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ON A TRIANGLE IN-AND-CIRCUMSCRIBED TO A QUARTIC CURVE.

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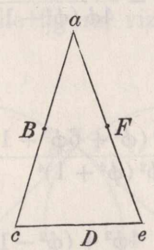
THE quartic curve $(x^2-a^2)^2+(y^2-b^2)^2=c^4$ presents a simple example of a triangle in-and-circumscribed to a single curve, viz. such that each angle of the triangle is situate on, and each side touches, the curve. Assuming that the triangle is symmetrically situate in regard to the axis of y, viz. if it be the isosceles triangle ace, the sides whereof touch the curve in the points B, D, F respectively, then we must have a single relation between the constants a, b, c of the curve; or if (as may



be done without loss of generality) we write $a=1$, then there must be a single relation between b and c. The relation in question is most conveniently expressed by putting b and c equal to certain functions of a parameter ϕ, which is in fact $=\tan^2\theta$, if θ be the angle at the base of the triangle; the equation of the curve is thus obtained in the form

$$(x^2-1)^2+\left(y^2-\frac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}\right)^2=1+\frac{(\phi^2-1)^4}{16\phi^2(\phi^2+1)^2};$$

and the coordinates of a, c, e, B, D, F are as follows:

$$
\begin{array}{llll}
\text{Coordinates of } a & \text{are} & 0, & \sqrt{\tfrac{1}{2}\phi}, \\
\quad ,, \quad c,\, e, & ,, & \pm\sqrt{2}, & -\sqrt{\tfrac{1}{2}\phi}, \\
\quad ,, \quad D, & ,, & 0, & -\sqrt{\tfrac{1}{2}\phi}, \\
\quad ,, \quad B,\, F & ,, & \pm\dfrac{\phi^2-1}{\sqrt{2}\,(\phi^2+1)}, & \dfrac{2\sqrt{\phi}}{\sqrt{2}\,(\phi^2+1)}.
\end{array}
$$

It is easy to verify that the points a, c, e, D are points of the curve, and it is obvious that the tangent at D is the horizontal line ce. It only remains to be shown that B and F are points of the curve, and that the tangents at these points are the lines ac and ea respectively. It is sufficient to consider one of the two points, say the point F; and taking its coordinates to be

$$\xi=\frac{\phi^2-1}{\sqrt{2}\,(\phi^2+1)},\quad \eta=\frac{2\sqrt{\phi}}{\sqrt{2}\,(\phi^2+1)},$$

we have to show that (ξ, η) is a point of the curve, and that the equation of the tangent at this point is $X\sqrt{\phi}+Y=\sqrt{\tfrac{1}{2}\phi}$, where (X, Y) are current coordinates.

First, to show that (ξ, η) is a point of the curve, the equation to be verified may be written

$$(\xi^2-1)^2+\left(\eta^2-\frac{\phi^4+4\phi^2-1}{4\phi\,(\phi^2+1)}\right)^2=\frac{(\phi^4+6\phi^2+1)^2}{16\phi^2\,(\phi^2+1)^2},$$

and we have

$$\xi^2-1=-\frac{\phi^4+6\phi^2+1}{2\,(\phi^2+1)^2},\qquad \eta^2-\frac{\phi^4+4\phi^2-1}{4\phi\,(\phi^2+1)}=-\frac{(\phi^2-1)(\phi^4+6\phi^2+1)}{4\phi\,(\phi^2+1)^2},$$

so that the equation becomes

$$\frac{(\phi^4+6\phi^2+1)^2}{4\,(\phi^2+1)^4}+\frac{(\phi^2-1)^2\,(\phi^4+6\phi^2+1)^2}{16\phi^2\,(\phi^2+1)^4}=\frac{(\phi^4+6\phi^2+1)^2}{16\phi^2\,(\phi^2+1)^2};$$

that is

$$4\phi^2+(\phi^2-1)^2=(\phi^2+1)^2,$$

which is right.

Next, the equation of the tangent at the point (ξ, η) is

$$(\xi^2-a^2)(\xi X-a^2)+(\eta^2-b^2)(\eta Y-b^2)-c^4=0;$$

that is

$$(\xi^2-1)(\xi X-1)+\left(\eta^2-\frac{\phi^4+4\phi^2-1}{4\phi\,(\phi^2+1)}\right)\left(\eta Y-\frac{\phi^4+4\phi^2-1}{4\phi\,(\phi^2+1)}\right)=\frac{(\phi^4+6\phi^2+1)^2}{16\phi^2\,(\phi^2+1)^2};$$

or, substituting for ξ, η, ξ^2-1, and $\eta^2-\dfrac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}$, their values, and throwing out a factor $\dfrac{\phi^4+6\phi^2+1}{16\phi^2(\phi^2+1)^2}$, the equation becomes

$$-8\phi^2\left(X\frac{\phi^2-1}{\sqrt{2}(\phi^2+1)}-1\right)-4\phi(\phi^2-1)\left(Y\frac{2\sqrt{\phi}}{\sqrt{2}(\phi^2+1)}-\frac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}\right)=\phi^4+6\phi^2+1,$$

or, what is the same thing,

$$-8\phi^2\{X(\phi^2-1)-\sqrt{2}(\phi^2+1)\}-(\phi^2-1)\{Y.8\phi\sqrt{\phi}-\sqrt{2}(\phi^4+4\phi^2-1)\}$$
$$=\sqrt{2}(\phi^2+1)(\phi^4+6\phi^2+1);$$

that is

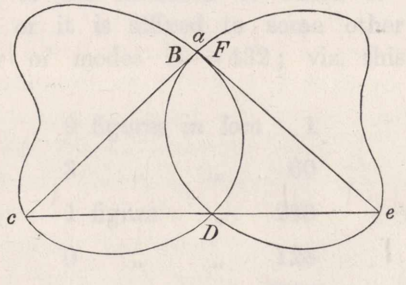
$$\begin{aligned}(\phi^2-1)(-8\phi^2X-8\phi\sqrt{\phi}\,Y)\\ &=\sqrt{2}(\phi^2+1)(\phi^4+6\phi^2+1)-\sqrt{2}(\phi^2+1).8\phi^2-\sqrt{2}(\phi^2-1)(\phi^4+4\phi^2-1),\\ &=\sqrt{2}(\phi^2+1)(\phi^2-1)^2-\sqrt{2}(\phi^2-1)(\phi^4+4\phi^2-1),\\ &=\sqrt{2}(\phi^2-1)\{(\phi^4-1)-(\phi^4+4\phi^2-1)\},\\ &=-4\sqrt{2}(\phi^2-1)\phi^2,\end{aligned}$$

whence, finally,

$$X\sqrt{\phi}+Y=\sqrt{\tfrac{1}{2}}\,\phi,$$

which is the required equation.

It may be remarked that for $\phi=1$, the equation of the curve is $(x^2-1)^2+(y^2-\frac{1}{2})^2=1$, which is the binodal form $a^2>b^2$, $c^4=a^4$. We have in this case $\xi=0$, $\eta=\sqrt{\frac{1}{2}}$, and the curve and triangle are as shown in the figure, viz. the base ce of the triangle, instead



of being a proper tangent, is a line through the node D. For any other value of ϕ, the curve consists of an exterior oval (pinched in at the sides and the top and bottom) and of an interior oval; the angles a, c, e lie in the exterior oval, the sides ac, ea touch the interior oval, and the base ce touches the exterior oval.

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If, to fix the ideas, we assume $\phi > 1$, then we have always $c^4 > a^4 < a^4 + b^4$: for $\phi = 1$ we have, as appears above, $b^2 = \frac{1}{2}$, which is $< a^2$; but for a certain value of ϕ between 3 and 4, b^2 becomes $= a^2$, and for any greater value of ϕ we have $b^2 > a^2$. The condition for the equality of a^2 and b^2 is

$$\phi^4 + 4\phi^2 - 1 = 4\phi(\phi^2+1), \text{ or } \phi^4 - 4\phi^3 + 4\phi^2 - 4\phi - 1 = 0\,;$$

this equation may be written $2\phi(\phi-2)(\phi^2+1) = (\phi^2-1)^2$, and we thence obtain

$$\frac{(\phi^2-1)^4}{16\phi^2(\phi^2+1)^2} = \tfrac{1}{4}(\phi-2)^2\,;$$

or the equation of the curve is $(x^2-1)^2 + (y^2-1)^2 = 1 + \frac{1}{4}(\phi-2)^2$, where ϕ is determined by the equation just referred to. The curve is in this case symmetrical in regard to the two axes; and there are in fact four triangles, each in-and-circumscribed to the curve.

Cambridge, June 16, 1865.