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NOTE ON BEZOUT'S METHOD OF ELIMINATION.

[From the *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. II. (1864), pp. 88, 89.]

LET U, U' be any two rational and integral functions of x of the same order; to fix the ideas let them be the cubic functions

$$U = ax^3 + bx^2 + cx + d,$$
$$U' = a'x^3 + b'x^2 + c'x + d'.$$

Write

$$A = \begin{vmatrix} U, & U' \\ a, & a' \end{vmatrix}, \quad P = \begin{vmatrix} U, & U' \\ a, & a' \end{vmatrix},$$

$$B = \begin{vmatrix} U, & U' \\ b, & b' \end{vmatrix}, \quad Q = \begin{vmatrix} U & , U' \\ ax+b, & a'x+b' \end{vmatrix},$$

$$C = \begin{vmatrix} U, & U' \\ c, & c' \end{vmatrix}, \quad R = \begin{vmatrix} U & , U' \\ ax^2+bx+c, & a'x^2+b'x+c' \end{vmatrix},$$

$$D = \begin{vmatrix} U, & U' \\ d, & d' \end{vmatrix}, \quad S = \begin{vmatrix} U & , U' \\ ax^3+bx^2+cx+d, & a'x^3+b'x^2+c'x+d' \end{vmatrix}, = \begin{vmatrix} U, & U' \\ U, & U' \end{vmatrix}, = 0,$$

then we have

$$P = A,$$
$$Q = Ax + B,$$
$$R = Ax^2 + Bx + C,$$
$$S = Ax^3 + Bx^2 + Cx + D, = 0,$$

and thence

$$\begin{aligned} A &= P, \\ B &= Q - Px, \\ C &= R - Qx, \\ D &= S - Rx, = - Rx. \end{aligned}$$

Let α be an arbitrary quantity and write

$$\Box z = \begin{vmatrix} U & , U' \\ a\alpha^3 + b\alpha^2 + c\alpha + d', & a'\alpha^3 + b'\alpha^2 + c'\alpha + d' \end{vmatrix};$$

we have it is clear

$$\begin{aligned} \Box &= A\alpha^3 + B\alpha^2 + C\alpha + D, \\ &= \alpha^3 P + \alpha^2 (Q - Px) + \alpha (R - Qx), = Rx, \\ &= (\alpha^3 - \alpha^2 x) P + (\alpha^2 - \alpha x) Q + (\alpha - x) R, \end{aligned}$$

and thence

$$\frac{\Box}{\alpha - x} = \alpha^2 P + \alpha Q + R.$$

The equations $P = 0$, $Q = 0$, $R = 0$ are respectively quadratic equations in x, the equations which are used in Bezout's method of elimination; and representing them by

$$\begin{aligned} P &= Lx^2 \ + Mx \ + N \ , = 0, \\ Q &= L'x^2 + M'x + N' , = 0, \\ R &= L''x^2 + M''x + N'', = 0, \end{aligned}$$

we have

$$\begin{vmatrix} L, & M, & N \\ L', & M', & N' \\ L'', & M'', & N'' \end{vmatrix} = 0$$

as the equation resulting from the elimination of x from the equations $U = 0$, $U' = 0$. The foregoing investigation shows that the functions P, Q, R are obtained as the coefficients of α^2, α, 1 in the development of

$$\frac{1}{\alpha - x} \begin{vmatrix} U & , U' \\ a\alpha^3 + b\alpha^2 + c\alpha + d, & a'\alpha^3 + b'\alpha^2 + c'\alpha + d' \end{vmatrix};$$

or more generally, taking U, U' to be any two functions of the order n, that the n functions P, Q, R, &c. each of the order $n-1$ are obtained as the coefficients of α^{n-1}, α^{n-2}, ... α, 1 in the development of

$$\frac{1}{\alpha - x} \begin{vmatrix} U, & U' \\ U_\alpha, & U_\alpha' \end{vmatrix},$$

where U_α, U_α' are what U, U' become when x is replaced therein by α: and we have thus a simple *à posteriori* verification of the form in which, several years ago, I presented Bezout's Method of Elimination.

2, *Stone Buildings, W.C., March* 5, 1863.