369.

ON A PROPERTY OF COMMUTANTS.

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I CALL to mind the definition of a commutant, viz. if in the symbol

$$\left[\begin{array}{ccc} \dagger & & \\ 1 & 1 & 1 \;(\theta) \\ 2 & 2 & 2 \\ \vdots & \vdots & \vdots \\ p & p & p \end{array}\right]$$

we permute independently in every possible manner the numbers 1, 2, ... p of each of the θ columns except the column marked (†), giving to each permutation its proper sign, + or −, according as the number of inversions is even or odd, thus

$$\pm_s \pm_t \ldots A_{\begin{array}{l} 1\, s_1\, t_1 \ldots (\theta) \\ 2\, s_2\, t_2 \\ \vdots \;\; \vdots \;\; \vdots \\ p\, s_p\, t_p \end{array}}$$

which is to be read as meaning

$$\pm_s \pm_t \ldots A_{1\, s_1\, t_1 \ldots} \quad A_{2\, s_2\, t_2 \ldots} \quad \ldots A_{p\, s_p\, t_p \ldots},$$

the sum of all the $(1 \,.\, 2 \,.\, 3 \ldots p)^{\theta-1}$ terms so obtained is the commutant denoted by the above-mentioned symbol. In the particular case $\theta = 2$, the commutant is of course a determinant: in this case, and generally if θ be even, it is immaterial which of the columns is left unpermuted, so that the (†) instead of being placed over any column may be placed on the left hand of the A; but when θ is odd, the function has different values according as one or another column is left unpermuted, and the position of the (†) is therefore material. It may be added that if *all* the columns are permuted, then, if θ be even, the sum is $1 \,.\, 2 \ldots p$ into the commutant obtained by leaving any one column unpermuted; but if θ is odd, then the sum is $= 0$.

The property in question is a generalization of a property of determinants, viz. we have

$$\left|\begin{array}{lll} 2\lambda\lambda' , & \lambda\mu' + \lambda'\mu, & \lambda\nu' + \lambda'\nu, \,.. \\ \lambda\mu' + \lambda'\mu, & 2\mu\mu' , & \mu\nu' + \mu'\nu, \,.. \\ \lambda\nu' + \lambda'\nu, & \mu\nu' + \mu'\nu, & 2\nu\nu' , \\ \vdots & & \end{array}\right| = 0$$

whenever the order of the determinant is greater than 2.

To enunciate the corresponding property of commutants, let

$$\left\{\begin{array}{ll} \lambda_{11}, & \lambda_{12} \,.. \\ \lambda_{21}, & \lambda_{22} \\ \vdots & \end{array}\right\}$$

or, in a notation analogous to that of a commutant,

$$\left[\begin{array}{l} {}^{\dagger}\lambda \; + \\ \;\;1\;1 \\ \;\;2\;2 \\ \;\;\vdots\;\;\vdots \\ \;\;p\;p \end{array}\right]$$

denote a function formed precisely in the manner of a determinant (or commutant of two columns), except that the several terms (instead of being taken with a sign + or − as above) are taken with the sign +: thus

$$\left\{\begin{array}{ll} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{array}\right\} \text{ or } \left[\begin{array}{l} {}^{\dagger}\lambda \; + \\ \;\;1\;1 \\ \;\;2\;2 \end{array}\right]$$

each denote

$$\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21}.$$

This being so, the theorem is that the commutant

$$\left[\begin{array}{l} A \\ \;\;1\;1\;1 \,..\, (\theta) \\ \;\;2\;2\;2 \\ \;\;\vdots\;\;\vdots\;\;\vdots \\ \;\;p\;p\;p \end{array}\right]$$

where

$$A_{rst\,..\,(\theta)} = \left\{\begin{array}{lll} \lambda_{1r}, & \lambda_{1s} \,..\, (\theta) & \\ \lambda_{2r}, & \lambda_{2s} & . \\ \vdots & & \\ \lambda_{\theta r}, & \lambda_{\theta s} & . \end{array}\right\} = \left[\begin{array}{l} {}^{\dagger}\lambda \; + \\ \;\;r\;1 \\ \;\;s\;2 \\ \;\;t\;3 \\ \;\;\vdots\;\;\vdots \\ \;\;.\;\;p \end{array}\right]$$

whenever $p > \theta$, is $= 0$.

To prove this, consider the general term of the commutant, viz. this is

$$\pm_s \pm_t \cdots A_{1s't'..}\, A_{2s''t''..} \cdots A_{ps^pt^p..}$$

the general term of $A_{rst..}$ is $\lambda_{ar}\lambda_{bs}\lambda_{ct}\,..$, where $a, b, c, ..$ represent some permutation of the numbers 1, 2, 3 . . θ. Substituting the like values for each of the factors $A_{1s't'..}$, $A_{2s''t''..}$, &c., the general term of the commutant is

$$= \pm_s \pm_t \ldots \lambda_{a'1}\lambda_{b's'}\lambda_{c't'}\,..\,\lambda_{a''2}\lambda_{b''s''}\lambda_{c''t''}\,..\,\lambda_{a^p p}\lambda_{b^p s^p}\lambda_{c^p t^p}\ldots$$

Taking the sum of this term with respect to the quantities $s', s'', .. s^p$, which denote any possible permutation of the numbers 1, 2 ... p; again, with respect to the quantities $t', t'', .. t^p$, which denote any possible permutation of the numbers 1, 2, ... p; and the like for each of the $(\theta-1)$ series of quantities, the sum in question is

$$\lambda_{a'1}\lambda_{a''2}\ldots\lambda_{a^p p}\,\Sigma \pm_s \lambda_{b's'}\lambda_{b''s''}\,..\,\lambda_{b^p s^p}\,\Sigma \pm_t \lambda_{c't'}\lambda_{c''t''}\ldots\lambda_{c^p t^p}\,..,$$

which is

$$= \lambda_{a'1}\lambda_{a''2}\ldots\lambda_{a^p p}\left[\begin{array}{l}\lambda^{\dagger} \\ \;b'\ 1 \\ \;b''\ 2 \\ \;\vdots\ \ \vdots \\ \;b^p\ p\end{array}\right]\left[\begin{array}{l}\lambda^{\dagger} \\ \;c'\ 1 \\ \;c''\ 2 \\ \;\vdots\ \ \vdots \\ \;c^p\ p\end{array}\right]\ldots;$$

but p being greater than θ, since the numbers $b', b'', \ldots b^p$ are all of them taken out of the series 1, 2 ... θ, some of these numbers must necessarily be equal to each other, and we have therefore

$$\left[\begin{array}{l}\lambda^{\dagger} \\ \;b'\ 1 \\ \;b''\ 2 \\ \;\vdots \\ \;b^p\ p\end{array}\right] = 0;$$

whence finally the commutant is $=0$.

In the case where $p=\theta=2$, we have for a determinant of the order 2 the theorem

$$\left|\begin{array}{cc} 2\lambda\lambda', & \lambda\mu'+\lambda'\mu \\ \lambda\mu'+\lambda'\mu, & 2\mu\mu' \end{array}\right| = -\left|\begin{array}{cc}\lambda, & \mu \\ \lambda', & \mu'\end{array}\right|^2;$$

and it is probable that there exists a corresponding theorem for the commutant

$$\left[\begin{array}{l} A^{\dagger} \\ \;1\ 1\ 1 .. (p) \\ \;2\ 2\ 2 \\ \;\vdots\ \ \vdots \\ \;p\ p\ p \end{array}\right],$$

where

$$A_{rst\,..\,(p)} = \left(\begin{array}{lll}\lambda_{1r}, & \lambda_{1s} \ldots (p) \\ \lambda_{2r}, & \lambda_{2s} & . \\ \cdot & & \\ \lambda_{pr}, & \lambda_{ps} & . \end{array}\right) = \left[\begin{array}{l} {}^{\dagger}\lambda^{+} \\ \;r\ 1 \\ \;s\ 2 \\ \;t\ 3 \\ \;\vdots \\ \;.\ p \end{array}\right],$$

but I have not ascertained what this theorem is.

Cambridge, October 26, 1865.