361.

ON QUARTIC CURVES.

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THE expression 'an oval' is used, in regard to the plane, to denote a closed curve without nodes or cusps; and, in regard to the sphere, it is assumed moreover that the oval is a curve which is not its own opposite, and does not meet the opposite curve[1]—that is, that the oval is one of a pair of non-intersecting twin ovals. I say that every spherical curve of the fourth order (or spherical quartic) without nodes or cusps may be considered as composed of an oval or ovals lying wholly in one hemisphere (that is, not cutting or touching the bounding circle of the hemisphere), and of the opposite oval or ovals lying wholly in the opposite hemisphere; or, disregarding the opposite curves, that it consists of an oval or ovals lying wholly in one hemisphere. And this being so, the quartic cone having its vertex at the centre of the sphere is met by a plane parallel to that of the bounding circle in a plane quartic curve consisting of an oval or ovals; and thence every plane quartic is either a finite curve consisting of an oval or ovals, or else the projection of such a curve.

Considering first the case of the plane, a line in general meets the oval in an *even* number of points (the number may of course be $=0$); hence as the point of contact of a tangent reckons for two points, the tangent at any point of the oval again intersects the oval in an even number of points (this number may of course be $=0$). The number of points of intersection by the tangent (the point of contact being always excluded) is either evenly even, and the point is then situate on a *convex* portion of the oval; or it is oddly even, and the point is then situate on a *concave* portion of the oval. Now imagine that the oval is (or is part of) a quartic curve; the number of points

[1] The notions of opposite curves, &c. are fully developed in the excellent Memoir of Möbius, "Ueber die Grundformen der Linien der dritter Ordnung," *Abh. der K. Sächs. Ges. zu Leipzig*, vol. I. (1852), to which I have elsewhere frequently referred.

of intersection by the tangent is $=0$ or else $=2$; and there is at least one portion of the oval for which the number of intersections is $=0$; for otherwise the oval would be *concave* at every point, which is impossible. Hence there is a tangent which does not meet the oval (except at the point of contact), and we may in the immediate neighbourhood of the tangent draw a line which does not meet the oval at all.

Precisely the same considerations apply to the case of an oval which is part of a spherical quartic, the tangent being of course a great circle; and the conclusion arrived at is that there exists a great circle which does not meet the oval at all; that is, the oval lies wholly in one hemisphere.

I remark that the demonstration would, as it ought to do, fail, if we attempted to apply it to an oval portion of a spherical sextic; the tangent circle meets the oval in a number of points which is $=0$, 2, or 4; and the number cannot be for every tangent circle whatever $=2$; but there is nothing to prevent it from being for every tangent circle whatever $=2$ or 4. Hence we cannot, for every spherical sextic, obtain a tangent circle not meeting the oval except at the point of contact; and consequently we do not obtain in the immediate neighbourhood of the tangent a circle which does not meet the oval at all. And in fact such circle does not in every case exist; that is, *the oval portion of a spherical sextic does not in every case lie in a hemisphere.*

It has been shown that the oval portion of a spherical quartic lies in a hemisphere; but we have to consider the case where the quartic consists of two or more ovals. To fix the ideas, let A, A' be a pair of opposite ovals, and B, B' another pair of opposite ovals, components of the same spherical quartic. If there exists a tangent circle of A which does not meet B, then there exists in the immediate neighbourhood of the tangent circle a circle which does not meet either A or B; and we may assume that A and B lie on the same side of this circle; for if B were on the side opposite to A, then B' would be on the same side with A; and we have only, instead of B, to consider the opposite oval B'. Hence we may consider that the ovals A and B lie on the same side of the circle; that is, we have a spherical quartic consisting of or comprising the ovals A and B in the same hemisphere: the two ovals are, it is clear, external each to the other.

But every tangent of A may meet B in two points; consider the whole spherical figure, and suppose that the tangent (or say, the tangent circle) of A, A' meets the ovals B, B' in the points K, L and the opposite points K', L': then considering the tangent circle as moving round A, A' until it returns to its original position, the points K, L, K', L' are always four distinct points; and K and some one (say L) of the two points L, L' will describe the same oval, say the oval B; while the opposite points K', L' will describe the opposite oval B'. We have here the oval A included in the oval B (and of course the opposite oval A' included in the opposite oval B'). But the oval B, *quà* portion of a spherical quartic, lies wholly in one hemisphere; hence the two ovals A, B lie wholly in one hemisphere. It is easy to see that there is not in this case any other portion of the spherical quartic, but that the two ovals A, B are the entire curve.

Reverting to the case where we have in one hemisphere the two ovals A, B external to each other, the spherical quartic may comprise as part of itself another oval C. The ovals A and B, *quà* ovals external to each other, have a common tangent circle (a double tangent of the spherical quartic) which cannot meet the oval C (for if it did we should have six points of intersection); hence in the immediate neighbourhood thereof we have a circle not meeting any one of the ovals A, B, C. We may consider A, B, C as lying on the same side of this circle; for if B were on the opposite side to A, then B' would be on the same side; and so if C be on the opposite side, then C' will be on the same side; that is, we have the three ovals A, B, C external to each other, and in the same hemisphere.

There may be a fourth oval, D, and it would be shown in a similar manner that we have then the four ovals A, B, C, D external to each other and in the same hemisphere. But there cannot be a fifth oval, E; the proof is precisely the same as for the theorem *in plano*; viz. taking within each of the five ovals a point, and through these points drawing a conic, the conic would meet each oval in two points, and therefore the plane quartic in ten points, which is impossible.

Passing from the sphere to the plane, the foregoing investigation shows that every plane quartic without nodes or cusps is either a finite curve, or else the projection of a finite curve, of one of the following forms:

1. a single oval.
2. two ovals external to each other.
3. two ovals, one inside the other.
4. three ovals external to each other.

5, 6. four ovals external to each other.

The last case has been called (5, 6) for the sake of the following subdivision, viz.:

5. the four ovals are so situate as to be intersected, each in two points, by the same ellipse.

6. they are so situate as not to be intersected by any one ellipse whatever—the distinction being similar to that which exists between four points, which may be either such as to have passing through them as well ellipses as hyperbolas, or else to have passing through them hyperbolas only.

I remark that the limitation of the theorem to the case of a quartic curve without nodes or cusps is necessary, at any rate as regards the nodes. We may in fact find a quartic curve having a single node which is met by every line in at least two real points, and which is therefore not the projection of any finite curve; for if we imagine two hyperbolas so situate that each branch of the one cuts each branch of the other, then it may be seen that there exists a quartic curve approaching everywhere very nearly to the system of two hyperbolas, but having, instead of the four nodes of the system, only a single node, which is such that every line meets it in at least two points.

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