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On quasi-isotropic tensor functions

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THE CLASS of such functions is considered, whose arguments and values are symmetric second-order tensors, and which has the following property: the value of the function of a certain argument is at least as symmetric with respect to the rotations as the argument itself. A general representation is given. The potential function is shown to have the property mentioned above only in the case when it is isotropic.

Rozpatrzono klasę funkcji, których argumentami i wartościami są tensory symetryczne drugiego rzędu, mających następującą własność: wartość funkcji od dowolnego argumentu jest co najmniej tak symetryczna względem obrotów jak ten argument. Podano ogólny wzór reprezentacyjny. Pokazano, że funkcja potencjalna ma wspomnianą własność tylko wtedy, gdy jest izotropowa.

Рассмотрен класс функций, аргументами и значенями которых являются симетричные тензоры второго ранга, обладающих следующим свойством: значение функции от любого аргумента по крайней мере так же симметрично относительно вращений как и этот аргумент. Дано общее выражение функций этого класса. Показано, что потенциальная функция обладает указанным свойством только тогда, когда она изотропна.

1. Formulation of the problem

IN THE THEORY of continua, mainly in mechanics and in related fields, a principal role is played by such functions f which ascribe symmetric second order tensors $\mathbf{f}(\boldsymbol{\omega})$ to second order symmetric tensors $\boldsymbol{\omega}$. Such functions are represented, for instance, by the constitutive laws of elastic bodies, viscous liquids etc. If a physical property of the body is independent on the direction in space, the function $\mathbf{f}$ must be *isotropic*. Isotropic functions are described by the classical representation theorems [1–6]

$$\mathbf{f}(\boldsymbol{\omega}) = \varphi_1(\boldsymbol{\omega})\mathbf{1}+\varphi_2(\boldsymbol{\omega})\boldsymbol{\omega}+\varphi_3(\boldsymbol{\omega})\boldsymbol{\omega}^2, \tag{1.1}$$

φ_i being the invariants.

In the case of an isotropic function, tensor $\mathbf{f}(\boldsymbol{\omega})$ is always coaxial with $\boldsymbol{\omega}$. This important property *is not* equivalent to isotropy. Consequently, it is reasonable and useful to consider the entire class of tensor functions having this property.

DEFINITION. *The function* $\boldsymbol{\omega} \rightarrow \mathbf{f}(\boldsymbol{\omega})$ *will be called quasi-isotropic if for each argument* $\boldsymbol{\omega}$ *all its eigenvectors are at the same time the eigenvectors of the values of* $\mathbf{f}(\boldsymbol{\omega})$.

Isotropic functions constitute a subclass of the quasi-isotropic functions. Our considerations are aimed at the analysis of quasi-isotropic functions and, in the first place, at those which are not isotropic. Taking this opportunity, certain new facts concerning the isotropic functions will be pointed out.

2. Representation theorem

The definition of quasi-isotropy may be formulated in the language of symmetry. The entire space of symmetric second order tensors over the Euclidean vector space E is denoted by $\mathscr{S}$, $\mathscr{S} \equiv \mathrm{sym}\, E\otimes E$. Element of $\mathscr{S}$ are denoted by $\boldsymbol{\omega}$, $\boldsymbol{\tau}$, ..., except for the orthogonal tensors for which the notationss $Q, \ldots$, are generally accepted. The orthogonal tensors realize the automorphisms (isometries) of the original space, $\mathbf{a} \to \mathbf{Qa}$, $\mathbf{a} \in E$ and constitute the orthogonal group O. This group acts in $\mathscr{S}$ according to the usual rule $\boldsymbol{\omega} \to \mathbf{Q} * \boldsymbol{\omega} \equiv \mathbf{Q}\boldsymbol{\omega}\mathbf{Q}^T$, $\mathbf{Q} * (\mathbf{a}\otimes\mathbf{b}) = \mathbf{Qa}\otimes\mathbf{Qb}$.

The symmetry group of tensor $\boldsymbol{\omega}$ is

(2.1) $$O_{\boldsymbol{\omega}} \equiv \{\mathbf{Q} \in O | \mathbf{Q} * \boldsymbol{\omega} = \boldsymbol{\omega}\}.$$

LEMMA 1. For the arbitrary tensors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ the following statements are equivalent:

1) Each eigenvector of tensor $\boldsymbol{\alpha}$ is the eigenvector of tensor $\boldsymbol{\beta}$.

2) $$O_{\boldsymbol{\beta}} \supset O_{\boldsymbol{\alpha}}.$$

P r o o f of 1) $\Rightarrow$ 2): Let us apply to $\boldsymbol{\alpha}$ the spectral resolution theorem,

(2.2) $$\boldsymbol{\alpha} = \alpha_1 \mathbf{n}_1\otimes\mathbf{n}_1 + \alpha_2 \mathbf{n}_2\otimes\mathbf{n}_2 + \alpha_3 \mathbf{n}_3\otimes\mathbf{n}_3.$$

If the statement 1) holds true, then

(2.3) $$\boldsymbol{\beta} = \beta_1 \mathbf{n}_1\otimes\mathbf{n}_1 + \beta_2 \mathbf{n}_2\otimes\mathbf{n}_2 + \beta_3 \mathbf{n}_3\otimes\mathbf{n}_3,$$

and for any prescribed $1 \leqslant i, k \leqslant 3$

(2.4) $$\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_k \Rightarrow \boldsymbol{\beta}_i = \boldsymbol{\beta}_k.$$

Now, if $\mathbf{Q} * \boldsymbol{\alpha} = \boldsymbol{\alpha}$, then also $\mathbf{Q} * \boldsymbol{\beta} = \boldsymbol{\beta}$.

P r o o f of 2) $\Rightarrow$ 1): Let us consider, for instance, the eigenvector $\mathbf{n}_1$ of tensor $\boldsymbol{\alpha}$. Consider the mirror reflection (following the approach used in [6]) $\mathbf{Qn}_1 = -\mathbf{n}_1$, $\mathbf{Qn}_2 = \mathbf{n}_2$, $\mathbf{Qn}_3 = \mathbf{n}_3$. It is obvious that $\mathbf{Q} \in O_{\boldsymbol{\alpha}}$ whence also $\mathbf{Q} \in O_{\boldsymbol{\beta}}$. Now $\boldsymbol{\beta}\mathbf{n}_1 = (\mathbf{Q} * \boldsymbol{\beta})\mathbf{n}_1 = -(\mathbf{Q} * \boldsymbol{\beta})\mathbf{Q}\mathbf{n}_1 = -\mathbf{Q}(\boldsymbol{\beta}\mathbf{n}_1)$ whence it follows that $\boldsymbol{\beta}\mathbf{n}_1 = \lambda\mathbf{n}_1$, *Q.E.D.*

It is seen that the definition of quasi-isotropic functions may also be formulated without using the notion of eigenvectors.

DEFINITION. *The "law"* $\boldsymbol{\omega} \to \mathbf{f}(\boldsymbol{\omega})$ *will be called quasi-isotropic if for each* $\boldsymbol{\omega}$ *the "effect"* $\mathbf{f}(\boldsymbol{\omega})$ *is at least as symmetric as the "cause"* $\boldsymbol{\omega}$, *that is*

(2.5) $$O_{\mathbf{f}(\boldsymbol{\omega})} \supset O_{\boldsymbol{\omega}}.$$

Let us introduce the important invariant in $\mathscr{S}$

(2.6) $$\mu(\boldsymbol{\omega}) \equiv \inf_{\substack{i\neq k \\ i,k}} |\omega_i - \omega_k|,$$

where ω_i are the eigenvalues of $\boldsymbol{\omega}$. Introduce the set of tensors with pairwise different eigenvalues, $\omega_1 \neq \omega_2 \neq \omega_3 \neq \omega_1$,

(2.7) $$\mathscr{A} \equiv \{\boldsymbol{\omega} \in \mathscr{S} | \mu(\boldsymbol{\omega}) \neq 0\},$$

and its complement

(2.8) $$\mathscr{B} = \{\boldsymbol{\omega} \in \mathscr{S} | \mu(\boldsymbol{\omega}) = 0\}.$$

In the set $\mathscr{B}$ we can distinguish the set of spherical tensors $\mathscr{B}_1$, $\omega_1 = \omega_2 = \omega_3 = \omega$ and its complement $\mathscr{B}_2$ The decompositions into disjoint sets

$$\mathscr{S} = \mathscr{A} \cup \mathscr{B} = \mathscr{A} \cup \mathscr{B}_1 \cup \mathscr{B}_2 \tag{2.9}$$

are of considerable importance for further considerations.

For each fixed $\boldsymbol{\omega} \in \mathscr{S}$ let us introduce the set

$$W(\boldsymbol{\omega}) = \{\boldsymbol{\tau} \in \mathscr{S} | O_{\tau} \supset O_{\omega}\}. \tag{2.10}$$

The condition (2.5) of the definition of quasi-isotropic functions may be written in an equivalent form:

$$\mathbf{f}(\boldsymbol{\omega}) \in W(\boldsymbol{\omega}). \tag{2.11}$$

Evidently, $W(\boldsymbol{\omega})$ is a linear subspace in $\mathscr{S}$ (not invariant to O) with the dimension

$$\dim W(\boldsymbol{\omega}) = \begin{cases} 3 & \text{for} \quad \boldsymbol{\omega} \in \mathscr{A}, \\ 2 & \text{for} \quad \boldsymbol{\omega} \in \mathscr{B}_2, \\ 1 & \text{for} \quad \boldsymbol{\omega} \in \mathscr{B}_1. \end{cases} \tag{2.12}$$

$\boldsymbol{\tau} \in W(\boldsymbol{\omega})$ may easily be written in an explicit form.

The system of isotropic functions $g_i \colon \mathscr{S} \to \mathscr{S}$, $i = 1, 2, 3$ is called a *generating system* if for each $\boldsymbol{\omega}$

$$W(\boldsymbol{\omega}) = \mathrm{Lin}(\mathbf{g}_1(\boldsymbol{\omega}), \mathbf{g}_2(\boldsymbol{\omega}), \mathbf{g}_3(\boldsymbol{\omega})). \tag{2.13}$$

This means that for $\boldsymbol{\omega} \in \mathscr{A}$ the tensors $\mathbf{g}_1(\boldsymbol{\omega})$, $\mathbf{g}_2(\boldsymbol{\omega})$, $\mathbf{g}_3(\boldsymbol{\omega})$ constitute a base in $W(\boldsymbol{\omega})$, and for $\boldsymbol{\omega} \in \mathscr{B}_2$ two of three tensors $\mathbf{g}_1(\boldsymbol{\omega})$, $\mathbf{g}_2(\boldsymbol{\omega})$, $\mathbf{g}_3(\boldsymbol{\omega})$ represent a base in $W(\boldsymbol{\omega})$, the third one being their linear combination. For $\boldsymbol{\omega} \in \mathscr{B}_1$ all $\mathbf{g}_i(\boldsymbol{\omega})$ are spherical and at least one of them is nonvanishing.

An example of generating systems is the well-known system

$$\mathbf{g}_1(\boldsymbol{\omega}) = \mathbf{1}, \quad \mathbf{g}_2(\boldsymbol{\omega}) = \boldsymbol{\omega}, \quad \mathbf{g}_3(\boldsymbol{\omega}) = \boldsymbol{\omega}^2. \tag{2.14}$$

As a matter of fact, if $\boldsymbol{\omega} \in \mathscr{A}$, then the equation

$$\alpha \mathbf{1} + \beta \boldsymbol{\omega} + \gamma \boldsymbol{\omega}^2 = \mathbf{0} \tag{2.15}$$

holds true only for $\alpha = \beta = \gamma = 0$, and if $\omega \in \mathscr{B}_2$ then the equation

$$\alpha \mathbf{1} + \beta \boldsymbol{\omega} = \mathbf{0} \tag{2.16}$$

holds true only for $\alpha = \beta = 0$.

Besides the standard system (2.14) there exist infinitely many other generating systems. One of them is the system proposed in [7],

$$\begin{gathered} \mathbf{g}_1(\boldsymbol{\omega}) = \mathbf{1}, \quad \mathbf{g}_2(\boldsymbol{\omega}) = \boldsymbol{\omega}^*, \\ \mathbf{g}_2(\boldsymbol{\omega}) = ((\boldsymbol{\omega}^*)^2)^* - \frac{\mathrm{tr}(\boldsymbol{\omega}^*)^3}{\mathrm{tr}(\boldsymbol{\omega}^*)^2} \boldsymbol{\omega}^*. \end{gathered} \tag{2.17}$$

Here $\boldsymbol{\alpha}^* \equiv \boldsymbol{\alpha} - \frac{1}{3}(\mathrm{tr}\boldsymbol{\alpha})\mathbf{1}$ is the deviator of $\boldsymbol{\alpha}$. The system has a useful orthogonality property

$$\mathbf{g}_i(\boldsymbol{\omega}) \cdot \mathbf{g}_k(\boldsymbol{\omega}) = 0 \quad \text{for} \quad i \neq k. \tag{2.18}$$

Other orthogonal generating systems were proposed in [7].

Now we are ready to formulate the theorem.

THEOREM 1 (REPRESENTATION): *Let* $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ *be an arbitrary, fixed generating set. The function* $\mathbf{f}: \mathscr{S} \to \mathscr{S}$ *is quasi-isotropic if and only if there exist such three functions* $\alpha_1, \alpha_2, \alpha_3: \mathscr{S} \to R$ *that for each* $\boldsymbol{\omega} \in \mathscr{S}$

$$\mathbf{f}(\boldsymbol{\omega}) = \alpha_1(\boldsymbol{\omega})\mathbf{g}_1(\boldsymbol{\omega}) + \alpha_2(\boldsymbol{\omega})\mathbf{g}_2(\boldsymbol{\omega}) + \alpha_3(\boldsymbol{\omega})\mathbf{g}_3(\boldsymbol{\omega}). \tag{2.19}$$

P r o o f. It is the immediate result of the definitions (2.11) and (2.13), *Q.E.D.*

In the particular case of Eq. (2.14) we obtain

$$\mathbf{f}(\boldsymbol{\omega}) = \alpha_1(\boldsymbol{\omega})\mathbf{1} + \alpha_2(\boldsymbol{\omega})\boldsymbol{\omega} + \alpha_3(\boldsymbol{\omega})\boldsymbol{\omega}^2. \tag{2.20}$$

The representation formula (1.1) is a particular case of Eq. (2.20).

Like in the classical case, the functions α_i (their existence being guaranteed by Theorem 1) are not, in general, uniquely determined. It should be stressed, however, that Eq. (2.19) is unique in $\mathscr{A}$,

$$\mathbf{f}|_{\mathscr{A}} \leftrightarrow (\alpha_1, \alpha_2, \alpha_3)|_{\mathscr{A}}. \tag{2.21}$$

Here $\eta|_{\mathscr{A}}$ means that the function η is considered only in $\mathscr{A}$. Significance of this statement follows from the fact that "almost all" $\mathscr{S}$ consists of $\mathscr{A}$ in the sense of Lemma 2.

The nonuniqueness mentioned above may be avoided. Let us take an arbitrary generating system $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$. Without any loss of generality it may be assumed that $\mathbf{g}_1(\boldsymbol{\omega})$, $\mathbf{g}_2(\boldsymbol{\omega})$ are a base for all $\boldsymbol{\omega} \in \mathscr{B}_2$, and $\mathbf{g}_1(\mathbf{1}) = \mathbf{1}$. Let us write

$$\mathbf{f}(\boldsymbol{\omega}) = \hat{\alpha}_1(\boldsymbol{\omega})\mathbf{g}_1(\boldsymbol{\omega}) + \hat{\alpha}_2(\boldsymbol{\omega})\mathbf{g}_2(\boldsymbol{\omega}) + \hat{\alpha}_3(\boldsymbol{\omega})\mathbf{g}_3(\boldsymbol{\omega}), \tag{2.22}$$

where

$$\begin{aligned} \hat{\alpha}_2(\boldsymbol{\omega}) &= 0 \quad \text{for} \quad \boldsymbol{\omega} \in \mathscr{B}_1 \\ \hat{\alpha}_3(\boldsymbol{\omega}) &= 0 \quad \text{for} \quad \boldsymbol{\omega} \in \mathscr{B}_1 \cup \mathscr{B}_2. \end{aligned} \tag{2.23}$$

It may easily be shown that $\hat{\alpha}_i$ are uniquely determined,

$$\mathbf{f} \to (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3). \tag{2.24}$$

On the other hand, the form (2.22) proves to be not convenient in applications.

3. Symmetry of quasi-isotropic functions

In order to avoid possible misunderstandings, let us recall that *the symmetry group of a tensor function* f is defined as

$$O_f \equiv \{\mathbf{Q} \in O | \mathbf{Q} * f(\boldsymbol{\omega}) = f(\mathbf{Q} * \boldsymbol{\omega}) \quad \text{for all} \quad \boldsymbol{\omega} \in \mathscr{S}\}, \tag{3.1}$$

while *the symmetry group of a scalar function* φ is

$$O_\varphi \equiv \{\mathbf{Q} \in O | \varphi(\mathbf{Q} * \boldsymbol{\omega}) = \varphi(\boldsymbol{\omega}) \quad \text{for all} \quad \boldsymbol{\omega} \in \mathscr{S}\}. \tag{3.2}$$

The following fact is most useful:

THEOREM 2 (ON THE SYMMETRY OF CAUSES AND EFFECTS). *Each symmetry of a "cause"* $\boldsymbol{\omega}$ *being the symmetry of the "law"* f *is also a symmetry of the "effect"* $\mathbf{f}(\boldsymbol{\omega})$

$$O_{\mathbf{f}(\boldsymbol{\omega})} \supset O_{\mathbf{f}} \cap O_{\boldsymbol{\omega}}. \tag{3.3}$$

Proof. If $\mathbf{Q} \in O_{\mathfrak{f}} \cap O_{\omega}$, then $\mathbf{Q} * \mathbf{f}(\boldsymbol{\omega}) = \mathbf{f}(\mathbf{Q} * \boldsymbol{\omega}) = f(\boldsymbol{\omega})$, *Q.E.D.* (This theorem remains true for the arbitrary "law" $f: \mathfrak{A} \to \mathfrak{B}$ where $\mathfrak{A}$ is an arbitrary set of "causes", $\mathfrak{B}$ — an arbitrary set of "effects", and the same group $\mathscr{G}$ acts in $\mathfrak{A}$ and in $\mathfrak{B}$).

For an isotropic function $O_{\mathfrak{f}} = O$, Eq. (3.3) assumes the form of Eq. (2.5), what means that each isotropic function is also quasi-isotropic, as it should be.

THEOREM 3. *For an arbitrary quasi-isotropic function*

$$O_{\mathfrak{f}} = O_{\hat{\alpha}_1} \cap O_{\hat{\alpha}_2} \cap O_{\hat{\alpha}_3} \supset O_{\alpha_1} \cap O_{\alpha_2} \cap O_{\alpha_3}. \tag{3.4}$$

Proof. If $\mathbf{Q} \in O_{\alpha_1} \cap O_{\alpha_2} \cap O_{\alpha_3}$, then

$$\mathbf{f}(\mathbf{Q} * \boldsymbol{\omega}) = \alpha_1(\mathbf{Q} * \boldsymbol{\omega}) \mathbf{g}_1(\mathbf{Q} * \boldsymbol{\omega}) + \dots \tag{3.5}$$
$$= \alpha_1(\boldsymbol{\omega}) \mathbf{Q} * \mathbf{g}_1(\boldsymbol{\omega}) + \dots = \mathbf{Q} * [\alpha_1(\boldsymbol{\omega}) \mathbf{g}_1(\boldsymbol{\omega}) + \dots] = \mathbf{Q} * \mathbf{f}(\boldsymbol{\omega}),$$

that is $\mathbf{Q} \in O_f$, and so $O_f \supset O_{\alpha_1} \cap O_{\alpha_2} \cap O_{\alpha_3}$. The same is true for $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$. If $\mathbf{Q} \in O_{\mathfrak{f}}$, then

$$\hat{\alpha}_1(\mathbf{Q} * \boldsymbol{\omega}) g_1(\mathbf{Q} * \boldsymbol{\omega}) + \dots = \mathbf{Q} * [\hat{\alpha}_1(\boldsymbol{\omega}) g_1(\boldsymbol{\omega}) + \dots], \tag{3.6}$$

whence, in view of Eq. (2.24), $\hat{\alpha}_i(\mathbf{Q} * \boldsymbol{\omega}) = \hat{\alpha}_i(\boldsymbol{\omega})$ and it follows that $\mathbf{Q} \in O_{\hat{\alpha}_i}$, $i = 1, 2, 3$, *Q.E.D.*

Let us mention, for the sake of completeness, the slightly modified classical representation theorem for isotropic functions.

THEOREM 4 (REPRESENTATION). *Let g_1, g_3, g_3, be an arbitrary, fixed generating system. Function f is isotropic if and only if there exist such three invariants $\psi_1, \psi_2, \psi_3, O_{\psi_i} = O$ that for every $\boldsymbol{\omega} \in \mathscr{S}$*

$$f(\boldsymbol{\omega}) = \psi_1(\boldsymbol{\omega}) g_1(\boldsymbol{\omega}) + \psi_2(\boldsymbol{\omega}) g_2(\boldsymbol{\omega}) + \psi_3(\boldsymbol{\omega}) g_3(\boldsymbol{\omega}). \tag{3.7}$$

Proof. Sufficiency is obvious. Necessity: if $\mathbf{f}$ is isotropic, then it is also quasi-isotropic, and the formulae (2.22) and (3.4) hold true. Since $O_f = O$, then also $O_{\hat{\alpha}_1} = O_{\hat{\alpha}_2} = O_{\hat{\alpha}_3} = O$. It follows that the invariants are, for instance, $\hat{\alpha}_i$, *Q.E.D.*

Formula (3.7) contains the classical formula (1.1) and all its possible modifications proposed in [4] or [7]. Let us observe that, in view of the nonuniqueness of α_i, Eq. (2.19) may hold true for isotropic functions $\mathbf{f}$, even for such α_i which are not invariant in $\mathscr{B}$.

4. Continuous quasi-isotropic functions

Tensorial operation $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \to \boldsymbol{\alpha} \cdot \boldsymbol{\beta} \equiv \alpha_{ij} \beta_{ij}$ is a correct scalar product in $\mathscr{S}$. It generates the norm

$$|\boldsymbol{\omega}| \equiv (\boldsymbol{\omega} \cdot \boldsymbol{\omega})^{1/2} \tag{4.1}$$

and the metric

$$\varrho(\boldsymbol{\omega}, \boldsymbol{\tau}) \equiv |\boldsymbol{\omega} - \boldsymbol{\tau}|. \tag{4.2}$$

The product $\boldsymbol{\alpha} \cdot \boldsymbol{\beta}$, and hence the norm $|\boldsymbol{\omega}|$ and the distance $\varrho(\boldsymbol{\alpha}, \boldsymbol{\beta})$ are invariant with respect to the group O.

All topological notions like continuity, differentiability etc. will be understood exclusively in the sense of the norm (4.1) (metric (4.2)). In particular, a subset $\mathscr{H} \in \mathscr{S}$ is called *open* if for each $\mathring{\boldsymbol{\omega}} \in \mathscr{H}$ there exists an *open* ball $|\boldsymbol{\omega}-\mathring{\boldsymbol{\omega}}| < \varepsilon$ contained entirely in $\mathscr{H}$. The subset $\mathscr{H} \subset \mathscr{S}$ is called dense in $\mathscr{S}$ if in each open ball $|\boldsymbol{\omega}-\mathring{\boldsymbol{\omega}}| < \varepsilon$ (for every $\mathring{\boldsymbol{\omega}} \in \mathscr{S}$, $\varepsilon > 0$) there exists at least one tensor $\boldsymbol{\omega} \in \mathscr{H}$.

LEMMA 2. The subset $\mathscr{A}$ is open and dense in $\mathscr{S}$.

Proof. *Openness*. Function $f: R^3 \to R, f(x_1, x_2, x_3) \equiv \inf\limits_{i \neq k} |x_i - x_k|$ is a function continuous with respect to the metric $(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2$. Consider an arbitrary tensor $\mathring{\boldsymbol{\omega}} \in \mathscr{A}$ with the eigenvalues $\mathring{\omega}_1 > \mathring{\omega}_2 > \mathring{\omega}_3$. In the neighbourhood of $\mathring{\boldsymbol{\omega}}$ the mapping $\boldsymbol{\omega} \to (\omega_1, \omega_2, \omega_3)$ is defined. The mapping is continuous (with respect to the metric (4.2) and the metric in R^3) since ω_i are the roots of the characteristic equation. Hence the invariant $\mu: \mathscr{S} \to R$ given by Eq. (2.6), as a composition of continuous functions, is a continuous function. Consequently, if $\mu(\mathring{\boldsymbol{\omega}}) \neq 0$, then $\mu(\boldsymbol{\omega}) > 0$ in a certain open ball. According to Eq. (2.7), this ball belongs to $\mathscr{A}$.

Density. Consider an arbitrary $\mathring{\boldsymbol{\omega}} \in \mathscr{B}$ and an arbitrary ball $|\boldsymbol{\omega}-\mathring{\boldsymbol{\omega}}| < \varepsilon$. If, for instance, $\mathring{\omega}_1 = \mathring{\omega}_2 > \omega_3 (< \omega_3)$, then the tensor $\boldsymbol{\omega}$ with the eigenvalues $\omega_1 = \mathring{\omega}_1 - \frac{\varepsilon}{2}\left(= \mathring{\omega}_1 + \frac{\varepsilon}{2}\right)$, $\omega_2 = \mathring{\omega}_2, \omega_3 = \mathring{\omega}_3$, and with eigenvectors identical with the corresponding eigen vectors $\mathring{\boldsymbol{\omega}}$, belongs to that ball. If $\mathring{\omega}_1 = \mathring{\omega}_2 = \mathring{\omega}_3 = \mathring{\omega}$, then the tensor with eigenvalues $\omega_1 = \mathring{\omega} - \frac{\varepsilon}{3}$, $\omega_2 = \mathring{\omega} + \frac{\varepsilon}{3}$, $\omega_3 = \mathring{\omega}$ belongs to the ball mentioned above, *Q.E.D.*

LEMMA 3. If a scalar continuous function $\varphi: \mathscr{S} \to R$ vanishes in $\mathscr{A}$, then it is identically equal to zero.

Proof. As a matter of fact, if for a certain $\mathring{\boldsymbol{\omega}} \in \mathscr{B}$, $\varphi(\mathring{\boldsymbol{\omega}}) \neq 0$, then, in view of continuity, $\varphi(\boldsymbol{\omega}) \neq 0$ for all $\boldsymbol{\omega}$ from within a certain ball $|\boldsymbol{\omega}-\mathring{\boldsymbol{\omega}}| < \varepsilon$. It is impossible since the set $\mathscr{A}$ is dense in $\mathscr{S}$, *Q.E.D.*

Consequently, we have arrived at the following important result connected with the continuity:

THEOREM 5. *For an arbitrary quasi-isotropic function* $\mathbf{f}$ (*in particular, for every isotropic function*) *there exists at the most one set of continuous coefficients* $\alpha_1, \alpha_2, \alpha_3$ *of the representation formula* (2.19).

Proof. If two such sets of continuous coefficients α_i, α_i' existed, we would have for every $\boldsymbol{\omega} \in \mathscr{S}$

$$[\alpha_1(\boldsymbol{\omega}) - \alpha_1'(\boldsymbol{\omega})]\mathbf{g}_1(\boldsymbol{\omega}) + \ldots = 0. \tag{4.3}$$

Since $\mathbf{g}_i(\boldsymbol{\omega})$ constitute a base in every $\boldsymbol{\omega} \in \mathscr{A}$, then

$$\psi_i(\boldsymbol{\omega}) \equiv \alpha_i(\boldsymbol{\omega}) - \alpha_i'(\boldsymbol{\omega}) = 0 \quad \text{on} \quad \mathscr{A}. \tag{4.4}$$

But ψ_i is a continuous function and, according to the previous lemma, $\psi_i(\boldsymbol{\omega}) = 0$ for all $\boldsymbol{\omega} \in \mathscr{S}$, *Q.E.D.*

If α_i and $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ are continuous, then obviously the quasi-isotropic function f is also continuous. Then

$$O_{\mathbf{f}} = O_{\alpha_1} \cap O_{\alpha_2} \cap O_{\alpha_3}. \tag{4.5}$$

Unfortunately, the continuity of α_i is not so tightly connected with the continuity of the generating system and of the function $\mathbf{f}$ itself. It was shown in [6] in connection with isotropic functions that such pathological continuous (or even differentiable!) functions $\mathbf{f}$ may be found for which it is not possible to select continuous φ_i according to the formula (1.1).

5. Potential functions

In certain applications (e.g. in elasticity) an important role is played by the *potential tensor functions.* These are the functions $\mathbf{f} : \mathscr{S} \to \mathscr{S}$ of the form

$$\mathbf{f} = \partial \Pi, \tag{5.1}$$

where $\Pi : \mathscr{S} \to R$ is a certain scalar function called the potential $\left(\text{in Cartesian notation } f_{ij} = \dfrac{\partial \Pi}{\partial \omega_{ij}}\right)$.

Potentiality imposes strong limitations upon the function. For a differentiable potential function f (twice differentiable Π), the following condition is obtained:

$$(\partial \mathbf{f})^T = \partial \mathbf{f}, \tag{5.2}$$

where the transposition symbol $(\ )^T$ denotes the permutation $(ijkl) \to (klij)$ (in the indicial notation Eq. (5.2) has the form $(\partial \mathbf{f})_{ijkl} = (\partial \mathbf{f})_{klij}$). $\partial \mathbf{f} = \partial^2 \Pi$ it follows from.

Let us demonstrate that *the potentiality practically eliminates the effect of proper quasi-isotropy* (quasi-isotropy without isotropy). Consider a proper subgroup $\mathscr{G} \subset O$, $\mathscr{G} \neq O$ and the functionally complete and functionally independent set of invariants of the group $\mathscr{G}$:

$$\begin{gathered} I_1, \ldots, I_k : \mathscr{S} \to R, \qquad k \leqslant 6, \\ I_i(\mathbf{Q} * \boldsymbol{\omega}) = I_i(\boldsymbol{\omega}) \quad \text{for all} \quad \boldsymbol{\omega} \in \mathscr{S}, \quad \mathbf{Q} \in \mathscr{G}. \end{gathered} \tag{5.3}$$

The set is assumed to be continuous and differentiable, and

$$I_1(\boldsymbol{\omega}) = \operatorname{tr} \boldsymbol{\omega}, \quad I_2(\boldsymbol{\omega}) = \frac{1}{2} \operatorname{tr} \boldsymbol{\omega}^2, \quad I_3(\boldsymbol{\omega}) = \frac{1}{3} \operatorname{tr} \boldsymbol{\omega}^3. \tag{5.4}$$

Functional independence means that for each $\boldsymbol{\omega}$ of a certain subset $\hat{\mathscr{S}} \subset \mathscr{S}$ dense in $\mathscr{S}$, the tensors

$$\begin{gathered} \partial I_1(\boldsymbol{\omega}) = 1, \quad \partial I_2(\boldsymbol{\omega}) = \boldsymbol{\omega}, \quad \partial I_3(\boldsymbol{\omega}) = \boldsymbol{\omega}^2, \\ \partial I_4(\boldsymbol{\omega}), \ldots, \partial I_k(\boldsymbol{\omega}) \end{gathered} \tag{5.5}$$

constitute a linearly independent system.

Let us consider a potential in the form

$$\Pi(\boldsymbol{\omega}) = \Phi[I_1(\boldsymbol{\omega}), \ldots, I_k(\boldsymbol{\omega})], \tag{5.6}$$

where $\Phi : R^k \to R$ is a continuous function possessing continuous derivatives $\partial \Phi / \partial I_i$, $i = 1, \ldots, k$.

THEOREM 7. *The potential function* $\mathbf{f} = \partial \Pi$ *with the potential* Eq. (5.6) *is quasi-isotropic only in the case when it is isotropic.*

P r o o f. On the one hand

$$\mathbf{f} = \partial \Pi = \frac{\partial \Phi}{\partial I_1} \partial I_1 + \ldots + \frac{\partial \Phi}{\partial I_k} \partial I_k. \tag{5.7}$$

On the other hand, if $\mathbf{f}$ is quasi-isotropic,

$$\mathbf{f} = \alpha_1 \mathbf{1} + \alpha_2 \boldsymbol{\omega} + \alpha_3 \boldsymbol{\omega}^2, \tag{5.8}$$

whence

$$\left(\alpha_1 - \frac{\partial \Phi}{\partial I_1}\right) \mathbf{1} + \left(\alpha_2 - \frac{\partial \Phi}{\partial I_2}\right) \boldsymbol{\omega} + \left(\alpha_3 - \frac{\partial \Phi}{\partial I_3}\right) \boldsymbol{\omega}^2 = \frac{\partial \Phi}{\partial I_4} \partial I_4 + \ldots + \frac{\partial \Phi}{\partial I_k} \partial I_k. \tag{5.9}$$

The set (5.5) being linearly independent in $\hat{\mathscr{S}}$, for each $\boldsymbol{\omega} \in \hat{\mathscr{S}}$ we have

$$\boldsymbol{\psi}_i(\boldsymbol{\omega}) \equiv \frac{\partial \Phi}{\partial I_i} = \mathbf{0}, \quad i = 4, \ldots, k. \tag{5.10}$$

Since ψ_i are continuous functions of $\boldsymbol{\omega}$ and $\hat{\mathscr{S}}$ is dense in $\mathscr{S}$, $\psi_i(\boldsymbol{\omega}) = 0$ for all $\boldsymbol{\omega} \in \mathscr{S}$, whence

$$\Pi(\boldsymbol{\omega}) = \Lambda(I_1, I_2, I_3). \tag{5.11}$$

In conclusion, both the potential and the function $\mathbf{f}$ must be isotropic, *Q.E.D.*

6. Examples of quasi-isotropic functions

Starting with the representation formulae (2.19) or (2.20) with arbitrary scalar functions α_i which are no invariants, it is possible to obtain an infinite number of various functions exhibiting the effect of proper quasi-isotropy, that is the functions which are quasi-isotropic without being isotropic.

In the linear case the number of possibilities is rather limited what follows from:

THEOREM 8. *Linear quasi-isotropic functions have the form*

$$\mathbf{f}(\boldsymbol{\omega}) = (\boldsymbol{\alpha} \cdot \boldsymbol{\omega} + \lambda \operatorname{tr} \boldsymbol{\omega}) \mathbf{1} + 2\mu \boldsymbol{\omega}, \tag{6.1}$$

where $\boldsymbol{\alpha}$ *is a parametric deviator*, $\operatorname{tr} \boldsymbol{\alpha} = 0$. *This function is orthotropic, that is*

$$O_f = O_\alpha. \tag{6.2}$$

It is isotropic if and only if $\boldsymbol{\alpha} = \mathbf{0}$.

P r o o f. Let us start with the representation theorem in its standard form (2.20). Linearity in $\boldsymbol{\omega}$ implies the conditions

$$\alpha_3(\boldsymbol{\omega}) \equiv 0, \quad \alpha_2(\boldsymbol{\omega}) \equiv 2\mu = \text{const}, \tag{6.3}$$

and

$$\alpha_1(\boldsymbol{\omega}) = \boldsymbol{\mu} \cdot \boldsymbol{\omega}, \tag{6.4}$$

where $\boldsymbol{\mu}$ is a certain constant tensor. Without any loss in generality it may be assumed that

$$\boldsymbol{\mu} = \boldsymbol{\alpha} + \lambda \mathbf{1}, \quad \operatorname{tr} \boldsymbol{\alpha} = 0 \tag{6.5}$$

what yields the formula (6.1).

Condition $\mathbf{Q} \in O_f; f(\mathbf{Q} * \boldsymbol{\omega}) = \mathbf{Q} * \mathbf{f}(\boldsymbol{\omega})$ for each $\boldsymbol{\omega} \in \mathscr{S}$ takes now the form

(6.6) $$\boldsymbol{\alpha} \cdot \boldsymbol{\omega} = \boldsymbol{\alpha} \cdot (\mathbf{Q} * \boldsymbol{\omega}) = (\mathbf{Q}^T * \boldsymbol{\alpha}) \cdot \boldsymbol{\omega}$$

valid for each $\boldsymbol{\omega} \in \mathscr{S}$, what means that

(6.7) $$\mathbf{Q}^T * \boldsymbol{\alpha} = \boldsymbol{\alpha},$$

whence $\mathbf{Q} \in O_\alpha$; Since $\mathrm{tr}\boldsymbol{\alpha} = 0$, the equality $O_\alpha = O$ holds true only for $\boldsymbol{\alpha} = \mathbf{0}$, *Q.E.D.*

Function (6.1) may be put in the form [1]

(6.8) $$\mathbf{f}(\boldsymbol{\omega}) = \mathbf{C} \cdot \boldsymbol{\omega},$$

where $\mathbf{C}$ is a fourth order tensor of the form

(6.9) $$\mathbf{C} \equiv \mathbf{1} \otimes \boldsymbol{\alpha} + \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I},$$

while $\mathbf{I} \cdot \boldsymbol{\omega} = \boldsymbol{\omega}$ for each $\boldsymbol{\omega} \in \mathscr{S}$. The function (6.8) is differentiable and

(6.10) $$\partial \mathbf{f} = \mathbf{C}.$$

It is not potential for $\boldsymbol{\alpha} \neq 0$ since the condition (5.2) is not fulfilled. Indeed,

(6.11) $$\mathbf{C}^T - \mathbf{C} = \boldsymbol{\alpha} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{\alpha} \neq \mathbf{0}.$$

Separation of the deviatoric and spherical components in Eq. (6.1) yields the set equivalent to Eq. (6.1)

(6.12) $$\mathbf{f}^*(\boldsymbol{\omega}) = 2\mu \boldsymbol{\omega}^*,$$

(6.13) $$\mathrm{tr}\,\mathbf{f}(\boldsymbol{\omega}) = (3\lambda + 2\mu)\mathrm{tr}\,\boldsymbol{\omega} + 3\boldsymbol{\alpha} \cdot \boldsymbol{\omega}^*.$$

The effect of proper quasi-isotropy in the linear case is seen to consist in adding to $\mathrm{tr}\,\mathbf{f}(\boldsymbol{\omega})$ the term $3\boldsymbol{\alpha} \cdot \boldsymbol{\omega}^*$ linearly dependent on the deviator $\boldsymbol{\omega}^*$.

If we assume that elastic materials are not necessarily hyperelastic [8], then Theorem 7 yields a general form of *the linearly elastic quasi-isotropic material*

(6.14) $$\boldsymbol{\sigma} = (\boldsymbol{\alpha} \cdot \boldsymbol{\epsilon} + \lambda \mathrm{tr}\,\boldsymbol{\epsilon})\mathbf{1} + 2\mu \boldsymbol{\epsilon}.$$

Here $\boldsymbol{\sigma}$ is the stress tensor and $\boldsymbol{\epsilon}$ — small deformation tensor. Equations (6.12) and (6.13) may be written in the equivalent form

(6.15) $$\boldsymbol{\sigma}^* = 2\mu \boldsymbol{\epsilon}^*,$$

(6.16) $$p = K\varepsilon + \boldsymbol{\alpha} \cdot \boldsymbol{\epsilon}^*,$$

with $p \equiv \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma}$, $\varepsilon \equiv \mathrm{tr}\,\boldsymbol{\epsilon}$, $K \equiv \lambda + \frac{2\mu}{3}$. In the spherical states $\boldsymbol{\sigma} = p\mathbf{1}$, $\boldsymbol{\epsilon} = \frac{1}{3}\varepsilon\mathbf{1}$ the material behaves like the classical Hookean material. In spite of the linearity, the following effect appears: the purely deviatoric state of stress is accompanied by the voluminal change $\varepsilon = -\frac{K}{2\mu}\ \boldsymbol{\sigma}$. It should be stressed that the material (6.14) has no elastic potential. Indeed, the stiffness tensor $\mathbf{C}$ determined by the formula (6.9) does not satisfy the symmetry condition, $\mathbf{C}^T \neq \mathbf{C}$.

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(1) In the Cartesian indicial notation $f_{ij}(\boldsymbol{\omega}) = C_{ijkl}\omega_{kl}$, $\mathrm{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$.

Replacing in Eq. (6.9) the strain tensor $\boldsymbol{\epsilon}$ with the stretching tensor $\mathbf{d}$, we obtain the constitutive equation of a material which might be called *the quasi-isotropic* Stokesian fluid

$$\boldsymbol{\sigma} = (\boldsymbol{\alpha} \cdot \mathbf{d} + \lambda \operatorname{tr} d)\mathbf{1} + 2\mu \mathbf{d}. \tag{6.17}$$

The tensor of anisotropy $\boldsymbol{\alpha}$ is defined here in a material particle in its actual configuration. Equation (6.17) seems to be more sensible than Eq. (6.14) though it requires a complementary constitutive assumption concerning $\boldsymbol{\alpha}$.

Let us consider *the hyperelastic material,*

$$\mathring{\boldsymbol{\sigma}} = \mathbf{H}(\boldsymbol{\sigma}) \cdot \mathbf{d}, \tag{6.18}$$

where $\mathring{\boldsymbol{\sigma}}$ is the Zaremba–Jaumann derivative and H — an isotropic fourth order function. The relation $\mathbf{d} \to \mathring{\boldsymbol{\sigma}}$ is quasi-isotropic when Eq. (6.18) has the following particular form:

$$\mathring{\boldsymbol{\sigma}} = [\boldsymbol{\alpha}(\boldsymbol{\sigma})\mathbf{d} + \lambda(\boldsymbol{\sigma}) \operatorname{tr} \mathbf{d}]\mathbf{1} + 2\mu(\boldsymbol{\sigma})\mathbf{d}. \tag{6.19}$$

Here λ, μ are the invariants of $\boldsymbol{\sigma}$, and $\boldsymbol{\alpha}$ is the isotropic, deviator-valued function. In compliance with Eq. (1.1) for $\operatorname{tr} \boldsymbol{\alpha} = 0$,

$$\boldsymbol{\alpha}(\boldsymbol{\sigma}) = \varphi_1(\boldsymbol{\sigma})\boldsymbol{\sigma}^* + \varphi_2(\boldsymbol{\sigma})(\boldsymbol{\sigma}^2)^*, \tag{6.20}$$

φ_1, φ_2 being the invariants of $\boldsymbol{\sigma}$.

To conclude let us give an example of a nonlinear quasi-isotropic relation. Consider *the rigid-plastic incompressible material with the flow law*

$$\boldsymbol{\sigma}^* = \lambda \mathbf{d}^*, \qquad \lambda = \frac{k}{\sqrt{\mathbf{d}^* \cdot \mathbf{P} \cdot \mathbf{d}^*}}, \tag{6.21}$$

where $\mathbf{P}$ is the fourth order tensor with the properties: $\boldsymbol{\alpha} \cdot \mathbf{P} \cdot \boldsymbol{\alpha} > 0$ for every $\boldsymbol{\alpha} \neq \mathbf{0}$, $k > 0$ is a material constant. Combining Eq. (6.21) with $\mathbf{P}$ we obtain

$$\boldsymbol{\sigma}^* \cdot \mathbf{P} \cdot \boldsymbol{\sigma}^* = k^2. \tag{6.22}$$

This is *the plasticity condition* of the material considered. The flow law (6.21) is not "associated" with the condition (6.22), and this fact makes the quasi-isotropy effect (6.21) possible. For $\mathbf{P} = \mathbf{I}$ an isotropic material is obtained; it is the rigid-plastic Levy–Mises material.

7. Generalization

The definition of a quasi-isotropic tensor function of arbitrary order and arbitrary number of tensor arguments may be based on Eq. (2.5) what has been done in [9].

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