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Some aspects of invariant theory in plasticity
Part I. New results relative to representation of isotropic and anisotropic tensor functions

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STARTING from the general method of representation of tensor functions, new results have been obtained in some specific cases. Particularly, more general than the ordinarily used representation of the fourth-order isotropic tensor function of a second-order symmetric tensor has been derived. It has been shown that list of generators proposed by SMITH [27] for a second-order symmetric tensor function contains a redundant element. Some yield conditions for initially orthotropic and transversely isotropic materials have been discussed.

Wychodząc z ogólnej metody reprezentacji funkcji tensorowych, otrzymano nowe rezultaty dla pewnych przypadków szczególnych. I tak, podano ogólniejszą niż zwykle stosowana, reprezentację izotropowej funkcji tensorowej czwartego rzędu zależnej od symetrycznego tensora drugiego rzędu. Wykazano, że lista generatorów, zaproponowana przez SMITHA [27] dla symetrycznej funkcji tensorowej drugiego rzędu, zawiera zbędny element. Podano pewne warunki plastyczności dla materiałów pierwotnie ortotropowych i transwersalnie izotropowych.

Исходя из общего метода представлений тензорных функций, получены новые результаты для некоторых частных случаев. Итак, приведено более общее, чем применяемое обычно, представление изотропной тензорной функции четвертого порядка, зависящей от симметричного тензора второго порядка. Показано, что список генераторов, предложенный Смитом [27] для симметричной тензорной функции второго порядка, содержит лишний элемент. Приведены некоторые условия пластичности для материалов первично ортотропных и трансверсально изотропных.

1. Introduction

THE CLASSICAL theory of invariants has penetrated many fields of continuum mechanics. The essential aspects of this theory oriented towards applications in continuum mechanics are presented by SPENCER [30]. Contemporaneous exposition of the invariant theory has been dealt with in the books [13, 32], see also [34]. As the title of this paper suggests, we are primarily concerned with applications of the invariant theory to plasticity. The invariant theory is here understood in the classical sense [30]. The review papers [5, 19, 31, 34] present essential aspects of the applications of the invariant theory to a description of inelastic, particularly plastic, behaviour of metals, soils and rocks.

The origin of the developments presented in Part I is directly connected with the contribution by DAFALIAS [11], cf. also [12]. This author derived the polynomial representation of the orthotropic fourth-order tensor function of a second-order symmetric tensor. Dafalias' derivation seems to me unduly complicated because he treats this particular problem as a problem in itself, not related to the available simpler results. Therefore, in Sect. 3 of Part I of the paper I shall demonstrate that the representation obtained by

Dafalias readily follows as a consequence of the general theory of representation of tensor functions. As one knows, such theory is well established, cf. Refs. [18, 23, 26, 30, 31].

The theory of representation of tensor functions suggests that the problem of representation of isotropic and anisotropic tensor, particularly vector fucntions, can always be reduced to the examination of suitable scalar functions. Thus representations of vector and tensor functions are always available provided that bases of appropriate quantities are known. These simple facts are often overlooked in papers on representation of specific vector and tensor functions.

The plan of the first part of the paper is as follows. Instead of adducing the general method of representation of tensor functions, given in [23, 30], in Subsect. 2.1 I shall illustrate it by deriving the representations of some constant tensors. Next, the representation of the isotropic tensor function of a second-order tensor is investigated. It is shown that the list of generators proposed by Smith [27] for a symmetric tensor function contains a redundant element. The representation of the isotropic fourth-order tensor function of a symmetric tensor is reexamined. It is shown that the representation commonly used as the most general [3, 25] is not such, see also Remark 2. Consequences for existing applications are briefly discussed. Section 3 is concerned with some yield criterions for initially orthotropic materials, compressibility being included. Here an essential role is played by the representation of the orthotropic fourth-order tensor function of a symmetric second-order tensor (plastic strain). This representation is here readily obtained, as a consequence of the available representation of a general orthotropic scalar function of appropriate arguments. An inclusion of terms linear in stresses gives, in the case of initial flow, yield criterions used for oriented polymeric materials [8–10, 21, 24] and rocks [29]. In Sect. 4 some yield criterions for initially transversely isotropic materials are studied.

2. New results relative to representation of some isotropic tensor functions

Before proceeding to the presentation of new results I shall first show, in subsection 2.1, how the general method of representation of tensor functions [23, 30] operates in the specific case of some constant isotropic and hemitropic tensors.

2.1. Some constant isotropic and hemitropic tensors

Isotropy is usually related to the full orthogonal group 0(3), while hemitropy is described by the proper orthogonal group S0(3), cf. Ref. [17].

Let us first derive the representation of constant isotropic tensors: $\mathbf{c} = (c_{ij})$, $\mathbf{C} = (C_{ijkl})$, under $i \leftrightarrow j$, $k \leftrightarrow l$ symmetry requirement and with complete pairwise symmetry $(ij) \leftrightarrow (kl)$. The indices run from 1 to 3. For the purpose we take a symmetric tensor $\mathbf{a} = (a_{ij})$ and next we form scalar functions $f_1 = c_{ij}a_{ij}$, $f_2 = C_{ijkl}a_{ij}a_{kl}$, which are to be isotropic, i.e. invariant under the group 0 (3). The isotropic integrity basis, being now also the functional basis, for the tensor $\mathbf{a}$ is given by $\mathrm{tr}\,\mathbf{a} = a_{ii}$, $\mathrm{tr}\,\mathbf{a}^2$, $\mathrm{tr}\,\mathbf{a}^3$. This basis has been derived, for instance, in [30]. It is well known that in this case the most general isotropic scalar function is a function of the elements of the isotropic basis. As particular cases the funcctions f_1 and f_2 result. Taking into account previously mentioned symmetry requirements and noting that in f_1 the tensor **a** enters linearly, while in f_2 only quadratic components of **a** are present, we readily obtain

$$f_1 = c_1 \operatorname{tr} \mathbf{a} = c_1 \delta_{ij} a_{ij}, \tag{2.1}$$

$$f_2 = c_2 \operatorname{tr}^2 \mathbf{a} + c_3 \operatorname{tr} \mathbf{a}^2 = (c_2 \delta_{ij} \delta_{kl} + c_3 I_{ijkl}) a_{ij} a_{kl}, \tag{2.2}$$

where $\mathbf{I} = (\delta_{ij})$ is the Kronecker's delta; c_1, c_2, c_3 are constants and

$$I_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \tag{2.3}$$

The relations (2.1)–(2.2) imply

$$c_{ij} = c_1 \delta_{ij}, \tag{2.4}$$

$$c_{ijkl} = c_2 \delta_{ij} \delta_{kl} + c_3 I_{ijkl}. \tag{2.5}$$

Nonexistence of a nontrivial, that is different from **0**, constant isotropic tensor $\mathbf{c}^1 = (c^1_{ij})$ such that $c^1_{ij} \neq c^1_{ji}$ for $i \neq j$ can similarly be proved if an unsymmetric tensor is dealt with instead of **a**.

Nonexistence of a nontrivial constant isotropic tensor of the third order: $\mathbf{C}^1 = (C^1_{ijk})$ readily follows if the isotropic invariant $f_3 = C^1_{ijk} u_i v_j w_k$ is considered. In this case the isotropic basis for vectors **u**, **v**, **w** is involved and then $C^1_{ijk} \equiv 0$.

On the other hand the hemitropic constant tensor of the third order exists. For the purpose we consider the hemitropic invariant $f_4 = C^2_{ijk} u_i v_j w_k$. The hemitropic integrity basis for vectors **u**, **v**, **w** is given by

$$u_i u_i,\ u_i v_i,\ u_i w_i,\ v_i v_i,\ v_i w_i,\ w_i w_i,\ e_{ijk} u_i v_j w_k, \tag{2.6}$$

where (e_{ijk}) stands for the Ricci's symbol. Hence we infer that

$$f_4 = e_{ijk} u_i v_j w_k. \tag{2.7}$$

We note that f_4 is the pseudo-scalar relative to the group 0(3), but a scalar under S0(3). From Eq. (2.7) we eventually obtain

$$C^2_{ijk} = e_{ijk}. \tag{2.8}$$

Representations of constant orthotropic and transversely isotropic tensors have been derived in [35].

In similar manner representations of arbitrary constant isotropic and anisotropic tensors can effectively be derived, provided that appropriate bases are known.

2.2. Isotropic tensor function of a second-order tensor

We proceed to solving the problem of the representation of an isotropic tensor function $f_{ij} = (\hat{f}_{ij} b_{kl})$. Both the function $\hat{\mathbf{f}}$ and the argument **b** may be unsymmetric. We can split **f** into the symmetric and skew-symmetric parts

$$\mathbf{f} = \mathbf{g} + \mathbf{l}, \quad \mathbf{g} = \frac{1}{2}(\mathbf{f} + \mathbf{f}^T), \quad \mathbf{l} = \frac{1}{2}(\mathbf{f} - \mathbf{f}^T). \tag{2.9}$$

Here "T" denotes transposition. Therefore it is sufficient to consider the isotropic representations of the symmetric function $\hat{\mathbf{g}}(\mathbf{b})$ and of the skew-symmetric function $\hat{\mathbf{l}}(\mathbf{b})$.

We set

$$\mathbf{b} = \mathbf{d}+\mathbf{e}, \quad \mathbf{d} = \frac{1}{2}(\mathbf{b}+\mathbf{b}^T), \quad \mathbf{e} = \frac{1}{2}(\mathbf{b}-\mathbf{b}^T). \tag{2.10}$$

Let us find the isotropic scalar function

$$f_5 = \hat{g}_{ij}(\mathbf{b})q_{ij}, \tag{2.11}$$

where $\mathbf{q}$ is a symmetric tensor. The functional basis for tensors $\mathbf{d}$, $\mathbf{e}$, $\mathbf{q}$ consists of 21 invariants [4]

$$\begin{gathered} \operatorname{tr}\mathbf{d}, \operatorname{tr}\mathbf{d}^2, \operatorname{tr}\mathbf{d}^3, \operatorname{tr}\mathbf{q}, \operatorname{tr}\mathbf{q}^2, \operatorname{tr}\mathbf{q}^3, \operatorname{tr}\mathbf{e}^2, \operatorname{tr}\mathbf{de}^2, \\ \operatorname{tr}\mathbf{d}^2\mathbf{e}^2, \operatorname{tr}\mathbf{d}^2\mathbf{e}^2\mathbf{de}, \operatorname{tr}\mathbf{dq}, \operatorname{tr}\mathbf{d}^2\mathbf{q}, \operatorname{tr}\mathbf{dq}^2, \operatorname{tr}\mathbf{d}^2\mathbf{q}^2, \operatorname{tr}\mathbf{qe}^2, \\ \operatorname{tr}\mathbf{q}^2\mathbf{e}^2, \operatorname{tr}\mathbf{q}^2\mathbf{e}^2\mathbf{qe}, \operatorname{tr}\mathbf{qde}, \operatorname{tr}\mathbf{q}^2\mathbf{de}, \operatorname{tr}\mathbf{qd}^2\mathbf{e}, \operatorname{tr}\mathbf{qe}^2\mathbf{de}. \end{gathered} \tag{2.12}$$

The general isotropic scalar function of $\mathbf{d}$, $\mathbf{e}$, $\mathbf{q}$ is an arbitrary function of the invariants (2.12). As a specific case the function f_5 results. Since $\mathbf{q}$ enters linearly, then this function has the form

$$f_5 = \alpha_1 \operatorname{tr}\mathbf{q}+\alpha_2 \operatorname{tr}\mathbf{dq}+\alpha_3 \operatorname{tr}\mathbf{d}^2\mathbf{q}+\alpha_4 \operatorname{tr}\mathbf{qe}^2+\bar{\alpha}_5 \operatorname{tr}\mathbf{qde}+\bar{\alpha}_6 \operatorname{tr}\mathbf{qd}^2\mathbf{e}+\bar{\alpha}_7 \operatorname{tr}\mathbf{qe}^2\mathbf{de}, \tag{2.13}$$

where the coefficients $\alpha_1, \ldots, \bar{\alpha}_7$ are arbitrary scalar functions of the invariants $\operatorname{tr}\mathbf{d}$, $\operatorname{tr}\mathbf{d}^2$, $\operatorname{tr}\mathbf{d}^3$, $\operatorname{tr}\mathbf{e}^2$, $\operatorname{tr}\mathbf{de}^2$, $\operatorname{tr}\mathbf{d}^2\mathbf{e}^2$, $\operatorname{tr}\mathbf{d}^2\mathbf{e}^2\mathbf{de}$, and Eq. (2.10) has to be taken into account. From Eq. (2.13) the representation of the symmetric function $\hat{\mathbf{g}}$ readily follows:

$$\hat{\mathbf{g}}(\mathbf{b}) = \alpha_1\mathbf{I}+\alpha_2\mathbf{d}+\alpha_3\mathbf{d}^2+\alpha_4\mathbf{e}^2+\alpha_5(\mathbf{de}-\mathbf{ed})+\alpha_6(\mathbf{d}^2\mathbf{e}-\mathbf{ed}^2)+\alpha_7(\mathbf{ede}^2-\mathbf{e}^2\mathbf{de}), \tag{2.14}$$

where

$$\alpha_5 = \frac{1}{2}\bar{\alpha}_5, \quad \alpha_6 = \frac{1}{2}\bar{\alpha}_6, \quad \alpha_7 = -\frac{1}{2}\bar{\alpha}_7.$$

According to SMITH [27] the representation of the tensor function $\hat{\mathbf{g}}$ should involve eight generators. However, BOEHLER [4] has proved that the functional basis for two symmetric tensors $\mathbf{A}_1$, $\mathbf{A}_2$ and a skew-symmetric tensor $\mathbf{W}$ contains one redundant invariant, namely $\operatorname{tr}\mathbf{A}_1\mathbf{A}_2\mathbf{W}^2$ (in Boehler's and Smith's notations). Hence our procedure of representation of tensor functions immediately implies that also the set of generators listed by SMITH [27] contains redundant terms. In the formula (4.5) of [27] redundant is the generator $\mathbf{WAW}$. The statement follows if the generalized Cayley– Hamilton theorem is used [5, 30].

It is worthwhile noting that the problem of a determination of minimal functional bases in more involved cases is still open. For instance, in the case of two second-order tensors, one of which is symmetric and the other is skew-symmetric, there seems to be no convincing proof what is the minimal isotropic functional basis.

If $\mathbf{e} = \mathbf{0}$, then from Eq. (2.14) we obtain the well-known representation of the second-order symmetric tensor function of a symmetric tensor.

For $\mathbf{d} = \mathbf{0}$ the relation (2.14) furnishes the representation of a symmetric function $\hat{\mathbf{g}}_1$ of a skew-symmetric tensor

$$\hat{\mathbf{g}}_1(\mathbf{e}) = \alpha_1^0 \mathbf{I} + \alpha_4^0 \mathbf{e}^2, \tag{2.15}$$

where the scalar functions α_1^0, α_4^0 depend on $\mathrm{tr}\,\mathbf{e}^2$.

The set of generators for the isotropic skew-symmetric tensor function $\hat{\mathbf{l}}(\mathbf{b})$ obtained in the above manner is the same as the set derived by SMITH [27].

We note that the representations considered in Subsect. 2.2, and/or their generalizations, can be useful in micropolar theories of elasticity and inelasticity.

2.3. Comments on isotropic fourth-order tensor function of a second-order symmetric tensor

We shall derive the representation of the isotropic tensor function

$$N_{ijkl} = \hat{N}_{ijkl}(\boldsymbol{\epsilon}) \tag{2.16}$$

of a second-order symmetric tensor $\boldsymbol{\epsilon}$. The usual symmetry requirement is imposed, that is $N_{ijkl} = N_{jikl} = N_{klij}$. It is important to derive the correct representation of the isotropic polynomial function $\hat{\mathbf{N}}$, because when studying the papers [3, 19, 25] I have noticed that in this respect a confusion is current. We observe that in the paper by RIVLIN and ERICKSEN [26] the representation of the fourth-order tensor function is not investigated.

Let us take a symmetric tensor $\mathbf{a} = (a_{ij})$ and consider the isotropic function of $\boldsymbol{\epsilon}$ and $\mathbf{a}$

$$f_6 = \hat{N}_{ijkl}(\boldsymbol{\epsilon}) a_{ij} a_{kl}. \tag{2.17}$$

The isotropic integrity basis, being also the functional basis, is given by ten invariants which are listed in the set (2.12), with obvious changes of notations and $\mathbf{e} = \mathbf{0}$. In the scalar function f_6 the tensor $\mathbf{a}$ appears solely through quadratic components. Therefore we have

$$\begin{aligned} f_6 = {} & \alpha_1 \mathrm{tr}^2\mathbf{a} + \bar{\alpha}_2 \mathrm{tr}\,\mathbf{a}^2 + \bar{\alpha}_3 \mathrm{tr}\,\mathbf{a}\,\mathrm{tr}\,\mathbf{a}\boldsymbol{\epsilon} + \bar{\alpha}_4 \mathrm{tr}\,\mathbf{a}^2\boldsymbol{\epsilon} + \alpha_5 \mathrm{tr}^2\mathbf{a}\boldsymbol{\epsilon} + \bar{\alpha}_6 \mathrm{tr}\,\mathbf{a}^2\boldsymbol{\epsilon}^2 \\ & + \bar{\alpha}_7 \mathrm{tr}\,\mathbf{a}\,\mathrm{tr}\,\mathbf{a}\boldsymbol{\epsilon}^2 + \alpha_8 \mathrm{tr}\,\mathbf{a}\boldsymbol{\epsilon}\,\mathrm{tr}\,\mathbf{a}\boldsymbol{\epsilon}^2 + \alpha_9 \mathrm{tr}^2\mathbf{a}\boldsymbol{\epsilon}^2, \end{aligned} \tag{2.18}$$

where, in the case of the polynomial representation, the coefficients $\alpha_1, \bar{\alpha}_2, \ldots, \alpha_9$ are polynomials in $\mathrm{tr}\,\boldsymbol{\epsilon}$, $\mathrm{tr}\,\boldsymbol{\epsilon}^2$, $\mathrm{tr}\,\boldsymbol{\epsilon}^3$. Taking account of the symmetry requirements imposed on $\mathbf{N}$, from Eq. (2.18) we readily obtain f_6 in the form (2.17) where

$$\begin{aligned} N_{ijkl} = \hat{N}_{ijkl}(\boldsymbol{\epsilon}) = {} & \alpha_1 \delta_{ij}\delta_{kl} + \alpha_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha_3(\delta_{ij}\varepsilon_{kl} + \delta_{kl}\varepsilon_{ij}) \\ & + \alpha_4(\delta_{ik}\varepsilon_{jl} + \delta_{il}\varepsilon_{jk} + \delta_{jk}\varepsilon_{il} + \delta_{jl}\varepsilon_{ik}) + \alpha_5 \varepsilon_{ij}\varepsilon_{kl} + \alpha_6(\delta_{ik}\varrho_{jl} + \delta_{il}\varrho_{jk}\delta_{jk} + \varrho_{il} + \delta_{jl}\varrho_{ik}) \\ & + \alpha_7(\delta_{ij}\varrho_{kl} + \delta_{kl}\varrho_{ij}) + \alpha_8(\varepsilon_{ij}\varrho_{kl} + \varepsilon_{kl}\varrho_{ij}) + \alpha_9 \varrho_{ij}\varrho_{kl}. \end{aligned} \tag{2.19}$$

Here

$$\alpha_2 = \bar{\alpha}_2/2, \quad \alpha_3 = \bar{\alpha}_3/2, \quad \alpha_4 = \bar{\alpha}_4/4, \quad \alpha_6 = \bar{\alpha}_6/4, \quad \alpha_7 = \bar{\alpha}_7/2, \quad \varrho_{ij} = \varepsilon_{ik}\varepsilon_{kj}.$$

In the papers [3, 19, 25] the authors affirm that the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are simply constants. From our considerations it results that all scalar coefficients appearing in Eq. (2.19), including $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are scalar functions of basic invariants of $\boldsymbol{\epsilon}$.

One more remark concerning the paper [3]. In the formula (2.7) of [3] the symmetry relation $N_{ijkl} = N_{klij}$ should imply $\nu = \pi$, $B = C$ and $G = H$.

Further comments on the representation (2.19) furnishes Remark 2, given after Sect. 4 of the present paper.

From the representation formula (2.19) we conclude that it is not justified to affirm that the tensor $\mathbf{N}$ can be written in the form, cf. Refs. [3, 25]

$$N_{ijkl} = I_{ijkl} + A_{ijkl}, \tag{2.20}$$

where

$$I_{ijkl} = \alpha_1 \delta_{ij} \delta_{kl} + \alpha_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{2.21}$$

is the constant isotropic tensor while solely A_{ijkl} collects terms depending on $\boldsymbol{\epsilon}$. We observe that only for a fixed $\boldsymbol{\epsilon}$ the tensor I_{ijkl} is the constant isotropic tensor. If $\boldsymbol{\epsilon}$ is not fixed, then I_{ijkl} depends on $\boldsymbol{\epsilon}$ by means of α_1 and α_2. One obviously can assume that α_1 and α_2 are constants, but this is an additional assumption not implied by the representation itself.

In the case of plastic incompressible flow considered by BALTOV and SAWCZUK [3], $\boldsymbol{\epsilon}$ stands for the plastic strain. For $\boldsymbol{\epsilon} = \mathbf{0}$ we have $\alpha_2 = \frac{1}{2}$. However, during plastic deformation this coefficient changes because for incompressible flow it generally depends on $\operatorname{tr} \boldsymbol{\epsilon}^2$ and $\operatorname{tr} \boldsymbol{\epsilon}^3$.

We note that applications of isotropic even-order tensors in nonlinear elasticity are discussed by OGDEN [20]. Our comments regarding the tensor function $\mathbf{N}(\boldsymbol{\epsilon})$ are also relevant to isotropic tensors $\mathscr{L}$, in Ogden's notation.

3. Yield conditions for initially orthotropic materials

In this section we first consider the general form of a yield condition for a material which can be regarded as orthotropic in a preferred reference configuration, cf. Refs. [11, 12]. We assume that the corresponding Cartesian coordinate system coincides with the system of the principal axes of orthotropy. We consider the scalar function

$$f(\boldsymbol{\sigma} - \boldsymbol{\alpha}, \boldsymbol{\epsilon}, w) = 0 \tag{3.1}$$

invariant under the coordinate transformations associated with orthotropic symmetry. Here $\boldsymbol{\sigma} = (\sigma_{ij})$ is the stress tensor whereas $\boldsymbol{\epsilon} = (\varepsilon_{ij})$ is the plastic part of the strain tensor; $i, j = 1, 2, 3$. The tensor $\boldsymbol{\alpha} = (\alpha_{ij})$ and the scalar w are parameters (internal variables) describing, respectively, the translation and the expansion or contraction of the yield surface. A choice of the scalar w depends on a particular class of materials. For metals it is usually assumed that w is represented by the plastic work [11, 12, 14]. On the other hand, for granular materials w can be the plastic volumetric strain (cf. [14, 15]).

The group of orthotropic symmetries, here denoted by Q, is a finite, or discrete group defined as follows [5, 30]:

$$Q = \{[Q_{ij}] | Q_{ij} = \pm 1 \text{ for } i = j \text{ and } Q_{ij} = 0 \text{ for } i \neq j\}. \tag{3.2}$$

In terms of crystallographic classes orthotropy corresponds to the rhombic-dipyramidal class of the rhombic system.

We put

$$\mathbf{t} = \boldsymbol{\sigma} - \boldsymbol{\alpha}. \tag{3.3}$$

Thus we have

$$f(\mathbf{t}, \boldsymbol{\epsilon}, w) = 0. \tag{3.4}$$

The following special case of Eq. (3.4) is of interest in applications:

$$F(\mathbf{t}, \boldsymbol{\epsilon}) - k(w) = 0. \tag{3.5}$$

In the sequel both the polynomial and nonpolynomial representations of the scalar function (3.5) are studied.

3.1. General polynomial representation of Eq. (3.5) and some specific cases

In the case of orthotropy the integrity basis for two symmetric second-order tensors $\mathbf{t}, \boldsymbol{\epsilon}$ consists of 23 basic invariants [1, 7]:

$$\begin{aligned} &I_1 = t_{11}, \quad I_2 = t_{22}, \quad I_3 = t_{33}, \quad I_4 = t_{23}^2, \quad I_5 = t_{31}^2, \quad I_6 = t_{12}^2, \\ &I_7 = t_{23} t_{31} t_{12}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &J_1 = \varepsilon_{11}, \quad J_2 = \varepsilon_{22}, \quad J_3 = \varepsilon_{33}, \quad J_4 = \varepsilon_{23}^2, \quad J_5 = \varepsilon_{31}^2, \quad J_6 = \varepsilon_{12}^2, \\ &J_7 = \varepsilon_{23} \varepsilon_{31} \varepsilon_{12}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &K_1 = t_{23} \varepsilon_{23}, \quad K_2 = t_{31} \varepsilon_{31}, \quad K_3 = t_{12} \varepsilon_{12}, \quad K_4 = t_{23} t_{31} \varepsilon_{12}, \\ &K_5 = t_{31} t_{12} \varepsilon_{23}, \quad K_6 = t_{12} t_{23} \varepsilon_{31}, \quad K_7 = t_{12} \varepsilon_{23} \varepsilon_{31}, \\ &K_8 = t_{23} \varepsilon_{31} \varepsilon_{12}, \quad K_9 = t_{31} \varepsilon_{12} \varepsilon_{23}. \end{aligned} \tag{3.8}$$

Thus the most general orthotropic scalar function (3.5) has the form

$$F(I_1, I_2, \ldots, K_9) - k(w) = 0, \tag{3.9}$$

where F is a polynomial function of the indicated arguments.

The scalar function (3.9) is too general to be applicable. Therefore we shall investigate several particular cases of the function (3.9). Let us suppose that the function F is independent of I_7 and is a polynomial comprehending solely linear and quadratic terms in stress components, with coefficients being polynomial functions of the invariants $J_1, \ldots, J_7$. Denoting this function by F_1 we have

$$F_1 - k(w) = F_2 + F_3 - k(w), \tag{3.10}$$

where

$$F_2 = a_1 I_1 + a_2 I_2 + a_3 I_3 + a_4 K_1 + a_5 K_2 + a_6 K_3 + a_7 K_7 + a_8 K_8 + a_9 K_9, \tag{3.11}$$

$$F_3 = b_1 I_1^2 + b_2 I_2^2 + \ldots + b_{51} K_8 K_9. \tag{3.12}$$

The exact form of the function F_3 is given in the paper [35]. The orthotropic scalar functions $a_1, \ldots, a_8$ and $b_1, \ldots, b_{51}$ are arbitrary polynomials in $J_1, \ldots, J_7$. Taking account of Eqs. (3.6) and (3.8) we obtain the final form of the scalar orthotropic function F_2, linear in components of the tensor $\mathbf{t}$

$$F_2 = \hat{h}_{ij}(\boldsymbol{\epsilon}) t_{ij}, \tag{3.13}$$

where

$$h_{11} = a_1, \quad h_{22} = a_2, \quad h_{33} = a_3,$$

$$\text{(3.14)} \quad h_{12} = h_{21} = \frac{1}{2}(a_6\varepsilon_{12}+a_7\varepsilon_{13}\varepsilon_{23}), \quad h_{13} = h_{31} = \frac{1}{2}(a_5\varepsilon_{13}+a_9\varepsilon_{12}\varepsilon_{23}),$$

$$h_{23} = h_{32} = \frac{1}{2}(a_4\varepsilon_{23}+a_8\varepsilon_{12}\varepsilon_{13}), \quad h_{ij} = \hat{h}_{ij}(\boldsymbol{\epsilon}).$$

DAFALIAS [11, 12] discusses the yield criterion when $\hat{\mathbf{h}} \equiv \mathbf{0}$. The inclusion of the terms linear in stresses seems to us important, as we shall see in the sequel.

The function F_3 has the form

$$\text{(3.15)} \qquad F_3 = \frac{1}{2}\hat{H}_{ijkl}(\boldsymbol{\epsilon})t_{ij}t_{kl}, \quad i,j,k,l = 1,2,3.$$

The components of the orthotropic fourth-order tensor function $\hat{\mathbf{H}}(\boldsymbol{\epsilon}) = (\hat{H}_{ijkl}(\boldsymbol{\epsilon}))$, such that $H_{ijkl} = H_{jikl} = H_{klij}$ are given in [35], see also [11, 34].

In Dafalias' approach the scalar function F_3 is known if first the representation of the fourth-order tensor function $\hat{\mathbf{H}}(\boldsymbol{\epsilon})$ is derived. Our approach leads to this representation quite naturally and easily. Further, our approach seems to be more advantageous in applications of the function (3.15) in yield conditions. Why? The function F_3 will usually be too general due to the presence of 21 functions $\hat{H}_{ijkl}$. Therefore further simplifications are needed. The form (3.12) of the function F_3 seems to be more appropriate for possible simplifications than the equivalent form given by the function (3.15). The same conclusion regards both initially isotropic and anisotropic materials.

The results as yet obtained indicate that a consistent approach to representation of tensor functions furnishes not merely representations of unknown functions but can also lead to improvements in the existing representations.

An alternative method to the representation of second-order anisotropic tensor functions has been proposed by BOEHLER [5, 6], see also [17].

3.2. Special cases of Eq. (3.10)

It is interesting to study the initial yield condition resulting from Eq. (3.10). In this case we have $\boldsymbol{\epsilon} = \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{t} = \boldsymbol{\sigma}$, $k(w) = k_0$, where k_0 is a material constant. We put

$$\text{(3.16)} \qquad \mathbf{h}^0 = \hat{\mathbf{h}}^0(\mathbf{0}), \quad \mathbf{H}^0 = \hat{\mathbf{H}}(\mathbf{0}).$$

From the form (3.14) and the explicit form of H_{ijkl} we readily infer that merely h_{11}^0, h_{22}^0, h_{33}^0 and nine constants H_{ijkl}^0 do not disappear, cf. [11, 34, 35]. Thus from the relation (3.10) we obtain

$$\text{(3.17)} \quad h_{11}^0\sigma_{11}+h_{22}^0\sigma_{22}+h_{33}^0\sigma_{33}+\frac{1}{2}(H_{1111}^0\sigma_{11}^2+H_{2222}^0\sigma_{22}^2+H_{3333}^0\sigma_{33}^0$$
$$+2H_{1122}^0\sigma_{11}\sigma_{22}+2H_{1133}^0\sigma_{11}\sigma_{33}+2H_{2233}^0\sigma_{22}\sigma_{33}+4H_{1212}^0\sigma_{12}^2$$
$$+4H_{1313}^0\sigma_{13}^2+4H_{2323}^0\sigma_{23}^2)-k_0 = 0.$$

If we set

$$(3.18)\qquad \begin{aligned} &\frac{1}{2k_0}H^0_{1111} = G+H+X^2, \quad \frac{1}{2k_0}H^0_{2222} = F+H+Y^2,\\ &\frac{1}{2k_0}H^0_{3333} = F+G+Z^2, \quad \frac{1}{k_0}H^0_{1122} = 2XY-2H,\\ &\frac{1}{k_0}H^0_{1133} = 2XY-2G, \quad \frac{1}{k_0}H^0_{2233} = 2YZ-2F,\\ &\frac{2}{k_0}H^0_{2323} = L, \quad \frac{2}{k_0}H^0_{1313} = M, \quad \frac{2}{k_0}H^0_{1212} = N, \end{aligned}$$

then the relation (3.17) gives

$$(3.19)\qquad F(\sigma_{22}-\sigma_{33})^2+G(\sigma_{33}-\sigma_{11})^2+H(\sigma_{11}-\sigma_{22})^2+L\sigma_{23}^2+M\sigma_{13}^2 \\ +N\sigma_{12}^2+(X\sigma_{11}+Y\sigma_{22}+Z\sigma_{33})^2+\frac{1}{k_0}(h^0_{11}\sigma_{11}+h^0_{22}\sigma_{22}+h^0_{33}\sigma_{33})-1 = 0.$$

The criterion (3.19) has been proposed for orthotropic compacting materials like porous limestone by SMITH and CHEATHAM [29]. The criterion (3.10) can thus be regarded as a direct generalization of the initial yield condition (3.19).

A deeper insight into the criterion (3.10) is gained if the deviatoric and normal components of the tensor **t** are used. We have

$$(3.20)\qquad t_{ij} = s_{ij}+\frac{1}{3}t_{kk}\delta_{ij},$$

where

$$(3.21)\qquad s_{ij} = \sigma^D_{ij}-\beta_{ij}, \quad \sigma^D_{ij} = \sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}, \quad \beta_{ij} = \alpha_{ij}-\frac{1}{3}\alpha_{kk}\delta_{ij}.$$

Therefore we eventually obtain

$$(3.22)\qquad F_1-k(w) = \frac{1}{2}H_{ijkl}s_{ij}s_{kl}-\frac{1}{18}H_{iikk}t^2_{mm}+\frac{1}{3}H_{ijkk}t_{ij}t_{mm}+h_{ij}t_{ij}-k(w) = 0.$$

The first term in Eq. (3.22) is the quadratic form in **s**. Hence only fifteen independent components of **H** enter into this form, cf. [11]. For $\boldsymbol{\epsilon} = \mathbf{0}$ the quadratic form reduces to the first six terms appearing in Eq. (3.19); compare with the Hill's yield condition [16]. The second and the third terms entering the condition (3.22) correspond to the term $(X\sigma_{11}+Y\sigma_{22}+Z\sigma_{33})^2$ which appears in Eq. (3.19). Comparing the initial yield condition (3.19) with the criterion (3.22) and taking into account the dependence of $\hat{\mathbf{h}}$ and $\hat{\mathbf{H}}$ on $\boldsymbol{\epsilon}$ we infer that strong interrelation exists between normal and shear behaviour during yielding. For instance, in the condition (3.19) merely three constants connected with normal stresses are present. Yielding activates also the shear stresses entering linearly, as the presence in Eq. (3.22) of the terms $h_{12}t_{12}$, $h_{13}t_{13}$ and $h_{23}t_{23}$ indicates.

Let us return to the condition (3.19) once again. We set $X = Y = Z = 0$ and $K^0_1 = h^0_{11}/k_0$, $K^0_2 = h^0_{22}/k_0$, $K^0_3 = h^0_{33}/k_0$; hence we obtain the following criterion

$$(3.23)\qquad F(\sigma_{22}-\sigma_{33})^2+G(\sigma_{33}-\sigma_{11})^2+H(\sigma_{11}-\sigma_{22})^2+L\sigma_{23}^2+M\sigma_{13}^2+N\sigma_{12}^2 +K_1^0\sigma_{11}+K_2^0\sigma_{22}+K_3^0\sigma_{33} = 1.$$

The condition (3.23) has been proposed by STASSI-D'ALIA [33] and rediscovered by CADDELL, RAGHAVA and ATKINS [9]. Afterwards it has been extensively used to describe the macroscopic, pressure dependent initial yield behaviour of oriented polymeric materials like polycarbonate, polyethylene and polypropylene, see Refs. [8, 10, 21, 24, 37]. In this case the material constants $F, G, \ldots, K_3^0$ are defined as follows:

$$(3.24)\qquad \begin{gathered} H+G=\frac{1}{T_x|C_x|},\quad F+H=\frac{1}{T_y|C_y|},\quad G+F=\frac{1}{T_z|C_z|},\\ K_1^0=\frac{|C_x|-T_x}{|C_x|T_x},\quad K_2^0=\frac{|C_y|-T_y}{|C_y|T_y},\quad K_3^0=\frac{|C_z|-T_z}{|C_z|T_z}, \end{gathered}$$

where T_x, T_y, T_z denote the tensile yield stresses at atmospheric pressure and room temperature in the 1, 2 and 3 directions, respectively. C_x, C_y and C_z are the corresponding compressive yield stresses.

Setting $H_{ijkk}=0$ in Eq. (3.22) one obtains

$$(3.25)\qquad \frac{1}{2}H_{ijkl}s_{ij}s_{kl}+h_{ij}t_{ij}-k(w)=0.$$

Only fifteen coefficients H_{ijkl} enter now the criterion (3.25) since $H_{ijkk}=0$. Therefore the condition (3.25) offers a direct generalization of the initial condition (3.23). As a first approximation one can assume $\hat{h}_{ij}(\boldsymbol{\epsilon})=0$ for $i\neq j$.

3.3. Briefly on nonpolynomial representation of $\hat{\mathbf{H}}$

We shall briefly comment on the nonpolynomial representation of the fourth-order tensor function $\hat{\mathbf{H}}(\boldsymbol{\epsilon})$. It means that we must find the orthotropic form-invariant tensor function

$$(3.26)\qquad H_{ijkl}=\hat{H}_{ijkl}(\boldsymbol{\epsilon}),\qquad H_{ijkl}=H_{jikl}=H_{klij}$$

and the representation of $\hat{\mathbf{H}}$ is not necessarily polynomial.

Our approach used previously permits to obtain the nonpolynomial representation of $\hat{\mathbf{H}}(\boldsymbol{\epsilon})$ in the same manner as the polynomial representation. The problem reduces to finding the orthotropic, nonpolynomial scalar function

$$(3.27)\qquad f_7=\frac{1}{2}\hat{H}_{ijkl}(\varepsilon)t_{ij}t_{kl}.$$

The functional basis for $\mathbf{t}$, $\boldsymbol{\epsilon}$ is given by [1]

$$I_1,\ldots,I_7,J_1,\ldots,J_7,K_1,K_2,K_3$$

and

$$\begin{aligned} K_4&=t_{12}t_{23}\varepsilon_{13}+t_{23}t_{13}\varepsilon_{12}+t_{13}t_{12}\varepsilon_{23},\\ K_5&=t_{13}\varepsilon_{12}\varepsilon_{23}+t_{12}\varepsilon_{23}\varepsilon_{13}+t_{23}\varepsilon_{13}\varepsilon_{12}. \end{aligned}$$

The nonpolynomial orthotropic scalar function (3.27) is obtained similarly as the polynomial function F_3. The components H_{ijkl} of the tensor $\mathbf{H}$ are now formally the same as previously, yet the scalar coefficients are scalar functions in the invariants $J_1, \ldots, J_7$, not necessarily polynomial. The components h_{ij} of the tensor $\hat{\mathbf{h}}(\boldsymbol{\epsilon})$ may now also be nonpolynomial functions.

Of interest seems to be the following criterion:

$$\left(\frac{1}{2} H_{ijkl} s_{ij} s_{kl}\right)^{1/n} + h_{ij} t_{ij} - k(w) = 0, \tag{3.28}$$

where $n > 1$ is a natural number. Since the deviator $\mathbf{s}$ appears in the first term, therefore necessarily $H_{iijk} = 0$, see Sect. 3.2. We observe that for $n > 1$ the criterion (3.28) represents always a nonpolynomial function even if polynomial representations of $\hat{\mathbf{h}}$ and $\hat{\mathbf{H}}$ are considered.

For the initial flow, Eq. (3.28) reduces to

$$[F(\sigma_{22}-\sigma_{33})^2 + G(\sigma_{33}-\sigma_{11})^2 + H(\sigma_{11}-\sigma_{22})^2 + L\sigma_{23}^2 + M\sigma_{13}^2 + N\sigma_{12}^2]^{1/n} + \frac{1}{k_0}(h_{11}^0 \sigma_{11} + h_{22}^0 \sigma_{22} + h_{33}^0 \sigma_{33}) - 1. \tag{3.29}$$

Setting $X = Y = Z = 0$ and taking k_0^n instead of k_0 in the relations (3.18) we obtain the relations between the constants $H_{ijkl}^0 = \hat{H}_{ijkl}(\mathbf{0})$ and the constants $F, \ldots, N$. The initial yield condition (3.29) has been proposed for anisotropic rocks and soils by PARISEAU [22]. Hence the criterion (3.28) represents a direct extension of this condition.

REMARK 1

The present remark is concerned with a representation of orthotropic functions in the form given by I-SHIH LIU [17], p. 1104.

Let f be either a scalar-valued, vector-valued, or tensor-valued orthotropic function (here we use notations of Ref. [17]). Further, let $\mathbf{v}$ denote a set of vectors, while $\mathbf{A}$ stands for a set of second-order tensors. I-Shih Liu claims that an orthotropic function $f(\mathbf{v}, \mathbf{A})$ can be represented by

$$f(\mathbf{v}, \mathbf{A}) = \tilde{f}(\mathbf{v}, \mathbf{A}, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2), \tag{3.30}$$

where $\tilde{f}$ is an isotropic function. However, I think that the problem is more subtle. Dropping the term $\mathbf{n}_3 \otimes \mathbf{n}_3$ we infer that in the case of orthotropy all basic invariants, listed for instance in [5, 6] and containing $\mathbf{M}_3 = \mathbf{n}_3 \otimes \mathbf{n}_3$, are redundant. Such conclusion is false. The representation of an orthotropic function can be obtained from

$$f(\mathbf{v}, \mathbf{A}) = \tilde{f}(\mathbf{v}, \mathbf{A}, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2, \mathbf{n}_3 \otimes \mathbf{n}_3) \tag{3.31}$$

and next we may use the identity $\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_2 + \mathbf{n}_3 \otimes \mathbf{n}_3 = \mathbf{I}$. In this manner an equivalent set of invariants and/or generators is obtained. In peculiar cases, but not in general, the number of basic invariants and/or generators can thus be reduced.

4. Some yield criterions for initially transversely isotropic materials

This section is concerned with yield conditions of the form (3.1) for transversely isotropic materials. The notion of transverse isotropy used in this paper has been defined in the Appendix.

In this case the transverse integrity basis for tensors $\mathbf{t}$, $\boldsymbol{\epsilon}$ is given by [1, 2, 30]

$$(4.1)\quad \begin{aligned} &i_1 = t_{\alpha\alpha}, \quad i_2 = t_{\alpha\beta}t_{\beta\alpha}, \quad i_3 = t_{33}, \quad i_4 = t_{3\alpha}t_{\alpha3}, \quad i_5 = t_{3\alpha}t_{\alpha\beta}t_{\alpha3},\\ &j_1 = \varepsilon_{\alpha\alpha}, \quad j_2 = \varepsilon_{\alpha\beta}\varepsilon_{\beta\alpha}, \quad j_3 = \varepsilon_{33}, \quad j_4 = \varepsilon_{3\alpha}\varepsilon_{\alpha3}, \quad j_5 = \varepsilon_{3\alpha}\varepsilon_{\alpha\beta}\varepsilon_{\beta3},\\ &k_1 = t_{\alpha\beta}\varepsilon_{\beta\alpha}, \quad k_2 = t_{3\alpha}\varepsilon_{\alpha3}, \quad k_3 = t_{3\alpha}t_{\alpha\beta}\varepsilon_{\beta3}, \quad k_4 = t_{3\alpha}\varepsilon_{\alpha\beta}\varepsilon_{\beta3},\\ &k_5 = t_{3\alpha}\varepsilon_{\alpha\beta}t_{\beta3}, \quad k_6 = \varepsilon_{3\alpha}t_{\alpha\beta}\varepsilon_{\beta3}, \quad k_7 = t_{3\alpha}t_{\alpha\beta}\varepsilon_{\beta\gamma}\varepsilon_{\gamma3}.\end{aligned}$$

Here Greek indices take values 1, 2. Like in the case of orthotropy, we want to derive the general form of the following yield criterion:

$$(4.2)\qquad F_4 = \frac{1}{2}\hat{M}_{ijkl}(\boldsymbol{\epsilon})t_{ij}t_{kl}+\hat{m}_{ij}(\boldsymbol{\epsilon})t_{ij}-k(w) = 0,$$

where $M_{ijkl} = M_{jikl} = M_{klij}$. The scalar function F_4 must be invariant under the group T_2. Considering the scalar function linear in $\mathbf{t}$

$$(4.3)\qquad F_5 = a_1 i_1+a_2 i_3+a_3 k_1+a_4 k_2+a_5 k_4+a_6 k_6,$$

we arrive at

$$(4.4)\qquad F_5 = \hat{m}_{ij}(\boldsymbol{\epsilon})t_{ij},$$

where

$$(4.5)\quad \begin{aligned} &m_{\alpha\beta} = a_1\delta_{\alpha\beta}+a_3\varepsilon_{\alpha\beta}+a_6\varepsilon_{\alpha3}\varepsilon_{3\beta},\\ &m_{\alpha3} = m_{3\alpha} = a_4\varepsilon_{\alpha3}+a_5\varepsilon_{\alpha\beta}\varepsilon_{\beta3}, \quad m_{33} = a_2; \quad m_{ij} = \hat{m}_{ij}(\boldsymbol{\epsilon}).\end{aligned}$$

The scalar coefficients $a_1, \ldots, a_6$ are polynomials in $j_1, \ldots, j_5$.

The polynomial representation of the function $\hat{\mathbf{M}}$ is given in the Appendix. Representations of tensor functions, form-invariant under the remaining transverse isotropy groups can be derived by a similar procedure. For this purpose the paper by Smith is indispensable. We observe that in the case of the transverse isotropy group T_2 new formulas given by SMITH [28] also result in 17 basic invariants for two symmetric second-order tensors. In general, when vectors, symmetric and skew-symmetric tensors are involved, the integrity basis derived in [28] contains fewer basic invariants than the basis listed in [2]. Further, it is interesting to note that according to SMITH [28] integrity bases of an arbitrary number of symmetric second order tensors are the same in the case of the transverse isotropy groups T_4, T_2 and T_6.

We pass to the initial flow. In this case we have $\boldsymbol{\epsilon} = \boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{t} = \boldsymbol{\sigma}$, while the only nonvanishing components of transversely isotropic tensors (m_{ij}), (M_{ijkl}) are

$$(4.6)\quad \begin{aligned} &m^0_{11} = m^0_{22}, \quad m^0_{33}, \quad M^0_{1111} = M^0_{2222}, \quad M^0_{3333}, \quad M^0_{1212},\\ &M^0_{1313} = M^0_{2323}, \quad M^0_{1122}, \quad M^0_{1133} = M^0_{2233}, \quad M^0_{1111} = 2M^0_{1212}+M^0_{1122},\end{aligned}$$

where $m^0_{ij} = \hat{m}_{ij}(\mathbf{0})$, $M^0_{ijkl} = \hat{M}_{ijkl}(\mathbf{0})$. The yield criterion (4.2) reduces to

$$(4.7)\qquad \frac{1}{2}\sigma^D_{ij}\sigma^D_{ij} = M_1+M_2\sigma_{33}+M_3\sigma^2_{33}+M_4\sigma_{3\alpha}\sigma_{\alpha3}+(M_5+M_6\sigma_{33})\sigma_{ii}+M_7(\sigma_{ii})^2,$$

where

$$(4.8)\qquad M_1 = \frac{k_0}{2M^0_{1212}}, \quad M_2 = \frac{m^0_{11}-m^0_{33}}{2M^0_{1212}},$$

(4.8) [cont.]
$$M_3 = \frac{M^0_{1212}+M^0_{1133}}{2M^0_{1212}} - \frac{M^0_{1122}+M^0_{3333}}{4M^0_{1212}}, \quad M_4 = 1-\frac{M^0_{1313}}{M^0_{1212}},$$
$$M_5 = -\frac{m^0_{11}}{2M^0_{1212}}, \quad M_6 = \frac{M^0_{1122}-M^0_{1133}}{2M^0_{1212}}, \quad M_7 = -\frac{1}{6}-\frac{M^0_{1122}}{4M^0_{1212}}.$$

The yield condition (4.7) has been used in [29].

Now we shall briefly discuss the nonpolynomial representation of the fourth-order tensor function $\hat{\mathbf{M}}_{ijkl}(\boldsymbol{\epsilon})$. The functional basis for tensors $\boldsymbol{\epsilon}$, $\mathbf{t}$ is obtained from the integrity basis (4.1) if the basic invariants k_3, k_4 are dropped, cf. [5, 7]. The nonpolynomial representation of $\hat{M}_{ijkl}(\boldsymbol{\epsilon})$ has the form (A.4)–(A.9), except that now the terms with the coefficients $\theta_{11}, \theta_{13}, \theta_{17}, \theta_{18}$ and θ_{23} disappear while the remaining θ are scalar functions in the invariants $j_1, \ldots, j_5$, not necessarily polynomial.

REMARK 2

Hitherto we have been deriving primarily second- and fourth-order isotropic and anisotropic tensor functions from appropriate scalar functions. Another approach is also possible. For instance, the second-order tensor function

$$f_{ij} = \hat{f}_{ij}(\mathscr{B}) \tag{4.9}$$

results from the vector function

$$v_i = f_{ij}u_j = \hat{f}_{ij}(\mathscr{B})u_j, \tag{4.10}$$

while the fourth-order tensor function

$$F_{ijkl} = \hat{F}_{ijkl}(\mathscr{B}), \tag{4.11}$$

can be derived from the second-order tensor function

$$F_{ij} = \hat{F}_{ijkl}(\mathscr{B})b_{kl}. \tag{4.12}$$

Here $\mathscr{B}$ stands for a set of arguments of the tensor function under consideration. Thus the following scheme can sometimes be useful when dealing with representations of tensor functions, see [35]:

scalar functions → vector functions → tensor functions.

It can readily be verified that the representation of the isotropic, fourth-order tensor function of a symmetric second-order tensor, given in [3] by the formula (2.7), is formally similar to the representation resulting from Eq. (4.12) if $\mathbf{b} = \mathbf{b}^T$ and if only i ↔ j and k ↔ l symmetry is required. However, now also *all* scalar coefficients depend on $\mathrm{tr}\boldsymbol{\epsilon}$, $\mathrm{tr}\boldsymbol{\epsilon}^2$ and $\mathrm{tr}\boldsymbol{\epsilon}^3$. If we additionally specify (ij) ↔ (kl) symmetry, then we arrive at the representation given by the formula (2.19) of our paper.

Though we have exclusively dealt with tensor functions of a single argument, an extension to more involved cases is straightforward, see [35].

Appendix

Fourth-order transversely isotropic tensor function of a symmetric tensor

From among the five groups which define the symmetry properties of materials which are referred to as being transversely isotropic, we consider only the group T_2, cf. [28, 30]. This group is generated by the following matrices:

(A.1) $$T_2 : \mathbf{Q}(\theta), \quad \mathbf{R}_1 = \mathrm{diag}(-1, 1, 1),$$

where

(A.2) $$\mathbf{Q}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 0 \leqslant \theta < 2\pi,$$

$\mathbf{R}_1$ corresponds to a reflection in the plane perpendicular to the x_1-axis.

Let us consider a transversely isotropic scalar function quadratic in components of **t**.

(A.3) $$F = \varrho_1 i_1^2 + \varrho_2 i_2 + \varrho_3 i_3^2 + \varrho_4 i_4 + \varrho_5 k_1^2 + \varrho_6 k_2^2 + \varrho_7 k_3 + \varrho_8 k_4^2 + \varrho_9 k_5 + \varrho_{10} k_6^2 + \varrho_{11} k_7 + \varrho_{12} i_1 i_3 + \varrho_{13} i_1 k_1 + \varrho_{14} i_1 k_2 + \varrho_{15} i_1 k_4 + \varrho_{16} i_1 k_6 + \varrho_{17} i_3 k_1 + \varrho_{18} i_3 k_2 + \varrho_{19} i_3 k_4 + \varrho_{20} i_3 k_6 + \varrho_{21} k_1 k_2 + \varrho_{22} k_1 k_4 + \varrho_{23} k_1 k_6 + \varrho_{24} k_2 k_4 + \varrho_{25} k_2 k_6 + \varrho_{26} k_4 k_6,$$

where $\varrho_1, \ldots, \varrho_{26}$ are polynomials in $j_1, \ldots, j_5$. Taking account of the relations (4.1), from Eq. (A.3) we eventually obtain

(A.4) $$M_{\beta\alpha\gamma\mu} = \theta_1 \delta_{\alpha\beta} \delta_{\gamma\mu} + \theta_2 (\delta_{\alpha\gamma} \delta_{\beta\mu} + \delta_{\alpha\mu} \delta_{\beta\gamma}) + \theta_3 \varepsilon_{\alpha\beta} \varepsilon_{\gamma\mu} + \theta_4 \varepsilon_{\alpha 3} \varepsilon_{\beta 3} \varepsilon_{\gamma 3} \varepsilon_{\mu 3} + \theta_5 (\delta_{\alpha\beta} \varepsilon_{\gamma\mu} + \delta_{\gamma\mu} \varepsilon_{\alpha\beta}) + \theta_6 (\delta_{\alpha\beta} \varepsilon_{\gamma 3} \varepsilon_{\mu 3} + \delta_{\gamma\mu} \varepsilon_{\alpha 3} \varepsilon_{\beta 3}) + \theta_7 (\varepsilon_{\alpha\beta} \varepsilon_{\gamma 3} \varepsilon_{\mu 3} + \varepsilon_{\gamma\mu} \varepsilon_{\alpha 3} \varepsilon_{\beta 3}),$$

(A.5) $$M_{\alpha\beta\gamma 3} = M_{\alpha\beta 3\gamma} = M_{\gamma 3\alpha\beta} = M_{3\gamma\alpha\beta} = \theta_8 (\delta_{\alpha\beta} \varepsilon_{\gamma 3} + \delta_{\alpha\gamma} \varepsilon_{\beta 3} + \delta_{\beta\gamma} \varepsilon_{\alpha 3}) + \theta_9 (\delta_{\alpha\beta} \varkappa_{\gamma 3} + \delta_{\alpha\gamma} \varkappa_{\beta 3} + \delta_{\beta\gamma} \varkappa_{\alpha 3}) + \theta_{10} (\varepsilon_{\alpha\beta} \varepsilon_{\gamma 3} + \varepsilon_{\alpha\gamma} \varepsilon_{\beta 3} + \varepsilon_{\beta\gamma} \varepsilon_{\alpha 3}) + \theta_{11} (\varepsilon_{\alpha\beta} \varkappa_{\gamma 3} + \varepsilon_{\alpha\gamma} \varkappa_{\beta 3} + \varepsilon_{\beta\gamma} \varkappa_{\alpha 3}) + \theta_{12} \varepsilon_{\alpha 3} \varepsilon_{\beta 3} \varepsilon_{\gamma 3} + \theta_{13} (\varepsilon_{\alpha 3} \varepsilon_{\beta 3} \varkappa_{\gamma 3} + \varepsilon_{\alpha 3} \varepsilon_{\gamma 3} \varkappa_{\beta 3} + \varepsilon_{\beta 3} \varepsilon_{\gamma 3} \varkappa_{\alpha 3}),$$

where $\varkappa_{\alpha 3} = \varepsilon_{\alpha\beta} \varepsilon_{\beta 3}$;

(A.6) $$M_{\alpha 3\beta 3} = M_{3\alpha\beta 3} = M_{\beta 3\alpha 3} = M_{3\beta\alpha 3} = \theta_{14} \delta_{\alpha\beta} + \theta_{15} \varepsilon_{\alpha\beta} + \theta_{16} \varepsilon_{\alpha 3} \varepsilon_{\beta 3} + \theta_{17} \varkappa_{\alpha 3} \varkappa_{\beta 3} + \theta_{18} (\varepsilon_{\alpha 3} \varkappa_{\beta 3} + \varepsilon_{\beta 3} \varkappa_{\alpha 3},$$

(A.7) $$M_{\alpha\beta 33} = M_{\beta\alpha 33} = M_{33\alpha\beta} = M_{33\beta\alpha} = \theta_{19} \delta_{\alpha\beta} + \theta_{20} \varepsilon_{\alpha\beta} + \theta_{21} \varepsilon_{\alpha 3} \varepsilon_{\beta 3},$$

(A.8) $$M_{\alpha 333} = M_{3\alpha 33} = M_{33\alpha 3} = M_{333\alpha} = \theta_{22} \varepsilon_{\alpha 3} + \theta_{23} \varkappa_{\alpha 3},$$

(A.9) $$M_{3333} = \theta_{24}.$$

The scalar coefficients $\theta_1, \ldots, \theta_{24}$ are polynomials in the invariants $j_1, \ldots, j_5$ and are related to $\varrho_1, \ldots, \varrho_{26}$.

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