Arch. Mech., 36, 2, pp. 163–172, Warszawa 1984

Fractional regularity of solutions in $L^{p,q}$ to the Navier–Stokes equations

G. ŁUKASZEWICZ (WARSZAWA)

WE CONSIDER the initial value problem for the Navier–Stokes equations in the infinite cylinde $S_T = R^3 \times [0, T)$ and study weak solutions of the problem belonging to the space $L^{p,q}(S_T) \equiv \equiv L^q(0, T; L^p(R^3))$. The aim of this paper is to estimate the Hausdorff dimension of the set $S = \{x \in R^3 : \operatorname{ess\,sup}_{t\in[0,T]} |u(x, t)| = \infty\}$ of possible singularities of the considered solutions.

Rozważamy zagadnienie początkowe dla równań Naviera–Stokesa w nieskończonym cylindrze $S_T = R^3 \times [0, T)$ i badamy słabe rozwiązania tego problemu należące do przestrzeni $L^{p,q}(S_T) \equiv \equiv L^q(0, T; L^p(R^3))$. Celem tej pracy jest oszacowanie wymiaru Hausdorffa zbioru $S = = \{x \in R^3 : \operatorname{ess\,sup}_{t\in[0,T]} |u(x, t)| = \infty\}$ możliwych osobliwości rozważanych rozwiązań.

Рассматривается начальная задача для уравнений Навье-Стокса для бесконечно цилиндра $S_T = R^3 \times [0, T)$ и исследуются слабые решения этой проблемы принадлежащие к пространству $L^{p,q}(S_T) \equiv L^q(0, T; L^p(R^3))$. Целью работы является оценка размерности Гаусдорфа множества $S = \{x \in R^3 : \operatorname{ess\,sup}_{t\in[0,T]} |u(x, t)| = \infty\}$ возможных особенностей рассматриваемых решений.

1. Introduction

THIS PAPER analyzes the fractional regularity of solutions of the initial value problem for the Navier–Stokes equations in the infinite cylinder $S_T = R^3 \times [0, T)$, $0 < T < \infty$. We consider the problem in its weak form (see definition 1.1 below). The initial data $g(x) = = (g_1(x), g_2(x), g_3(x))$ is taken from the space $L^r(R^3)$ of functions for which

$$||g||_{L^r(R^3)} \equiv \sum_{i=1}^{3} \Big(\int_{R^3} |g_i(x)|^r dx \Big)^{1/r} < \infty \qquad (r > 1).$$

The solutions $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ belong to the space $L^{p,q}(S_T)$ of functions for which

$$||u||_{L^{p,q}(S_T)} \equiv \sum_{i=1}^{3} \Big(\int_0^T \Big(\int_{R^3} |u_i(x, t)|^p dx \Big)^{q/p} dt \Big)^{1/q} < \infty \qquad (p, q \geqslant 2).$$

In this paper we consider solutions which have the following property: for almost every $t \in [0, T]$ each of them, say $u(x, t)$, can, when considered as a function $x \to u(x, t)$, be modified on a set of three-dimensional Lebesgue measure zero to become a continuous function on R^3.

We will assume that the modification of u has been done. This paper aims at proving the following:

THEOREM 1.1. *Suppose that*

(1.1) $$u \in L^{p,q}(S_T)$$

is a weak solution of the Navier–Stokes equations with initial data g such that $g \in L^r(R^3)$ *and* $Dg \in L^{r_1}(R^3)$ (Dg — the derivative of g) *with* $3/p+2/q > 3/r > 0$, $6/p+4/q > 3/r_1 > 0$.

If

(1.2) $$6 < p < q$$

and the equations

(1.3) $$\begin{aligned} p/(p-2)[5-A+a/q-2/q-13/p] &= 3+\varepsilon_1, \\ Ap/2+2(q-p)/q+ap/q &= 3+\varepsilon_2 \end{aligned}$$

hold for some positive A, a, ε_1, ε_2 *then the Hausdorff dimension of the set*

$$S = \Big\{x \in R^3 : \sup_{t\in[0,T]} \operatorname{ess} \Big(\sum_{i=1}^{3} u_i(x,t)^2\Big)^{1/2} = \infty\Big\}$$

does not exceed a.

This paper was inspired by the research of SCHEFFER [9], as well as of FABES, JONES and RIVIERE [3] (for other results of this nature see [1, 10, 11, 12, 13, 16]). The work [9] presents a similar result concerning the fractional regularity of Lerey solutions of the initial value problem for the Navier–Stokes equations in the infinite cylinder $R^3 \times [0, \infty)$. In this paper we consider weak solutions of the Navier–Stokes equations which are not Lerey solutions. They are, however, sufficiently smooth to satisfy an integro-differential equation of the same form as Lerey solutions do. From the very integro-differential equation, following the method used in [9], we derive a suitable estimate for the considered solutions.

Now, we precise the notion of a weak solution.

DEFINITION 1.1. *A function* $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ *is a weak solution of the Navier–Stokes equations with initial data g if the following conditions hold*:

(a) $u(x,t) \in L^{p,q}(S_T)$ *for some* p, q *with* $p, q \geqslant 2$;

(b) $g(x) \in L^r(R^3)$, $r \geqslant 1$ *with* $\operatorname{div}(g) = 0$ *in the sense of distribution*;

(c) $$\int_0^T \int_{R^3} u_i(x,t)\big(s_{i,t}(x,t) + \Delta s_i(x,t)\big)dx\,dt + \int_0^T \int_{R^3} u_j(x,t)u_i(x,t)s_{i,j}(x,t)\,dx\,dt = -\int_{R^3} g_i(x)s_i(x,0)\,dx$$

for all functions $s(x,t) = (s_1(x,t), s_2(x,t), s_3(x,t))$ *such that* $s_i(x,t)$ *belong to the space* $S(R^4)$ *of rapidly decreasing functions on* R^4, $s_i(x,t) = 0$ *for* $t \geqslant T$ *and* $\operatorname{div}(s)(\cdot, t) = 0$ *for all* t;

(d) *for almost every* $t \in [0, T]$, $\operatorname{div}(u)(\cdot, t) = 0$ *in the sense of distribution.*

Here, as in other contexts, we use the summation convention for repeated indices; differential operators are written: $u_{i,j} = (\partial/\partial x_j) u_i$; $u_{i,t} = (\partial/\partial t) u_i$; $\mathrm{div}(u) = u_{i,i}$; $\Delta u_i = u_{i,jj}$; $u_{i,jk} = (u_{i,j})_{,k}$; $\mathrm{D}u = \{u_{i,j}\}$, $1 \leqslant i,j \leqslant 3$. We denote by $|\cdot|$ the Euclidean norm. If a and b are real numbers with $a < b$, then we set $[a, b] = \{t: a \leqslant t \leqslant b\}$; $R^+ = \{t: t > 0\}$. If $x \in R^3$ and $r > 0$, then $B(x, r)$ is $\{y \in R^3: |x-y| \leqslant r\}$.

We denote by Q the fundamental solution of the heat equation running back in time, more precisely $Q: R^3 \times \{t: t < 0\} \to R^+$ is defined by $Q(x, t) = (-4\pi t)^{-3/2}\exp(|x|^2/4t)$.

Several absolute constants in this paper are denoted by the letter C without bothering to distinguish them with subscripts. If a constant depends only on a parameter N, we write it as $C\ (N)$.

In Sect. 2 we formulate an imbedding lemma (Lemma 2.2) and prove some inequalities in $L^{p,q}$ spaces being useful in further considerations. In Sect. 3 we define the notions of Hausdorff measure and dimension and use them to prove a property of functions from $L^{p,q}$ (Lemma 3.1). We show in Sect. 4 that solving the Navier–Stokes equations in a weak form is equivalent to solving a certain integro-differential equation. In Sect. 5 we use the integro-differential equation to get a basic inequality for the function u and prove Theorem 1.1.

2. $L^{p,q}$ inequalities

In this section we formulate imbedding lemmas and lemmas of a technical character which we shall use to prove the main theorem 1.1.

LEMMA 2.1. Suppose $f \in L^p(R^3)$ and $\mathrm{D}f \in L^{p/2}(R^3)$. If $p > 6$, then $f \in L^z(R^3)$, $z > p$, with

$$||f||_{L^z(R^3)} \leqslant C(||f||_{L^p(R^3)} + ||\mathrm{D}f||_{L^{p/2}(R^3)}). \tag{2.1}$$

Moreover, f can, when considered as a function, be redefined on a set of zero measure to become a continuous function on R^3.

Proof. The proof follows from interpolation inequalities [5], [6].

LEMMA 2.2. Suppose $u \in L^{p,q}(S_T)$ and $\mathrm{D}u \in L^{p/2,q/2}(S_T)$.
If $p > 6$, $q > 2$, then $u \in L^{z,q/2}(S_T)$, $z > p$, with

$$||u||_{L^{z,q/2}(S_T)} \leqslant C(||u||_{L^{p,q}(S_T)} + ||\mathrm{D}u||_{L^{p/2,q/2}(S_T)}).$$

Proof. The proof easily follows from the inequality (2.1).

LEMMA 2.3. Suppose $u \in L^{p,q}(S_T)$, $\mathrm{D}u \in L^{p/2,q/2}(S_T)$ and set

$$f(x, t) = \int_0^t \int_{R^3} |u(y, s)| \cdot |\mathrm{D}u(y, s)|(|x-y|^2 + |t-s|)^{-3/2} dy ds. \tag{2.2}$$

If $6 < p < q$, then $f \in L^{p,q}(S_T)$ with

$$||f||_{L^{p,q}(S_T)} \leqslant C(||u||_{L^{p,q}(S_T)} + ||\mathrm{D}u||_{L^{p/2,q/2}(S_T)})||\mathrm{D}u||_{L^{p/2,q/2}(S_T)}.$$

Proof. We use the following imbedding theorem, the proof of which can be found in [2, 15].

THEOREM (Imbedding). *Suppose* $g \in L^{p_1}(R^d)$ *and set*

$$Tg(x) = \int_{R^d} g(y)|x-y|^{-(d-\alpha)}dy \quad \text{where} \quad 0 < \alpha < d, \quad x \in R^d.$$

If $1 < p_1 < p < \infty$, $1/p_1 - \alpha/d = 1/p$, *then* T *is continuous from* $L^{p_1}(R^d)$ *into* $L^p(R^d)$.

We proceed to prove our lemma. For any θ, $0 < \theta < 1$

$$(|x|^2+t)^{-3/2} \leqslant C|x|^{-3\theta}t^{-3(1-\theta)/2},$$

from Eq. (2.2) we have

$$|f(x,t)| \leqslant C\int_0^t (t-s)^{-3(1-\theta)/2}\Big(\int_{R^3} |x-y|^{-3\theta}|u(y,s)|\cdot|\mathrm{D}u(y,s)|dy\Big)ds.$$

As a function of y, $|u(y,s|)\cdot|\mathrm{D}u(y,s)|$ belongs to $L^{zp/(2z+p)}(R^3)$, $z \geqslant p$, for almost every s with

$$||u(\cdot,s)\mathrm{D}u(\cdot,s)||_{L^{zp/(2z+p)}(R^3)} \leqslant ||u(\cdot,s)||_{L^z(R^3)}||\mathrm{D}u(\cdot,s)||_{L^{p/2}(R^3)}.$$

Hence, by the imbedding theorem, with $\theta = 1-(z+p)/zp$,

$$0 < (2z+p)/zp - 3(1-\theta)/3 = 1/p,$$

we have

$$(2.3) \qquad ||f(\cdot,t)||_{L^p(R^3)} \leqslant C\int_0^t (t-s)^{(-3/2)(z+p)zp}||u(\cdot,s)||_{L^z(R^3)}||\mathrm{D}u((\cdot,s)||_{L^{p/2}(R^3)}ds.$$

For $q_1 = (1/q+1-(3/2)(z+p)/zp)^{-1}$ with sufficiently large z we have, by the Hölder inequality,

$$\Big(\int_0^T (||u(\cdot,t)||_{L^z(R^3)}||\mathrm{D}u(\cdot,t)||_{L^{p/2}(R^3)})^{q_1}dt\Big)^{1/q_1} \leqslant C||u||_{L^{z,q/2}(S_T)}||\mathrm{D}u||_{L^{p/2,q/2}(S_T)}.$$

We apply the imbedding theorem to the inequality (2.3) with $0 < 1/q_1 - 1 + (3/2)\cdot(z+p)/ /zp = 1/q$ to get

$$||f||_{L^{p,q}(S_T)} \leqslant C\cdot||u||_{L^{z,q/2}(S_T)}||\mathrm{D}u||_{L^{p/2,q/2}(S_T)}.$$

Now we use lemma 2.2 to complete the proof of the lemma.

LEMMA 2.4. Suppose $u \in L^{p,q}(S_T)$, $\mathrm{D}u \in L^{p/2,q/2}(S_T)$ and set

$$f(x,t) = \int_E |u(y,s)|\cdot|\mathrm{D}u(y,s)|(|x-y|^2+|t-s|)^{-3/2}dyds,$$

where

$$E = (R^3\times[0,t]) - \big(B(x,2^{-N})\times([t-2^{-2N},t]\times R^+)\big).$$

If $6 < p < q$, N is a positive integer, then for every $(x,t) \in S_T$ we have $|f(x,t)| \leqslant C(N) < \infty$.

P r o o f. Observe that $E = ((R^3 - B(x,2^{-N}))\times[0,t]) \cup$
$\cup((R^3 - B(x,2^{-N}))\times[0,\max(0,t-2^{-2N})]) \cup (B(x,2^{-N})\times[0,\max(0,t-2^{-2N})]) \equiv$
$\equiv E_1 \cup E_2 \cup E_3$. Hence

$$|f(x,t)| \leqslant C\sum_{i=1}^{3}\int_{E_i}|u(y,s)|\cdot|\mathrm{D}u(y,s)|\cdot|x-y|^{-3\theta_i}|t-s|^{-3(1-\theta_i)/2}dyds \equiv C(I_1+I_2+I_3)$$

for any $0 < \theta_i < 1$ $(i = 1, 2, 3)$. We can choose the numbers θ_i in such a way that the integrals I_i are finite.

By the Young and Hölder inequalities we have

$$I_1 \leqslant \int_0^t\int_{R^3}|\mathrm{D}u|^{p/2}dyds+\int_0^t\int_{R^3-B(x,2^{-N})}(|u|\,|x-y|^{-3\theta_1}|t-s|^{-3(1-\theta_1)/2})^{p/(p-2)}dyds$$

$$\leqslant T^{(q-p)/q}(\|\mathrm{D}u\|_{L^{p/2,q/2}(S_T)})^{p/2}+\int_0^t\int_{R^3}|u|^pdyds$$

$$+\int_0^t\int_{R^3-B(x,2^{-N})}|x-y|^{-3\theta_1p/(p-3)}|t-s|^{-3(1-\theta_1)p/(2(p-3))}dyds < \infty$$

with θ_1 such that $3\theta_1p/(p-3) > 3$ and $3(1-\theta_1)p/(2(p-3)) < 1$.

The integrals I_2 and I_3 are estimate in the same way.

LEMMA 2.5. Let $6 < p < q$, $A \in R$, $a > 0$ and $(x,t) \in S_T$.

If

$$\int_0^T\Big(\int_{B(x,2^{-n})}|u(x,t)|^pdx\Big)^{q/p}dt < 2^{-an} \tag{2.4}$$

and

$$\int_0^T\Big(\int_{B(x,2^{-n})}|\mathrm{D}u(x,t)|^{p/2}dx\Big)^{q/p}dt < 2^{-an}, \tag{2.5}$$

then we have

$$\int_{\max(0,t-2^{-n})}^{t}\int_{B(x,2^{-n})}|u(x,t)|\cdot|\mathrm{D}u(x,t)|dxdt \tag{2.6}$$

$$\leqslant C2^{-np/(p-2)[5-A+a/q-2/q-13/p]}+C2^{-n[Ap/2+2(q-p)/q+ap/q]}.$$

Proof. We set $I = [t-2^{-2n}, t]\cap R^+$, $B = B(x, 2^{-n})$ and compute

$$\int_I\int_B|u(x,t)|\cdot|\mathrm{D}u(x,t)|dxdt \leqslant 2^{Anp/(p-2)}\int_I\int_B|u(x,t)|^{p/(p-2)}dxdt$$

$$+2^{-Anp/2}\int_I\int_B|\mathrm{D}u(x,t)|^{p/2}dxdt \equiv 2^{Anp/(p-2)}K_1+2^{-Anp/2}K_2.$$

By the Hölder inequality

$$K_1 \leqslant C2^{-3n(p-3)/(p-2)}\int_I\Big(\int_B|u(x,t)|^pdx\Big)^{1/(p-2)}dt$$

$$\leqslant C2^{-3n(p-3)/(p-2)}2^{-2n(q(p-2)-p)/(q(p-2))}\times\Big(\int_I\Big(\int_B|u(x,t)|^pdx\Big)^{q/p}dt\Big)^{p/(q(p-2))}$$

$$\leqslant C2^{-3n(p-3)/(p-2)}2^{-2n(q(p-2)-p)/(q(p-2))}2^{-nap/(q(p-2))}$$

and

$$K_2 \leqslant 2^{-2n(q-p)/q}\Big(\int_I \Big(\int_B |\mathrm{D}u(x,t)|^{p/2}dx\Big)^{q/p}dt\Big)^{p/q} \leqslant 2^{-2n(q-p)/q}2^{-anp/q}.$$

Summing up the above calculations we get the inequality (2.6).

LEMMA 2.6. Suppose $g \in L^r(R^3)$ and set

$$f(x,t) = \int_{R^3} Q(x-y, -t)g(y)dy, \quad ((x,t) \in S_T)). \tag{2.7}$$

If $3/p+2/q > 3/r > 0$, $p,q > 1$, then $f \in L^{p,q}(S_T)$ with

$$||f||_{L^{p,q}(S_T)} \leqslant CT^{1/q+3/(2p)-3/(2r)}||g||_{L^r(R^3)}.$$

Proof. Since

$$||Q(\cdot, -t)||_{L^s(R^3)} \leqslant Ct^{-3/2+3/(2s)},$$

if s is chosen so that $0 < 1/p = 1/s+1/r-1$, then by the Young inequality

$$||f(\cdot, t)||_{L^p(R^3)} \leqslant Ct^{-3/2+3/(2s)}||g||_{L^r(R^3)}.$$

If $q(1-1/s) < 2/3$,

$$||f||_{L^{p,q}(S_T)} \leqslant CT^{1/q-3/2(1-1/s)}||g||_{L^r(R^3)}.$$

Hence

$$||f||_{L^{p,q}(S_T)} \leqslant CT^{1/q+3/(2p)-3/(2r)}||g||_{L^r(R^3)}, \quad 3/p+2/q > 3/r.$$

3. Hausdorff measure and dimension

The basic facts about Hausdorff measure and dimension can be found in [4, 7]. We recall the definitions for convenience.

Let X be a metric space and $a > 0$. The a — dimensional Hausdorff measure of a subset $Y \subset X$ is

$$\mu_a(Y) = \sup_{\varepsilon>0}\mu_{a,\varepsilon}(Y) = \lim_{\varepsilon\to 0}\mu_{a,\varepsilon}(Y), \tag{3.1}$$

where

$$\mu_{a,\varepsilon}(Y) = \inf\sum_j (\operatorname{diam} B_j)^a, \tag{3.2}$$

the infimum being taken over all the coverings of Y by balls B_j such that diam B_j(= diammeter of B_j) $\leqslant \varepsilon$.

It is clear that $\mu_{a,\varepsilon}(Y) \leqslant \varepsilon^{a-a_0}\mu_{a_0,\varepsilon}(Y)$ for $a > a_0$, so if $\mu_{a_0}(Y) < \infty$ for some $0 < a_0 < \infty$, then $\mu_a(Y) = 0$ for all $a > a_0$. In this case the number

$$\inf\{a : \mu_a(Y) = 0\} = \inf\{a : \mu_a(Y) < \infty\}$$

is called the Hausdorff dimension of Y.

The set function $\mu_a(\cdot)$ is countably additive on the Borel subsets of X [4, 7].

LEMMA 3.1. For $a > 0$ and $u \in L^{p,q}(S_T)$, $q > p$, let $A_a(u)$ be the set of those $x \in R^3$ such that there exists m_x with

$$\int_0^T \Big(\int_{B(x, 2^{-m})} |u(x, t)|^p dx \Big)^{q/p} dt \leqslant 2^{-am} \quad \text{for all} \quad m \geqslant m_x.$$

Then the Hausdorff dimension of $R^3 - A_a(u)$ is $\leqslant a$.

Proof. By definition of $A_a(u)$, for any $\varepsilon > 0$ and $x \in R^3 - A_a(u)$ there exists a ball $B(x, \varepsilon')$, $\varepsilon' \leqslant \varepsilon$, such that

$$\int_0^T \Big(\int_{B(x, \varepsilon')} |u(x, t)|^p dx \Big)^{q/p} dt > 2^{-a} \big(\operatorname{diam} B(x, \varepsilon')\big)^a. \tag{3.3}$$

The family of all such balls covers $R^3 - A_a(u)$ in the sense of Vitali. By the Vitali covering theorem [8] there exists a subfamily $\{B(x_j, \varepsilon_j) : j \in J\}$ such that the $B(x_j, \varepsilon_j)$ are mutually disjoined, J is at most countable and

$$R^3 - A_a(u) \subset \bigcup_{j=1}^{\infty} B(x_j, 5\varepsilon_j). \tag{3.4}$$

By virtue of Eqs. (3.1), (3.2), (3.3), and (3.4) it follows that

$$\mu_a\big(R^3 - A_a(u)\big) = \sup_{\varepsilon > 0} \mu_{a, 5\varepsilon}\big(R^3 - A_a(u)\big)$$

and

$$\mu_{a, 5\varepsilon}\big(R^3 - A_a(u)\big) \leqslant \sum_{j=1}^{\infty} \big(\operatorname{diam} B(x_j, 5\varepsilon_j)\big)^a$$

$$= 5^a \sum_{j=1}^{\infty} \big(\operatorname{diam} B(x_j, \varepsilon_j)\big)^a < 10^a \sum_{j=1}^{\infty} \int_0^T \Big(\int_{B(x_j, \varepsilon_j)} |u(x, t)|^p dx \Big)^{q/p} dt$$

$$\leqslant 10^a \int_0^T \Big(\int_{R^3} |u(x, t)|^p dx \Big)^{q/p} dt < \infty \quad \text{independently of} \quad \varepsilon > 0$$

so the Hausdorff dimension of $R^3 - A_a(u)$ does not exceed a.

LEMMA 3.2. For $a > 0$, $u \in L^{p,q}(S_T)$, $\mathrm{D}u \in L^{p/2, q/2}(S_T)$, $6 < p < q$, let $A_a(u, \mathrm{D}u)$ be the set of those $x \in R^3$ for which there exists m_x such that for all $n \geqslant m_x$ Eqs. (2.4) and (2.5) hold. Then the Hausdorff dimension of $R^3 - A_a(u, \mathrm{D}u)$ is $\leqslant a$.

Proof. The proof immediately follows from lemma 3.1.

4. Equivalence of weak solution of the Navier–Stokes equations and solution of certain integro-differential equation

In this section we prove that if a function $g(x)$ satisfies suitable conditions, then u is a weak solution of the Navier–Stokes equations with the initial value g if and only if u is a solution of a certain integro-differential equation.

We define the function W with the domain $R^3 \times \{t: t < 0\}$ and range R^+ as follows:

$$W(x, t) = -(4\pi)^{-1} \int_{R^3} Q(y, t)|x-y|^{-1} dy.$$

We have the following:

LEMMA 4.1. Suppose $g \in L^r(R^3)$, $Dg \in L^{r_1}(R^3)$ and $\text{div}(g) = 0$ in the sense of distribution. If $6 < p < q$, $3/p+2/q > 3/r > 0$, $6/p+4/q > 3/r_1 > 0$, then $u \in L^{p,q}(S_T)$ is a weak solution of the Navier–Stokes equations with the initial value g if and only if u is a solution of the integro-differential equation

$$(4.1) \quad u_i(x, t) = \int_{R^3} g_i(y) Q(y-x, -t) dy - \int_0^t \int_{R^3} u_j(y, s) u_{i,j}(y, s) Q(y-x, s-t) dy ds$$
$$+ \int_0^t \int_{R^3} u_j(y, s) u_{k,j}(y, s) W_{,ik}(y-x, s-t) dy ds \quad (i = 1, 2, 3).$$

Proof. It is proved in [3] that if $g \in L^r(R^3)$, $1 \leqslant r < \infty$ and $\text{div}(g) = 0$ in the sense of distribution, then $u \in L^{p,q}(S_T)$, $p, q > 2$, $p < \infty$, is a weak solution of the Navier–Stokes equations with the initial value g if and only if u is a solution of the integral equation

$$(4.2) \quad u_i(x, t) = \int_{R^3} g_i(y) Q(y-x, -t) dy + \int_0^t \int_{R^3} u_j(y, s) u_i(y, s) Q_{,j}(y-x, s-t) dy ds$$
$$- \int_0^t \int_{R^3} u_j(y, s) u_k(y, s) W_{,ijk}(y-x, s-t) dy ds \quad (i = 1, 2, 3).$$

Further, it is proved in [3] that if $u \in L^{p,q}(S_T)$ with $2/q+3/p \leqslant 1$, $2 < p, q < \infty$ is a weak solution of the equation (4.2) with

$$(4.3) \quad D_x^\alpha \int_{R^3} g_i(y) Q(y-x, -t) dy \in L^{p/(|\alpha|+1), q/(|\alpha|+1)}(S_T),$$

whenever $|\alpha| \leqslant 1$, then also $D_x^\alpha u \in L^{p/(|\alpha|+1), q/(|\alpha|+1)}(S_T)$ for $|\alpha| \leqslant 1$.

By virtue of lemma 2.6, Eq. (4.3) is fulfilled if $g \in L^r(R^3)$, $Dg \in L^{r_1}(R^3)$ with $3/p+$ $+2/q > 3/r > 0$, $6/p+4/q > 3/r_1 > 0$.

To obtain the proof of the lemma observe that

$$(4.4) \quad |Q(y, s)| \leqslant C(|y|^2-s)^{-3/2}, \quad |W_{,ik}(y, s)| \leqslant C(|y|^2-s)^{-3/2}$$

so by virtue of lemma 2.3 we can integrate by parts in Eq. (4.2) to get Eq. (4.1).

5. Estimates of Hausdorff measures of the set of singularities of a solution to the Naviei –Stokes equations

In this section we prove Theorem 1.1 about the set of singularities of a solution to t Navier–Stokes equations. The primary role in our considerations is played by Eq. (4. (fulfilled by a solution of the Navier–Stokes equations, provided it exists, see remaı 5.1 below) from which we derive the basic inequality for the function u.

It follows from lemma 2.1 that if $u \in L^{p,q}(S_T)$, $\mathrm{D}u \in L^{p/2,q/2}(S_T)$ $(p > 6)$, then for almost every $t \in [0, T]$ $u(x, t)$ can, when considered as a function $x \to u(x, t)$, be modified on a set of three-dimensional Lebesgue measure zero to become a continuous function on R^3.

We assume that the modification of u has been done.

LEMMA 5.1. Suppose $u \in L^{p,q}(S_T)$, $\mathrm{D}u \in L^{p/2,q/2}(S_T)$, $6 < p < q$, and let a and A be any positive reals and N any positive integer. If u satisfies Eq. (4.1), then for all $x \in A_a(u, \mathrm{D}u)$ and almost all $t \in [0, T]$

$$(5.1) \quad |u(x,t)| \leqslant \left|\int_{R^3} g(y)Q(x-y, -t)dy\right| + C(N) + C\sum_{n=N}^{\infty} 2^{3n}2^{-np/(p-2)[5-A+a/q-2/q-13/p]} + C\sum_{n=N}^{\infty} 2^{3n}2^{-n[Ap/2+2(q-p)/q+ap/q]}.$$

Proof. Denote by E the set $R^3 \times [0, t] - (B(x, 2^{-N}) \times ([t-2^{-2N}, t] \cap R^+))$.
From Eqs. (4.1) and (4.4)

$$|u(x,t)| \leqslant \left|\int_{R_3} g(y)Q(x-y, -t)dy\right| + 2\int_E |u(y,s)| \cdot |\mathrm{D}u(y,s)|(|x-y|^2+|t-s|)^{-3/2}dyds + C\sum_{n=N}^{\infty} 2^{3n} \int_{\max(0,t-2^{-2n})} \int_{B(x,2^{-n})} |u(y,s)| \cdot |\mathrm{D}u(y,s)|dyds.$$

By virtue of lemmas 2.4, 2.5 and 3.2 we get easily the inequality (5.1). We are ready to prove Theorem 1.1.

Consider (1.1) and (1.2). By virtue of lemma 4.1 u satisfies Eq. (4.1) and $\mathrm{D}u \in L^{p/2,q/2}(S_T)$. Let $x \in A_a(u, \mathrm{D}u)$. From lemma 5.1 and Eqs. (1.3) we conclude that for any positive integer $N \geqslant m_x$ and for almost all $t \in [0, T]$

$$|u(x,t)| \leqslant \left|\int_{R^3} g(y)Q(y-x, -t)dy\right| + C(N) + C\sum_{n=N}^{\infty} 2^{-\min(\varepsilon_1,\varepsilon_2)n} < \infty.$$

The theorem follows by virtue of lemma 3.2 and the inclusion $S \subset R^3 - A_a(u, \mathrm{D}u)$.

REMARK 5.1. It is proved in [3] that if $3/p+2q \leqslant 1$ with $3 < p < \infty$, g belongs to $L^r(R^3)$ with $3/p+2/q > 3/r > 0$ and $\mathrm{div}(g) = 0$ in the sense of distribution, then the Navier–Stokes equations with initial data g have a weak solution $u \in L^{p,q}(S_T)$, at least for $0 < T < T_0$, $T_0 = T_0(p, q, r, g)$.

REMARK 5.2. The assumption (see Eq. (1.1)): $\mathrm{D}g \in L^{r_1}(R^3)$, $6/p+4/q > 3/r_1 > 0$ guarantee the regularity of $u(\mathrm{D}u \in L^{p/2,q/2}(S_T))$.

Observe that in the case of Lerey solutions we have the regularity of $u(\mathrm{D}u \in L^{2,2}(S_T))$ without such assumption. In this case we have from Eq. (2.7) (cf. Eq. (4.1)):

$$\int_0^T \int_{R^3} |f_{,k}(x,t)|^2 dxdt = \int_0^T \int_{R^3} x_k^2 \exp(-2|x|^2 t)|F(g)(x)|^2 dxdt = \int_{R^3} |F(g)(x)|^2 \left\{\int_0^T x_k^2 \exp(-2|x|^2 t)dt\right\} dx \leqslant 1/2\|g\|^2_{L^2(R^3)},$$

where F denotes the Fourier transform

$$F(g)(x) = (F(g_j)(x)) = \left(\int_{R^3} g_j(y)\exp\left(i\sum_{k=1}^{3} x_k y_k\right)dy\right).$$

Lerey solutions which belong to $L^{p,q}(S_T)$ were studied in [14].

References

1. L. CAFFARELLI, R. KOHN, L. NIRENBERG, *Partial regularity of suitable weak solutions of the Navier–Stokes equations*, Comm. in Pure and Appl. Math., **35**, 6, 771–831, November 1982.
2. A. P. CALDERON, A. ZYGMUND, *On the existence of certain singular integrals*, Acta Math., **88**, 85–139, 1952.
3. E. B. FABES, B. F. JONES, N. M. RIVIERE, *The initial value problem for the Navier–Stokes equations with data in L^p*, Arch. Rational Mech. Anal., **45**, 222–240, 1972.
4. H. FEDERER, *Geometric measure theory*, Springer–Verlag, New York 1969.
5. E. GAGLIARDO, *Ulterori proprietà di alcune classi di funzioni in più variabli*, Ricerche di Mat., **8**, 24–51, 1959.
6. L. NIRENBERG, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. di Pisa, ser. III, 13, Fasc. II, 115–162, 1959.
7. C. A. ROGERS, *Hausdorff measures*, Cambridge 1970.
8. S. SAKS, *Theory of the integral*, Warsaw 1937.
9. V. SCHEFFER, *Turbulence and Hausdorff dimension*, in: Turbulence and Navier–Stokes Equations, R. TEMAM ed., Lecture Notes in Math., vol. **565**, Springer-Verlag, 94–112, 1976.
10. V. SCHEFFER, *Partial regularity of solutions to the Navier–Stokes equations*, Pacific Journ. of Math., **66**, 535–552, 1976.
11. V. SCHEFFER, *Hausdorff measure and the Navier–Stokes equations*, Comm. in Mathematical Physics, **55**, 97–111, 1977.
12. V. SCHEFFER, *The Navier–Stokes equations in space dimension four*, Comm. in Mathematical Physics, **61**, 41–68, 1978.
13. V. SCHEFFER, *The Navier–Stokes equations on a bounded domain*, Comm. in Mathematical Physics, **73**, 1–42, 1980.
14. J. SERRIN, *The initial value problem for the Navier–Stokes equations*, Nonlinear Problems, R. E. Langer, University of Wisconsin Press, 69–83, 1963.
15. E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.
16. R. TEMAM, *Navier–Stokes equations and nonlinear functional analysis*, Université de Paris-Sund, 1982.

NSTITUTE OF MECHANICS
UNIVERSITY OF WARSAW.

Received April 6, 1983.