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NOTE ON A QUARTIC SURFACE.

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IT would, I think, be worth while to study in detail the quartic surface which is the envelope of a sphere having its centre on a given conic, and passing through a given point. The equations of the conic being $z=0$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the coordinates of a point on the conic may be taken to be $x=a\cos\theta$, $y=b\sin\theta$, $z=0$, whence, if (α, β, γ) be the coordinates of the given point, the equation of the sphere is

$$(x-a\cos\theta)^2+(y-b\sin\theta)^2+z^2=(\alpha-a\cos\theta)^2+(\beta-b\sin\theta)^2+\gamma^2,$$

or, what is the same thing,

$$x^2+y^2+z^2-\alpha^2-\beta^2-\gamma^2-2(x-\alpha)a\cos\theta-2(y-\beta)b\sin\theta=0\,;$$

and hence the equation of the surface is at once seen to be

$$(x^2+y^2+z^2-\alpha^2-\beta^2-\gamma^2)^2=4a^2(x-\alpha)^2+4b^2(y-\beta)^2.$$

If $a=b$ (that is, if the conic be a circle), then we may without loss of generality write $\beta=0$, and the equation then is

$$(x^2+y^2+z^2-\alpha^2-\gamma^2)^2=4a^2\{(x-\alpha)^2+y^2\}.$$

This may be written

$$(x^2+y^2+z^2-\alpha^2-\gamma^2-2a^2)^2=-8a^2\alpha\left(x-\frac{a^2+2\alpha^2+\gamma^2-z^2}{2\alpha}\right),$$

which, considering z as a constant, is of the form

$$(x^2+y^2-\alpha)^2=16A(x-m)\,;$$

that is, the section of the surface by a plane parallel to the plane of the conic is a Cartesian.

If a and b are unequal, but if we still have $\beta = 0$, the equation of the surface is

$$(x^2 + y^2 + z^2 - \alpha^2 - \gamma^2)^2 = 4a^2 (x - \alpha)^2 + 4b^2y^2.$$

There are here two planes parallel to the plane of the conic, each of them meeting the surface in a pair of circles. In fact, writing $x^2 + y^2 = \rho$, and therefore also $y^2 = \rho - x^2$, putting moreover $z^2 - \alpha^2 - \gamma^2 = k$, we have

$$(\rho + k)^2 = 4a^2x^2 - 8a^2\alpha x + 4a^2\alpha^2 + 4b^2 (\rho - x^2);$$

that is,

$$\rho^2 + 4 (b^2 - a^2) x^2 + k^2 - 4a^2\alpha^2 + 8a^2\alpha x + (2k - 4b^2) \rho = 0,$$

or, as this may also be written,

$$(1,\ 4(b^2 - a^2),\ \ k^2 - 4a^2\alpha^2,\ \ 4a^2\alpha,\ \ k - 2b^2,\ \ 0 \between \rho,\ x,\ 1)^2 = 0,$$

which is of the form

$$(a,\ \ b\ ,\ \ c\ ,\ \ f\ ,\ \ g\ ,\ \ 0 \between \rho,\ x,\ 1)^2 = 0;$$

and the left-hand side will break up into factors, each of the form $\rho + Ax + B$ (so that, equating either factor to zero, we have $\rho + Ax + B = 0$, that is, $x^2 + y^2 + Ax + B = 0$, the equation of a circle), if only

$$abc - af^2 - bg^2 = 0.$$

Writing this under the form $b (ac - g^2) - af^2 = 0$, and substituting for a, b, c, f, g their values, we have

$$b = 4 (b^2 - a^2),\quad ac - g^2 = k^2 - 4a^2\alpha^2 - (k - 2b^2)^2,\ = 4 (b^2k - b^4 - a^2\alpha^2),\quad af^2 = 16a^4\alpha^2,$$

and therefore the condition is

$$(b^2 - a^2) (b^2k - b^4 - a^2\alpha^2) - a^4\alpha^2 = 0;$$

that is,

$$b^2 \{(b^2 - a^2) (k - b^2) - a^2\alpha^2\} = 0.$$

If $b^2 = 0$, the surface is a pair of spheres; rejecting this factor, we have $(b^2 - a^2)(k^2 - b^2) - a^2\alpha^2 = 0$; or putting for k its value, the condition becomes

$$(b^2 - a^2) (z^2 - \alpha^2 - \gamma^2 - b^2) - a^2\alpha^2 = 0;$$

that is, for each of the values of z given by this equation, the section by a plane parallel to the plane of the conic will be a pair of circles.

The planes in question will coincide with the plane of the conic, if only

$$(b^2 - a^2) (\alpha^2 + \gamma^2 + b^2) + a^2\alpha^2 = 0,$$

or, what is the same thing,

$$b^2\alpha^2 - (a^2 - b^2) \gamma^2 = b^2 (a^2 - b^2);$$

that is, if the point $(\alpha, 0, \gamma)$ be situated on the hyperbola $y = 0$, $\dfrac{x^2}{a^2 - b^2} - \dfrac{z^2}{b^2} = 1$. The hyperbola in question and the ellipse $z = 0$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, are, it is clear, conics in planes at right angles to each other, having the transverse axes coincident in direction, and being such that each curve passes through the foci of the other curve; or, what is the same thing, they are a pair of focal conics of a system of confocal ellipsoids.

The surface in the case in question, viz. when the parameters a, b, α, β are connected by the equation

$$\frac{\alpha^2}{a^2-b^2}-\frac{\gamma^2}{b^2}=1,$$

is in fact the "Cyclide" of Dupin. It is to be noticed that we have here

$$(\alpha - a\cos\theta)^2 + b^2\sin^2\theta + \gamma^2 = \alpha^2+\gamma^2+b^2 - 2a\alpha\cos\theta + (a^2-b^2)\cos^2\theta;$$

which, observing that $\alpha^2+\gamma^2+b^2$ is $=\dfrac{a^2\alpha^2}{a^2-b^2}$, gives

$$(\alpha - a\cos\theta)^2 + b^2\sin^2\theta + \gamma^2 = \left(\sqrt{a^2-b^2}\cos\theta - \frac{a\alpha}{\sqrt{a^2-b^2}}\right)^2;$$

so that the radius of the variable sphere is

$$=\sqrt{a^2-b^2}\cos\theta - \frac{a\alpha}{\sqrt{a^2-b^2}}.$$

If the variable sphere, instead of passing through the point $(\alpha, 0, \gamma)$ on the hyperbola, be drawn so as to touch a sphere of radius l, having its centre at the point in question, then the radius of the variable sphere would be

$$=\sqrt{a^2-b^2}\cos\theta - \frac{a\alpha}{\sqrt{a^2-b^2}} - l,$$

which is in fact

$$=\sqrt{a^2-b^2}\cos\theta - \frac{a\alpha'}{\sqrt{a^2-b^2}},$$

if only $\alpha' = \alpha + \dfrac{l\sqrt{a^2-b^2}}{a}$; hence if γ' be the corresponding value of γ, the variable sphere passes through the point $(\alpha', 0, \gamma')$ on the hyperbola, and the envelope is still a cyclide. The cyclide as derived from the foregoing investigation is thus the envelope of a sphere having its centre on the ellipse, and touching a fixed sphere having its centre on the hyperbola. It also appears that there are, having their centres on the hyperbola, an infinite series of spheres each touched by the spheres which have their centre on the ellipse; if, instead of one of these spheres we take any four of them, this will imply that the centre of the variable sphere is on the ellipse, and it is thus seen that the cyclide as obtained above is identical with the cyclide according to the original definition, viz. as the envelope of a sphere touching four given spheres.

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