrangean) vector $\mathbf{v}_0$ in E_0 is unaffected by an observer transformation while an (Eulerian) vector $\mathbf{v}$ transforms according to $\mathbf{v}^* = \mathbf{Q}(t)\mathbf{v}$. (Recall the footnote ([3])). The second approach is more general in that it is appropriate for tensor *fields*. This we now describe.

The set of Lagrangean line elements $\mathbf{dX}$ at a point $\mathbf{X}$ in the reference configuration $\mathscr{B}_0$ spans a (three-dimensional) vector space which we denote by $T_{\mathbf{X}}(\mathscr{B}_0)$. It is called the *tangent space* of (the manifold) $\mathscr{B}_0$ at $\mathbf{X}$. Similarly, $T_{\mathbf{x}}(\mathscr{B}_t)$ denotes the tangent space of (the manifold) $\mathscr{B}_t$ at $\mathbf{x}$ and is spanned by the set of Eulerian line elements $\mathbf{dx}$.

In general the tangent spaces $T_{\mathbf{X}}(\mathscr{B}_0)$ and $T_{\mathbf{x}}(\mathscr{B}_t)$ are distinct and they are also distinct from the translation spaces E_0 and E_t, respectively (although there are isomorphisms between these spaces).

Let $T_{\mathbf{X}}(\mathscr{B}_0)^m \times T_{\mathbf{x}}(\mathscr{B}_t)^n$ denote the Cartesian product

$$\underbrace{T_{\mathbf{X}}(\mathscr{B}_0)\times \ldots \times T_{\mathbf{X}}(\mathscr{B}_0)}_{m \text{ times}} \times \underbrace{T_{\mathbf{x}}(\mathscr{B}_t)\times \ldots \times T_{\mathbf{x}}(\mathscr{B}_t)}_{n \text{ times}},$$

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and $\mathscr{L}(T_{\mathbf{X}}(\mathscr{B}_0)^m \times T_{\mathbf{x}}(\mathscr{B}_t)^n, \mathsf{R})$ denote the set of real $(m+n)$-multilinear mappings over $T_{\mathbf{X}}(\mathscr{B}_0)$ and $T_{\mathbf{x}}(\mathscr{B}_t)$.

A *Lagrangean tensor of order* n at $\mathbf{X}$ is defined to be a member of the space $\mathscr{L}(T_{\mathbf{X}}(\mathscr{B}_0)^n, \mathsf{R})$. Similarly, an *Eulerian tensor of order n* at $\mathbf{x}$ is a member of $\mathscr{L}(T_{\mathbf{x}}(\mathscr{B}_t)^n, \mathsf{R})$. A *two-point tensor of Lagrangean order m and Eulerian order n* is contained in $\mathscr{L}(T_{\mathbf{X}}(\mathscr{B}_0)^m \times \times T_{\mathbf{x}}(\mathscr{B}_t)^n, \mathsf{R})$.

Lagrangean and Eulerian *vectors* at $\mathbf{X}$ and $\mathbf{x}$, respectively, constitute the spaces $\mathscr{L}(T_{\mathbf{X}}(\mathscr{B}_0), \mathsf{R})$ and $\mathscr{L}(T_{\mathbf{x}}(\mathscr{B}_t), \mathsf{R})$. Strictly these are the *dual spaces* of $T_{\mathbf{X}}(\mathscr{B}_0)$ and $T_{\mathbf{x}}(\mathscr{B}_t)$, respectively, but, in order to avoid unnecessary complication, *we do not distinguish between vector spaces and their duals in this paper*. Note that $T_{\mathbf{X}}(\mathscr{B}_0)$ is unaffected by an observer transformation but $T_{\mathbf{x}}(\mathscr{B}_t)$ changes to $T_{\mathbf{x}^*}(\mathscr{B}^*_{t^*})$, where $\mathscr{B}^*_{t^*} = \chi^*_{t^*}(B)$.

Let $\mathbf{T}(\mathbf{x}, t) \in \mathscr{L}(T_{\mathbf{x}}(\mathscr{B}_t)^n, \mathsf{R})$ denote an Eulerian tensor of order n at $\mathbf{x}$. The corresponding multilinear form may be written

$$\mathbf{T}(\mathbf{x}, t)(\mathbf{dx}^{(1)}, \ldots, \mathbf{dx}^{(n)}),$$

where $(\mathbf{dx}^{(1)}, \ldots, \mathbf{dx}^{(n)}) \in T_{\mathbf{x}}(\mathscr{B}_t)^n$ and $\mathbf{dx}^{(k)} \in T_{\mathbf{x}}(\mathscr{B}_t)$ $(k = 1, \ldots, n)$.

We say that $\mathbf{T}(\mathbf{x}, t)$ is an *objective Eulerian tensor* of order n if its value $\mathbf{T}^*(\mathbf{x}^*, t^*)$ observed by 0^* is such that

$$\mathbf{T}^*(\mathbf{x}^*, t^*)(\mathbf{dx}^{(1)*}, \ldots, \mathbf{dx}^{(n)*}) = \mathbf{T}(\mathbf{x}, t)(\mathbf{dx}^{(1)}, \ldots, \mathbf{dx}^{(n)}), \tag{3.1}$$

where $\mathbf{dx}^{(k)*} = \mathbf{Q}(t)\,\mathbf{dx}^{(k)}$ and $\mathbf{dx}^{(k)}$ is arbitrary $(k = 1, \ldots, n)$. For $n = 1$, this yields the transformation rule

$$\mathbf{v}^*(\mathbf{x}^*, t^*) = \mathbf{Q}(t)\mathbf{v}(\mathbf{x}, t) \tag{3.2}$$

for an objective Eulerian vector, while for $n = 2$ Eq. (3.1) is reducible to

$$\mathbf{T}^*(\mathbf{x}^*, t^*) = \mathbf{Q}(t)\mathbf{T}(x, t)\mathbf{Q}(t)^T. \tag{3.3}$$

The relations (3.2) and (3.3) are common in the literature (see [4] and [5], for example) but there is no corresponding simple representation for $n \geqslant 3$. However, it is instructive to examine the component form of Eq. (3.1) with respect to a rectangular Cartesian basis, $\{\mathbf{e}_i\}$ say. Then, with

$$\mathbf{T}(\mathbf{x}, t) = T_{ijk\ldots}(\mathbf{x}, t)\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes \ldots,$$
$$\mathbf{T}^*(\mathbf{x}^*, t^*) = T^*_{ijk\ldots}(\mathbf{x}^*, t^*)\mathbf{e}^*_i\otimes\mathbf{e}^*_j\otimes\mathbf{e}^*_k\otimes \ldots$$

and $\mathbf{e}^*_i = \mathbf{Q}(t)\mathbf{e}_i$, the objectivity statement (3.1) becomes

$$T^*_{ijk\ldots}(\mathbf{x}^*, t^*) = T_{ijk\ldots}(\mathbf{x}, t). \tag{3.4}$$

We now turn to Lagrangean tensors. Let $\mathbf{T}_0(\mathbf{X}, t) \in \mathscr{L}(T_{\mathbf{X}}(\mathscr{B}_0)^n, \mathsf{R})$ denote a Lagrangean tensor of order n. It is said to be objective if

$$\mathbf{T}^*_0(\mathbf{X}, t^*)(\mathbf{dX}^{(1)}, \ldots, \mathbf{dX}^{(n)}) = \mathbf{T}_0(\mathbf{X}, t)(\mathbf{dX}^{(1)}, \ldots, \mathbf{dX}^{(n)}) \tag{3.5}$$

for all Lagrangean line elements $\mathbf{dX}^{(k)}$ $(k = 1, \ldots, n)$ (recall that $\mathbf{X}$ and $\mathbf{dX}^{(k)}$ are not affected by an observer transformation). More simply, this is expressible as

$$\mathbf{T}^*_0(\mathbf{X}, t^*) = \mathbf{T}_0(\mathbf{X}, t). \tag{3.6}$$

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The Eulerian tensor $\mathbf{T}(\mathbf{x}, t)$ is the value at $\mathbf{x} \in \mathscr{B}_t$ of the tensor field $\mathbf{T}(\cdot, t)$ defined over $\mathscr{B}_t$. This is the Eulerian description of the field. But, through the motion $\mathbf{x} = \boldsymbol{\varkappa}(\mathbf{X}, t)$, it may be given a Lagrangean description $\mathbf{T}\{\boldsymbol{\varkappa}(\mathbf{X}, t), t\}$ corresponding to a field defined over $\mathscr{B}_0$. It remains, however, an Eulerian tensor field. Equally, through the inverse $\mathbf{X} = \boldsymbol{\varkappa}^{-1}(\mathbf{x}, t)$ a Lagrangean tensor field may be given an Eulerian description. Thus it is important to distinguish between a Lagrangean (respectively Eulerian) tensor field and the Lagrangean (respectively Eulerian) description of a tensor field.

The situation is different in respect of scalar fields. If $\phi(\cdot, t)$ is a scalar field defined over $\mathscr{B}_t$, the corresponding field $\phi_0(\cdot, t)$ over $\mathscr{B}_0$ is specified by

$$\phi_0(\mathbf{X}, t) = \phi\{\boldsymbol{\varkappa}(\mathbf{X}, t), t\}, \qquad \mathbf{X} \in \mathscr{B}_0 \tag{3.7}$$

and conversely

$$\phi(\mathbf{x}, t) = \phi_0\{\boldsymbol{\varkappa}^{-1}(\mathbf{x}, t), t\}, \qquad \mathbf{x} \in \mathscr{B}_t. \tag{3.8}$$

Hence, a Lagrangean (respectively Eulerian) scalar field is simply Lagrangean (respectively Eulerian) description of a scalar field.

A scalar field is objective if

$$\phi^*(\mathbf{x}^*, t^*) = \phi(\mathbf{x}, t) \tag{3.9}$$

or, equivalently,

$$\phi_0^*(\mathbf{X}, t^*) = \phi_0(\mathbf{X}, t) \tag{3.10}$$

in view of Eqs. (3.7) and (3.8). We note that the scalar field $\det \mathbf{A}$ is objective but the particle speed $|\dot{\boldsymbol{\chi}}(\mathbf{X}, t)|$ is not. It follows from Eqs. (3.9) and (3.4) that the components $T_{ijk\ldots}(\mathbf{x}, t)$ of an objective Eulerian tensor are objective scalars.

We have already noted that in standard texts such as [4] and [5] no distinction is made between Eulerian and Lagrangean fields and a field is said to be objective if it satifies the appropriate one of the (Eulerian) transformation rules (3.1)–(3.3) or (3.9). It then follows that Lagrangian fields such as $\mathbf{A}^T\mathbf{A}$ with the transformation rule (3.6) are, by default, not regarded as objective. On the other hand, HILL [1] regards as objective only those fields for which the rule (3.6) holds. The distinction we have made between Eulerian and Lagrangean fields reconciles these two views which are essentially alternative manifestations of a single definition of objectivity. We shall expand on this point in Sect. 4.

For two-point tensors a definition of objectivity intermediate between the relations (3.1) and (3.5) is required. With an Eulerian description we suppose $\mathbf{T}(\mathbf{x}, t) \in \mathscr{L}(T_{\mathbf{X}}(\mathscr{B}_0)^m \times T_{\mathbf{x}}(\mathscr{B}_t)^n, \mathrm{R})$. This is an objective two-point tensor if

$$\mathbf{T}^*(\mathbf{x}^*, t^*)(\mathbf{dX}^{(1)}, \ldots, \mathbf{dX}^{(m)}, \mathbf{dx}^{(1)*}, \ldots, \mathbf{dx}^{(n)*}) = \mathbf{T}(\mathbf{x}, t)(\mathbf{dX}^{(1)}, \ldots, \mathbf{dX}^{(m)}, \mathbf{dx}^{(1)}, \ldots, \mathbf{dx}^{(n)}) \tag{3.11}$$

for all $\mathbf{dX}^{(k)} \in T_{\mathbf{X}}(\mathscr{B}_0)$, $k = 1, \ldots, m$, and all $\mathbf{dx}^{(l)} \in T_{\mathbf{x}}(\mathscr{B}_t)$, $l = 1, \ldots, n$, where $\mathbf{dx}^{(l)*} = \mathbf{Q}(t)\,\mathbf{dx}^{(l)}$. The definitions (3.1) and (3.5) are embraced by the relations (3.11) if we set either $m = 0$ or $n = 0$. It follows from the rule (2.8) that the deformation gradient $\mathbf{A}(\mathbf{X}, t)$ is an objective two-point tensor corresponding to $m = n = 1$ in the relations (3.11) (but note that $\mathbf{A}(\mathbf{X}, t)$ is given a Lagrangean description here).

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We remark that the characterization (3.11) of an objective tensor field is invariant under a change of reference configuration although the fields themselves change.

In Section 2 we observed that the transformation (2.5) can be regarded by 0 as a rigid-body motion superposed on the motion (2.2). Since we are restricting attention to proper orthogonal $\mathbf{Q}(t)$, it is appropriate to examine the consequences of this viewpoint in relation to the definition of objectivity. According to a single observer 0, the tensor $\mathbf{T}(x, t)$ is objective if it is invariant under a superposed rigid-body motion (1.3) in the sense that

$$\tag{3.12} \mathbf{T}(\mathbf{x}^*, t)(\mathbf{dX}^{(1)}, \ldots, \mathbf{dX}^{(m)}, \mathbf{dx}^{(1)*}, \ldots, \mathbf{dx}^{(n)*}) = \mathbf{T}(\mathbf{x}, t)(\mathbf{dX}^{(1)}, \ldots, \mathbf{dX}^{(m)}, \mathbf{dx}^{(1)}, \ldots, \mathbf{dx}^{(n)})$$

with $\mathbf{dX}^{(k)}$ and $\mathbf{dx}^{(l)*}$ defined as for the relations (3.11). This approach is conceptually simpler than that involving changes of observer and is entirely equivalent to it provided $\mathbf{Q}(t)$ is proper orthogonal. A direct correspondence between the relations (3.11) and (3.12) is established by setting $\mathbf{T}^*(\mathbf{x}^*, t^*) = \mathbf{T}(\mathbf{x}^*, t)$ with $t^* = t-a$.

4. Induced objectivity

We have seen in Eqs. (3.7) and (3.8) that for a given motion and choice of reference configuration each Eulerian (respectively Lagrangean) scalar field is associated with a unique Lagrangean (respectively Eulerian) scalar field. Lagrangean and Eulerian vector and tensor fields may also be associated through the deformation (by means of the deformation tensors $\mathbf{A}$ and $\mathbf{B}$ in particular), but not uniquely. For example, if $\mathbf{v}$ is an Eulerian vector field, then $\mathbf{A}^T\mathbf{v}$ and $\mathbf{B}^T\mathbf{v}$ are Lagrangean vector fields. Respectively they are covariant and contravariant in character since

$$(\mathbf{A}^T\mathbf{v}) \cdot \mathbf{dX} = \mathbf{v} \cdot (\mathbf{A}\mathbf{dX}) = \mathbf{v} \cdot \mathbf{dx} = v_i dx^i,$$

$$(\mathbf{B}^T\mathbf{v}) \cdot \mathrm{Grad} = \mathbf{v} \cdot (\mathbf{B}\mathrm{Grad}) = \mathbf{v} \cdot \mathrm{grad} = v^i \frac{\partial}{\partial x^i},$$

where v_i and v^i, respectively, are covariant and contravariant components of $\mathbf{v}$ with respect to a general curvilinear basis.

We adapt the terminology of HILL [1] and refer to $\mathbf{A}^T\mathbf{v}$ and $\mathbf{B}^T\mathbf{v}$ as (covariant and contravariant) *induced Lagrangean fields of* $\mathbf{v}$. Similarly, if $\mathbf{v}_0$ is a Lagrangean vector field, then $\mathbf{A}\mathbf{v}_0$ and $\mathbf{B}\mathbf{v}_0$ are *induced Eulerian fields of* $\mathbf{v}_0$. Note that $\mathbf{A}^T\mathbf{A}\mathbf{v}_0$ is Lagrangean and $\mathbf{A}\mathbf{A}^T\mathbf{v}$ is Eulerian. A second-order Eulerian tensor field $\mathbf{T}$ has induced Lagrangean fields $\mathbf{A}^T\mathbf{T}\mathbf{A}$, $\mathbf{A}^T\mathbf{T}\mathbf{B}$, $\mathbf{B}^T\mathbf{T}\mathbf{A}$, $\mathbf{B}^T\mathbf{T}\mathbf{B}$. More generally, if $\mathbf{T}$ and $\mathbf{T}_0$ are Eulerian and Lagrangean tensor fields respectively of order n, then the equation

$$\tag{4.1} \mathbf{T}(\mathbf{x}, t)(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}) = \mathbf{T}_0(\mathbf{X}, t)(\mathbf{v}_0^{(1)}, \ldots, \mathbf{v}_0^{(n)})$$

with $\mathbf{x} = \boldsymbol{\varkappa}(\mathbf{X}, t)$ defines 2^n possible induced Lagrangean (respectively Eulerian) fields of $\mathbf{T}$ (respectively $\mathbf{T}_0$), where the vector fields $\mathbf{v}_0^{(k)} \in T_{\mathbf{X}}(\mathscr{B}_0)$ and $\mathbf{v}^{(k)} \in T_{\mathbf{x}}(\mathscr{B}_t)$ are connected through either $\mathbf{v}^{(k)} = \mathbf{A}\mathbf{v}_0^{(k)}$ or $\mathbf{v}^{(k)} = \mathbf{B}\mathbf{v}_0^{(k)}$ for each $k (k = 1, \ldots, n)$(7).

The relation (4.1) may be generalized to include two-point tensor fields but in order to avoid introducing further notations we do not do this here. However, for illustration,

(7) We emphasize that we are making no distinction between a vector space and its dual in this paper.

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we note that if $\mathbf{T}$ is an Eulerian tensor field of order two, then $\mathbf{A}^T\mathbf{T}$, $\mathbf{B}^T\mathbf{T}$, $\mathbf{TA}$, $\mathbf{TB}$ are induced two-points fields of $\mathbf{T}$.

From the definition (3.12) and a generalization of Eq. (4.1), it follows immediately that *a tensor field is objective if and only if each of its induced fields is objective*.

With the help of Eq. (1.2), differentiation of Eq. (3.6) yields

$$\dot{\mathbf{T}}_0^*(\mathbf{X}, t^*) = \dot{\mathbf{T}}_0(\mathbf{X}, t), \tag{4.2}$$

where the dot indicates time differentiation at fixed $\mathbf{X}$. Thus *the rate of change of an objective Lagrangean tensor is an objective Lagrangean tensor*. Equally, all induced tensors of $\dot{\mathbf{T}}_0(\mathbf{X}, t)$ are objective, but the *time derivative* (either at fixed $\mathbf{X}$ or fixed $\mathbf{x}$) *of an Eulerian or two-point tensor field is not objective*. For example, if $\mathbf{v}$ is an objective Eulerian vector field, then $\mathbf{v}^* = \mathbf{Qv}$, $\dot{\mathbf{v}}^* = \mathbf{Q}\dot{\mathbf{v}}+\dot{\mathbf{Q}}\mathbf{v}$ and $\dot{\mathbf{v}}$ is therefore not objective. But, since $\mathbf{A}^T\mathbf{v}$ is an objective Lagrangean vector field,

$$\frac{\partial}{\partial t}(\mathbf{A}^T\mathbf{v}) = \mathbf{A}^T(\dot{\mathbf{v}}+\mathbf{\Gamma}^T\mathbf{v})$$

is objective and therefore the induced Eulerian vector field $\dot{\mathbf{v}}+\mathbf{\Gamma}^T\mathbf{v}$ is objective. Similarly for

$$\frac{\partial}{\partial t}(\mathbf{B}^T\mathbf{v}) = \mathbf{B}^T(\dot{\mathbf{v}}-\mathbf{\Gamma}\mathbf{v}).$$

More generally, for constant fields $\mathbf{v}_0^{(k)}$, differentiation of Eq. (4.1) with the use of $\dot{\mathbf{A}} = \mathbf{\Gamma A}$ and $\dot{\mathbf{B}} = -\mathbf{\Gamma}^T\mathbf{B}$ yields

$$\check{\mathbf{T}}(\mathbf{x}, t)(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}) = \dot{\mathbf{T}}_0(\mathbf{X}, t)(\mathbf{v}_0^{(1)}, \ldots, \mathbf{v}_0^{(n)}) \tag{4.3}$$

which defines the Eulerian tensor field $\check{\mathbf{T}}(\mathbf{x}, t)$. Two examples of $\check{\mathbf{T}}(\mathbf{x}, t)$ are

$$\dot{\mathbf{T}}+\mathbf{\Gamma}^T\mathbf{T}+\mathbf{T\Gamma}$$

and

$$\dot{\mathbf{T}}-\mathbf{\Gamma T}-\mathbf{T\Gamma}^T,$$

when $\mathbf{T}$ is of second order. These are induced Eulerian fields of the time derivatives of $\mathbf{A}^T\mathbf{TA}$ and $\mathbf{B}^T\mathbf{TB}$, respectively. In the context of continuum mechanics they are often, referred to as "convected" derivatives of $\mathbf{T}$, while in the language of differentiable manifolds [2], they are essentially Lie derivatives of $\mathbf{T}$ with respect to the velocity $\mathbf{v}$. The result that objective tensors have objective Lie derivatives, given in [2], is equivalent to Eq. (4.2) with Eq. (3.6).

5. Application to conjugate stress analysis

Let $\mathbf{T}$ denote the Kirchhoff stress tensor (the product of the scalar det $\mathbf{A}$ and the Cauchy stress tensor) and $\mathbf{\Sigma}$ the Eulerian strain-rate (both symmetric second-order Eulerian tensor fields). Then the expression $\mathrm{tr}(\mathbf{T\Sigma})$ represents the rate of working of the stresses on the material of the body B per unit volume of the reference configuration $\mathscr{B}_0$. It may be rewritten as

$$\mathrm{tr}(\mathbf{B}^T\mathbf{TB}\dot{\mathbf{E}}), \tag{5.1}$$

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where $\mathbf{E}$ is the Green strain tensor (2.15) and the Lagrangean tensor $\mathbf{B}^T\mathbf{T}\mathbf{B}$ is the (second) Piola–Kirchhoff stress tensor.

More generally, let $\mathbf{E}_0$ be a (symmetric) objective Lagrangean strain tensor in the sense of [1] so that $\mathbf{E}_0$ is coaxial with the right stretch tensor $\mathbf{U}$ and is expressible as an isotropic tensor function of $\mathbf{U}$ through $\mathbf{E}_0 = \mathbf{G}_0(\mathbf{U})$, where $\mathbf{G}_0$ is a suitably behaved function satisfying

$$\mathbf{G}_0(\mathbf{P}\mathbf{U}\mathbf{P}^T) = \mathbf{P}\mathbf{G}_0(\mathbf{U})\mathbf{P}^T \tag{5.2}$$

for all proper orthogonal Lagrangean tensors $\mathbf{P}$. Then there exists an objective (symmetric) Lagrangean tensor $\mathbf{T}_0$ such that

$$\mathrm{tr}(\mathbf{T}_0\dot{\mathbf{E}}_0) = \mathrm{tr}(\mathbf{T}\mathbf{\Sigma}), \tag{5.3}$$

and, following [1], $\mathbf{T}_0$ and $\mathbf{E}_0$ are said to be *conjugate stress and strain tensors*.

It is natural to regard such strain tensors as functions of $\mathbf{U}$, rather than $\mathbf{V}$, since, through Eq. (2.15), the expression (5.1) is expressible as a linear form in $\dot{\mathbf{U}}$, as is the left-hand side of Eq. (5.3). Nevertheless, consideration of the Eulerian strain tensor $\mathbf{F} = \mathbf{R}\mathbf{E}_0\mathbf{R}^T \equiv \mathbf{G}_0(\mathbf{V})$, where $\dot{\mathbf{F}}$ is not objective, is instructive. Substitution of $\mathbf{E}_0 = \mathbf{R}^T\mathbf{F}\mathbf{R}$ in Eq. (5.3) leads to

$$\mathrm{tr}(\mathbf{T}_0\dot{\mathbf{E}}_0) = \mathrm{tr}(\mathbf{R}\mathbf{T}_0\mathbf{R}^T\dot{\mathbf{F}}) + \mathrm{tr}\{(\mathbf{T}_0\mathbf{F}_0 - \mathbf{E}_0\mathbf{T}_0)\mathbf{R}^T\dot{\mathbf{R}}\} \tag{5.4}$$

after some rearrangement of terms. It follows that the Eulerian strain tensor $\mathbf{F}$ has a conjugate stress tensor, namely $\mathbf{R}\mathbf{T}_0\mathbf{R}^T$, if and only if the latter term in Eq. (5.4) vanishes identically for all $\dot{\mathbf{R}}$. Since $\mathbf{R}^T\dot{\mathbf{R}}$ is antisymmetric, this condition is met if and only if

$$\mathbf{T}_0\mathbf{E}_0 = \mathbf{E}_0\mathbf{T}_0, \tag{5.5}$$

i.e. $\mathbf{T}_0$ is coaxial with $\mathbf{E}_0$ for all deformations from the reference configuration $\mathscr{B}_0$. In the context of elasticity theory this means that the material is isotropic relative to $\mathscr{B}_0$.

Let W be the strain-energy function of an elastic material per unit volume in $\mathscr{B}_0$ so that

$$\dot{W} = \mathrm{tr}(\mathbf{T}_0\dot{\mathbf{E}}_0)$$

for any conjugate pair $(\mathbf{T}_0, \mathbf{E}_0)$. Thus W can be regarded as a function of $\mathbf{E}_0$. We write $W(\mathbf{E}_0)$ but, of course, the precise form of the function is dependent on the choice of $\mathbf{E}_0$. The stress $\mathbf{T}_0$ is given by

$$\mathbf{T}_0 = \frac{\partial W}{\partial \mathbf{E}_0}.$$

Objectivity of W, and hence of $\mathbf{T}_0$, follows automatically from that of $\mathbf{E}_0$ when $\mathbf{E}_0$ is an objective Lagrangean strain tensor. This implies that W is indifferent to superposed rigid motions of the material after deformation (as is required).

Equally, W may be expressed as a function of $\mathbf{F}$ but the objectivity of W is not then automatic and the restriction

$$W(\mathbf{Q}\mathbf{F}\mathbf{Q}^T) = W(\mathbf{F})$$

for all proper orthogonal $\mathbf{Q}$ must be imposed. And, in general, $\partial W/\partial\mathbf{F}$ is not a stress tensor. This illustrates one well-known advantage of the Lagrangean over the Eulerian viewpoint.

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Received September, 12, 1983.

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