365.

ON THE INTERSECTIONS OF A PENCIL OF FOUR LINES BY A PENCIL OF TWO LINES.

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PLÜCKER has considered ("Analytisch-geometrische Aphorismen," *Crelle*, vol. XI. (1834) pp. 26—32) the theory of the eight points which are the intersections of a pencil of four lines by any two lines, or say the intersections of a pencil of *four* lines by a pencil of *two* lines: viz., the eight points may be connected two together by twelve new lines; the twelve lines meet two together in forty-two new points; and of these, six lie on a line through the centre of the two-line pencil, twelve lie four together on three lines through the centre of the four-line pencil, and twenty-four lie two together on twelve lines, also through the centre of the four-line pencil.

The first and third of these theorems, viz. (1) that the six points lie on a line through the centre of the two-line pencil, and (3) that the twenty-four points lie two together on twelve lines through the centre of the four-line pencil, belong to the more simple theory of the intersections of a pencil of *three* lines by a pencil of *two* lines; the second theorem, viz. (2) the twelve points lie four together on three lines through the centre of the four-line pencil, is the only one which properly belongs to the theory of the intersections of a pencil of *four* lines by a pencil of *two* lines. The theorem in question (proved analytically by Plücker) may be proved geometrically by means of two fundamental theorems of the geometry of position: these are the theorem of two triangles in perspective, and Pascal's theorem for a line-pair. I proceed to show how this is.

Consider a pencil of two lines meeting a pencil of four lines in the eight points (a, b, c, d), (a', b', c', d'); so that the two lines are $abcd$, $a'b'c'd'$, meeting suppose in

Q; and the four lines are aa', bb', cc', dd', meeting suppose in P; then the twelve points are

$$a'd \,.\, c'b, \quad ad' \,.\, cb', \quad a'c \,.\, d'b, \quad ac' \,.\, db' \quad \text{lying in a line through } P,$$

$$a'b \,.\, d'c, \quad ab' \,.\, dc', \quad a'd \,.\, b'c, \quad ad' \,.\, bc' \quad \text{,, \quad ,,} \quad ,$$

$$a'c \,.\, b'd, \quad ac' \,.\, bd', \quad a'b \,.\, c'd, \quad ab' \,.\, cd' \quad \text{,, \quad ,,} \quad ;$$

where the combinations are most easily formed as follows; viz., for the first four points starting from the arrangement $\begin{matrix} a & c \\ d & b \end{matrix}$ (or any other arrangement having the diagonals $ab \,.\, cd$), and thence writing down the four expressions

$$\begin{matrix} a'c', & a\,c, & a'c, & ac' \\ d\,b, & d'b', & d'b, & db', \end{matrix}$$

we read off from these the symbols of the four points; and the like for the other two sets of four points.

Now, considering the points (a, b, c) and (a', b', c'), the points $ab' \,.\, a'b$, $ac' \,.\, a'c$, $bc' \,.\, b'c$ lie in a line through Q; and similarly the points $ab' \,.\, a'b$, $ad' \,.\, a'd$, $bd' \,.\, b'd$ lie in a line through Q; which lines, inasmuch as they each contain the points Q and $ab' \,.\, a'b$, must be one and the same line; considering the combinations (b, c, d), (b', c', d'), the line in question also passes through $cd' \,.\, c'd$; that is, the six points $ab' \,.\, a'b$, $ac' \,.\, a'c$, $ad' \,.\, a'd$, $bc' \,.\, b'c$, $bd' \,.\, b'd$, $cd' \,.\, c'd$ lie in a line through Q, which is in fact the before-mentioned first theorem. Hence the points $ab' \,.\, a'b$ and $cd' \,.\, c'd$ lie in a line through Q; or, calling these points M and N respectively, the triangles Maa', Mbb', Ncc', Ndd' are in perspective. Hence, considering the two triangles Maa', Ndd' (or, if we please, the complementary set Mbb', Ncc'), the corresponding sides are

$$\begin{matrix} Ma, \; Nd & \text{meeting in} & ab' \,.\, dc', \\ Ma', \; Nd' & \text{,,} & a'b \,.\, d'c, \\ aa', \; dd' & \text{,,} & P \; ; \end{matrix}$$

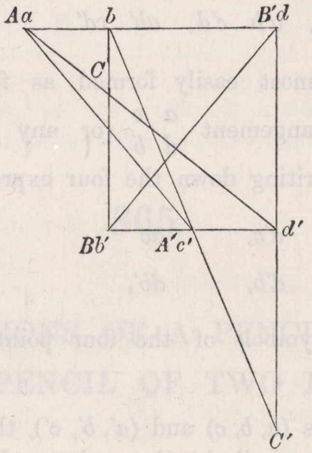
that is, the points $ab' \,.\, dc'$, $a'b \,.\, d'c$ lie in a line through P.

Similarly $ad' \,.\, a'd$ and $bc' \,.\, b'c$ lie in a line through Q; or, calling these points H, I respectively, the triangles Haa', Hdd', Ibb', Icc' are in perspective; and considering the combination Hdd', Ibb' (or, if we please, the complementary set Haa', Icc'), the corresponding sides are

$$\begin{matrix} Ha, \; Ib & \text{meeting in} & ad' \,.\, bc', \\ Ha', \; Ib' & \text{,,} & a'd \,.\, cb', \\ aa', \; bb' & \text{,,} & P \; ; \end{matrix}$$

that is, the points $a'd \,.\, c'b$, $ad' \,.\, cb'$ lie in a line through P.

It remains to be shown that the two lines through P, viz. the line containing $ab' . dc'$ and $a'b . d'c$, and the line containing $ad' . bc'$ and $a'd . cb'$, are one and the same line. This will be the case if, for instance, $ab' . dc'$ and $ad' . bc'$ also lie in a line through P.



We have the points (a, b, d) in a line, and the points (b', c', d') in a line; the points a, d, b', c' are also called A, B', B, A' respectively; ad', bb' meet in C, and bc', dd' meet in C'; hence, considering the hexagon $ad'db'bc'$, the lines

$$\begin{array}{lll} ad',\ b'b & \text{meet in} & C\ , \\ d'd,\ bc' & \quad ,, & C'\ , \\ db',\ ca' & \quad ,, & AA'\ .\ BB'\ ; \end{array}$$

and hence these three points lie in a line; or, what is the same thing, the lines AA', BB', and CC' meet in a point; that is, the triangles ABC, $A'B'C'$ are in perspective: the corresponding sides are

$$\begin{array}{llll} AB,\ A'B', & \text{that is, } ab',\ c'd, & \text{meeting in} & ab'\ .\ c'd, \\ BC,\ B'C' & \quad ,, \quad b'b,\ d'd, & \quad ,, & P\ , \\ CA,\ C'A' & \quad ,, \quad ad',\ bc', & \quad ,, & ad'\ .\ bc'\ ; \end{array}$$

and these three points lie in a line; that is, the points $ab' . dc'$ and $ad' . bc'$ lie in a line through P. Hence the line through $ab' . dc'$ and $a'b . d'c$ and the line through $ad' . bc'$ and $a'd . cb'$ are one and the same line; that is,

the points $ab' . dc'$, $a'b . d'c$, $ad' . bc'$, $a'd . b'c$ lie in a line through P.

This proves the existence of one of the lines through P; and that of the other two lines follows from the symmetry of the figure; it thus appears that the twelve points lie four together on three lines through P.

Cambridge, April 11, 1865.