349.

ON A CASE OF THE INVOLUTION OF CUBIC CURVES.

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THE present memoir relates to the involution

$$xyz + k(x+y+z)^2(\lambda x + \mu y + \nu z) = 0,$$

viz. treating x, y, z as coordinates, and k as a variable parameter, this equation represents the series of cubic curves passing through the intersections of the two cubics

$$xyz = 0, \quad (x+y+z)^2(\lambda x + \mu y + \nu z) = 0\,;$$

or, what is the same thing, the line $x+y+z=0$ meets any cubic of the series in three points the tangents at which are $x=0$, $y=0$, $z=0$, and these tangents again meet the cubic in three points lying on the line $\lambda x + \mu y + \nu z = 0$; so that in the language which I have used elsewhere, the lines $x+y+z=0$, $\lambda x + \mu y + \nu z = 0$ are in regard to the cubic a primary and a satellite line respectively. The investigation (which is a development of two short papers already published in the *Philosophical Magazine*)[1] was undertaken in order to applying it to the explanation and discussion of Plücker's Classification of Curves of the Third Order; but such application will properly be made in a separate memoir, *On the Classification of Cubic Curves*, and it has also appeared to me convenient to give therein the discussion of the geometrical forms of certain loci which present themselves in the present memoir.

I remark that the involution intended to be here considered is a case of the more general one $U + kV = 0$, where $U=0$, $V=0$ are any two cubic curves whatever. It appears from my memoir *On the Theory of Involution*, [348], that the equation, Disc^t. $(U+kV)=0$, which determines the critic values of k is in the general case of the order 12; the

[1] On the Cubic Centres of a Line with respect to Three Lines and a Line.—*Phil. Mag.* t. XX. pp. 418—423 (1860), [257]. *Ditto*, Second Paper, t. XXII. pp. 433—436 (1861), [315].

special case is however in the present memoir treated irrespectively of the general one, and the equation for the critic values of k is found to be of the order 3; this of course means that the equation of the order 12 breaks up into two equations of the orders 9 and 3 respectively, but I have not attempted to show how the decomposition and reduction arise. Moreover, in the general case the equation, Disc^t. Disc^t. $(U+kV)=0$, which is the condition for the existence of a twofold critic value, presents itself in the form $RQ^3P^2=0$, where $R=0$ is the condition that the two cubics $(U=0,\ V=0)$ shall touch each other; $Q=0$ the condition that there shall be in the involution $U+kV=0$ a curve having (not a mere node but) a cusp; and $P=0$ the condition that there shall be a curve having two nodes, or (what is the same thing) breaking up into a line and conic. But in the special case, which, as already noticed, is here considered irrespectively of the general one, the equation Disc^t. Disc^t. $(U+kV)=0$, for the existence of a twofold critic value presents itself in the reduced form $Q=0$, giving the condition, that corresponding to the twofold critic value there shall be a curve having (not a mere node but) a cusp.

Article Nos. 1 to 18, *Explanations, Definitions, and Results.*

1. I consider the involution

$$xyz+k(x+y+z)^2(\lambda x+\mu y+\nu z)=0,$$

where $x=0$, $y=0$, $z=0$, $x+y+z=0$ may be considered as representing any four lines no three of which meet in a point, and $\lambda x+\mu y+\nu z=0$, as representing any fifth line whatever: k is a variable parameter. The lines $x+y+z=0$, $\lambda x+\mu y+\nu z=0$, are a primary line and a satellite line of any cubic of the series, viz. the tangents $x=0$, $y=0$, $z=0$, at the points of intersection with the primary line $x+y+z=0$, meet the cubic in three points lying on the satellite line $\lambda x+\mu y+\nu z=0$.

2. A critic value of k is a value for which the corresponding curve

$$xyz+k(x+y+z)^2(\lambda x+\mu y+\nu z)=0$$

has a node; and such node, or say rather the site of such node, is a critic centre.

3. The critic values of k are in effect determined by a cubic equation, and the coordinates of the critic centre are then given rationally in terms of k; there are consequently three critic values of k; and the same number of critic centres, and of nodal curves: it is however found to be convenient to express as well the critic value of k, as the coordinates of the critic centre, rationally in terms of an auxiliary parameter θ which is given by a cubic equation.

4. The cubic equation in k (or what is the same thing, that in θ) may have a twofold root (pair of equal roots); or, say rather, it may have a twofold root and a one-with-twofold root: corresponding to the twofold value of k we have a twofold critic

centre, which is not a mere node but a cusp on the cubic, or instead of a merely nodal cubic we have a cuspidal cubic; and corresponding to the one-with-twofold value of k we have a one-with-twofold critic centre, being of course a mere node on the nodal cubic.

5. In the case in question of a twofold and one-with-twofold value of k, the line $\lambda x + \mu y + \nu z = 0$, or say the satellite line, envelopes a curve which might be termed the twofold and one-with-twofold envelope, but which is spoken of simply as the envelope.

The locus of the twofold centre is a curve which is called the twofold centre locus.

The locus of the one-with-twofold centre is a curve which is called the one-with-twofold centre locus.

These definitions premised, the following results may be stated;

6. The equation in θ may be represented in the three equivalent forms

$$\frac{1}{\theta+\lambda}+\frac{1}{\theta+\mu}+\frac{1}{\theta+\nu}-\frac{2}{\theta}=0,$$

$$\frac{\lambda}{\theta+\lambda}+\frac{\mu}{\theta+\mu}+\frac{\nu}{\theta+\nu}-1=0,$$

$$\theta^3-\theta(\mu\nu+\nu\lambda+\lambda\mu)-2\lambda\mu\nu=0.$$

7. The critic value of k and the coordinates of the critic centre are then given by the equations

$$k=\frac{-\frac{1}{4}\theta^2}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)},$$

$$x : y : z : x+y+z : \lambda x+\mu y+\nu z=\frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu} : \frac{2}{\theta} : 1.$$

8. The condition for a twofold and one-with-twofold value of k is

$$\lambda^{-\frac{1}{3}}+\mu^{-\frac{1}{3}}+\nu^{-\frac{1}{3}}=0,$$

or, what is the same thing,

$$(\mu\nu+\nu\lambda+\lambda\mu)^3-27\lambda^2\mu^2\nu^2=0,$$

which equations may either of them be considered as the line-equation of the envelope. The equation in the coordinates (x, y, z), or point-equation of the envelope is

$$\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}=0,$$

or, in its rationalised form,

$$x^4+y^4+z^4-4(yz^3+y^3z+zx^3+z^3x+xy^3+x^3y)$$
$$+6(y^2z^2+z^2x^2+x^2y^2)-124(x^2yz+xy^2z+xyz^2)=0.$$

9. The equation of the twofold centre locus is

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0,$$

or, in its rationalised form,

$$x^2+y^2+z^2-2yz-2zx-2xy=0;$$

the curve is therefore a conic, and it may be spoken of as the twofold centre conic.

10. The equation of the one-with-twofold centre locus is

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz=0,$$

the curve is therefore a cubic, and it may be spoken of as the one-with-twofold centre cubic.

11. The before-mentioned equation $\lambda^{-\frac{1}{3}}+\mu^{-\frac{1}{3}}+\nu^{-\frac{1}{3}}=0$ is satisfied by

$$\lambda : \mu : \nu = \alpha^{-3} : \beta^{-3} : \gamma^{-3},$$

where $\alpha+\beta+\gamma=0$, and it is very convenient to introduce these quantities α, β, γ into the formulæ.

12. The equation of the satellite line giving a twofold and one-with-twofold centre is

$$\frac{x}{\alpha^3}+\frac{y}{\beta^3}+\frac{z}{\gamma^3}=0;$$

the coordinates of the point of contact with the envelope are $x : y : z = \alpha^4 : \beta^4 : \gamma^4$.

The equation in θ gives $\theta_1=\theta_2=-\frac{1}{\alpha\beta\gamma}$ for the values corresponding to the twofold centre; and $\theta_3=\frac{2}{\alpha\beta\gamma}$ for the value corresponding to the one-with-twofold centre.

The coordinates of the twofold centre, or cusp, are $x : y : z = \alpha^2 : \beta^2 : \gamma^2$.

The coordinates of the one-with-twofold centre, or node, are

$$x : y : z = \alpha^2(\beta-\gamma) : \beta^2(\gamma-\alpha) : \gamma^2(\alpha-\beta).$$

The equation of the tangent at the cusp is

$$(\beta-\gamma)\frac{x}{\alpha}+(\gamma-\alpha)\frac{y}{\beta}+(\alpha-\beta)\frac{z}{\gamma}=0.$$

The equation of the line joining the cusp and the node, which line is also one of the tangents at the node is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0.$$

The equation of the other tangent at the node is

$$(2\beta\gamma+\alpha^2)\frac{x}{\alpha}+(2\gamma\alpha+\beta^2)\frac{y}{\beta}+(2\alpha\beta+\gamma^2)\frac{z}{\gamma}=0.$$

13. Considering the critic centre corresponding to a root θ of the cubic equation, the equation of the line joining the other two critic centres is

$$\frac{\lambda x}{\theta+\lambda}+\frac{\mu y}{\theta+\mu}+\frac{\nu z}{\theta+\nu}=0,$$

which is the polar of the critic centre in regard to the twofold centre conic. The critic centres are consequently conjugate poles in regard to the twofold centre conic.

14. The equation of the tangents at the critic centre considered as a node of the corresponding cubic curve is

$$\left(\theta+4\lambda,\ \theta+4\mu,\ \theta+4\nu,\ -\theta-\frac{2\mu\nu}{\theta},\ -\theta-\frac{2\nu\lambda}{\theta},\ -\theta-\frac{2\lambda\mu}{\theta}\right)(x,\ y,\ z)^2=0.$$

15. The last-mentioned formulæ lead to some which involve the three critic centres viz. if $X=0$, $Y=0$, $Z=0$ are the equations of the sides of the triangle formed by the critic centres, then the equations of the tangents at the three critic centres respectively are of the form

$$\begin{aligned} . \quad BY^2+CZ^2&=0,\\ AX^2 \quad . \quad +CZ^2&=0,\\ AX^2+BY^2 \quad . \quad &=0, \end{aligned}$$

so that the tangents in question are in fact the tangents from the three nodes respectively to the conic

$$AX^2+BY^2+CZ^2=0:$$

the three nodes or critic centres being thus conjugate poles in regard to the conic, this is called "the three centre conic."

16. The equation of a nodal cubic is also expressible in a simple form in terms of the new coordinates X, Y, Z. In the formulæ for these transformations, and indeed throughout the memoir, the three roots of the equation in θ are represented by θ_1, θ_2, θ_3, and I write also

$$\begin{aligned} l_1=\theta_2-\theta_3,\quad l_2&=\theta_3-\theta_1,\quad l_3=\theta_1-\theta_2,\\ \Theta_1&=(\theta_1+\lambda)(\theta_1+\mu)(\theta_1+\nu),\\ \Theta_2&=(\theta_2+\lambda)(\theta_2+\mu)(\theta_2+\nu),\\ \Theta_3&=(\theta_3+\lambda)(\theta_3+\mu)(\theta_3+\nu). \end{aligned}$$

17. If $\lambda a+\mu b+\nu c=0$, that is, if (a, b, c) are the coordinates of a point on the line $\lambda x+\mu y+\nu z=0$, then the critic centres lie on the cubic

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-\frac{2(a+b+c)}{x+y+z}=0,$$

or, what is the same thing, this curve is the locus of the critic centres corresponding to the several lines $\lambda x+\mu y+\nu z=0$ through the point (a, b, c).

18. In particular, taking in succession for the point (a, b, c) the point of intersection of the line $\lambda x + \mu y + \nu z = 0$, with the lines $x = 0$, $y = 0$, $z = 0$, $x + y + z = 0$, the critic centres lie on the conics

$$\frac{\nu}{y} - \frac{\mu}{z} - \frac{2(\nu - \mu)}{x + y + z} = 0,$$

$$\frac{\lambda}{z} - \frac{\nu}{x} - \frac{2(\lambda - \nu)}{x + y + z} = 0,$$

$$\frac{\mu}{x} - \frac{\lambda}{y} - \frac{2(\mu - \lambda)}{x + y + z} = 0,$$

$$\frac{\mu - \nu}{x} + \frac{\nu - \lambda}{y} + \frac{\lambda - \mu}{z} = 0,$$

which are useful for the construction of the critic centres for a given line $\lambda x + \mu y + \nu z = 0$. The last of the four conics passes through the point $(1, 1, 1)$ which is the harmonic of the line $x + y + z = 0$ in regard to the triangle formed by the lines $x = 0$, $y = 0$, $z = 0$; and I call it the harmonic conic.

Article Nos. 19 to 21. *General Formulæ for the Critic Centres.*

19. I consider the involution

$$xyz + k(x + y + z)^2 (\lambda x + \mu y + \nu z) = 0.$$

Writing the equation in the form

$$k(\lambda x + \mu y + \nu z) = \frac{-xyz}{(x + y + z)^2},$$

and differentiating with regard to x, y, z respectively, we obtain

$$-k\lambda (x + y + z)^3 = yz(-x + y + z),$$

$$-k\mu (x + y + z)^3 = zx(\ \ x - y + z),$$

$$-k\nu (x + y + z)^3 = xy(\ \ x + y - z),$$

which determine the coordinate ratios $x : y : z$ of the node or critic centre; and the corresponding value of k.

20. Writing the equations under the form

$$\frac{-k(x + y + z)^3}{xyz} = \frac{-x + y + z}{\lambda x} = \frac{x - y + z}{\mu y} = \frac{x + y - z}{\nu z} = \frac{2}{\theta},$$

where θ is an auxiliary parameter to be determined, we find

$$x\left(-1 - \frac{2\lambda}{\theta}\right) + y + z = 0,$$

that is

$$x+y+z=2x\left(1+\frac{\lambda}{\theta}\right),$$

and consequently

$$x+y+z=2x\left(1+\frac{\lambda}{\theta}\right)=2y\left(1+\frac{\mu}{\theta}\right)=2z\left(1+\frac{\nu}{\theta}\right),$$

or, what is the same thing,

$$x+y+z : x : y : z=\frac{2}{\theta} : \frac{1}{\lambda+\theta} : \frac{1}{\mu+\theta} : \frac{1}{\nu+\theta},$$

and we have thence

$$\frac{1}{\theta+\lambda}+\frac{1}{\theta+\mu}+\frac{1}{\theta+\nu}-\frac{2}{\theta}=0,$$

an equation which may also be written in the form

$$\frac{\lambda}{\theta+\lambda}+\frac{\mu}{\theta+\mu}+\frac{\nu}{\theta+\nu}-1=0,$$

or in the form

$$\theta^3-\theta(\mu\nu+\nu\lambda+\lambda\mu)-2\lambda\mu\nu=0;$$

and we then have

$$k=-\frac{2}{\theta}\frac{1}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}\div\left(\frac{2}{\theta}\right)^3,$$

$$=-\frac{\frac{1}{4}\theta^2}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}.$$

21. We see that θ is determined by a cubic equation, and that the ratios $x : y : z$ and the parameter k are rational functions of θ. There are thus three nodes or critic centres, and the like number of nodal curves and of critic values of k.

The form secondly obtained for the equation in θ shows that we may write

$$x : y : z : x+y+z : \lambda x+\mu y+\nu z=\frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu} : \frac{2}{\theta} : 1.$$

Article Nos. 22 to 32, *relating to a Twofold and a One-with-Twofold Centre.*

22. If k has a twofold and a one-with-twofold value, then θ will have also a twofold and a one-with-twofold value; and conversely. The equation in θ will have a twofold and a one-with-twofold root if

$$(\mu\nu+\nu\lambda+\lambda\mu)^3-27\lambda^2\mu^2\nu^2=0;$$

or, what is the same thing, if

$$\mu\nu+\nu\lambda+\lambda\mu-3(\lambda\mu\nu)^{\frac{2}{3}}=0,$$

or if

$$(\mu\nu)^{\frac{1}{3}}+(\nu\lambda)^{\frac{1}{3}}+(\lambda\mu)^{\frac{1}{3}}=0,$$

or finally if

$$\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0,$$

so that the condition is satisfied if $\lambda = \alpha^{-3}$, $\mu = \beta^{-3}$, $\nu = \gamma^{-3}$ where $\alpha + \beta + \gamma = 0$. In fact with these values the equation in θ becomes

$$(\alpha\beta\gamma\theta)^3 - 3\alpha\beta\gamma\theta - 2 = 0,$$

that is

$$(\alpha\beta\gamma\theta + 1)^2 (\alpha\beta\gamma\theta - 2) = 0,$$

so that the twofold value is $\theta_1 = \theta_2 = -\dfrac{1}{\alpha\beta\gamma}$; and the one-with-twofold value is $\theta_3 = \dfrac{2}{\alpha\beta\gamma}$.

23. It is throughout assumed that the quantities α, β, γ satisfy the condition $\alpha + \beta + \gamma = 0$. The result just obtained shows that the line

$$\frac{x}{\alpha^3} + \frac{y}{\beta^3} + \frac{z}{\gamma^3} = 0,$$

is a twofold and one-with-twofold satellite line. From this equation, considering α, β, γ as variable parameters satisfying the condition $\alpha + \beta + \gamma = 0$, we find at once the equation of the curve enveloped by the line in question, which curve is called simply the envelope—viz. the coordinates of the point of contact are found to be $x : y : z = \alpha^4 : \beta^4 : \gamma^4$, and thence the equation of the envelope is

$$\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z} = 0,$$

or rationalising, it is

$$x^4 + y^4 + z^4 - 4(yz^3 + y^3z + zx^3 + z^3x + xy^3 + x^3y)$$
$$+ 6(y^2z^2 + z^2x^2 + x^2y^2) - 124(x^2yz + y^2zx + z^2xy) = 0.$$

The before-mentioned equation $\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0$, or

$$(\mu\nu + \nu\lambda + \lambda\mu)^3 - 27\,\lambda^2\mu^2\nu^2 = 0,$$

may be considered as the tangential equation; the envelope is thus of the order 4, and the class 6.

24. It is easy to show that the curve has three nodes the coordinates whereof are $(-4, 1, 1)$, $(1, -4, 1)$, $(1, 1, -4)$; and this being known, the equation may be transformed so as to put the nodes in evidence. I effect the transformation synthetically as follows, viz. writing $x + y + z = \sigma$, $yz + zx + xy = q$, $xyz = r$, the equation of the curve is

$$\begin{aligned}
&(\sigma^4 - 2q\sigma^2 + 2q^2 + 4r\sigma) \\
-4&(\qquad q\sigma^2 - 2q^2 - \;r\sigma) \\
+6&(\qquad\qquad q^2 - 2r\sigma) \\
-124&(\qquad\qquad\qquad r\sigma) = 0,
\end{aligned}$$

viz. it is

$$\sigma^4 - 6q\sigma^2 + 16q^2 - 128r\sigma = 0,$$

which is

$$(7\sigma^2 + 4q)^2 - 16\sigma(3\sigma^3 + 4q\sigma + 8r) = 0,$$

or, putting for a moment $l = -\frac{6}{5}$, and therefore $5(l-2) = -16$, it is

$$(7\sigma^2 + 4q)^2 + (l-2)\,5\sigma(3\sigma^3 + 4q\sigma + 8r) = 0.$$

Now writing

$$x' = \sigma + 2x \quad = 3x + y + z,$$

$$y' = \sigma + 2y \quad = x + 3y + z,$$

$$z' = \sigma + 2z \quad = x + y + 3z,$$

we find

$$y'z' + z'x' + x'y' = 7\sigma^2 + 4q,$$

$$x' + y' + z' = 5\sigma,$$

$$x'y'z' \quad = 3\sigma^3 + 4q\sigma + 8r,$$

so that the equation is

$$(y'z' + z'x' + x'y')^2 + (l-2)\,x'y'z'(x' + y' + z') = 0,$$

that is

$$y'^2z'^2 + z'^2x'^2 + x'^2y'^2 + lx'y'z'(x' + y' + z') = 0\,;$$

or, putting for l its value, the equation is

$$5(y'^2z'^2 + z'^2x'^2 + x'^2y'^2) - 6x'y'z'(x' + y' + z') = 0\,;$$

or, as this may also be written,

$$(5,\ 5,\ 5,\ -3,\ -3,\ -3)\left(\frac{1}{x'},\ \frac{1}{y'},\ \frac{1}{z'}\right)^2 = 0\,;$$

a form which shows that the curve has three nodes at the angles of the triangle

$$x' = 0, \quad y' = 0, \quad z' = 0.$$

25. It is easy to see that the curve is touched by the lines $x = 0$, $y = 0$, $z = 0$ at their intersections with the lines $y - z = 0$, $z - x = 0$, $x - y = 0$ respectively, or (what is the same thing) in the points (0, 1, 1), (1, 0, 1), (1, 1, 0) respectively. It may be added that the line $y - z = 0$ meets the curve in the node $(-4, 1, 1)$, being of course a point of twofold intersection, in the point (0, 1, 1) on the line $x = 0$, and besides in the point (16, 1, 1): and the like for the lines $z - x = 0$ and $x - y = 0$.

26. It may be noticed that although any line passing through one of the nodes is in a sense a tangent to the envelope, yet that it is not a proper tangent and does not give rise to a twofold centre. It is in fact shown (*post*, Nos. 73 and 74) that the critic centres for a line $\lambda x + \mu y + \nu z = 0$ passing through the point $(-4, 1, 1)$ are three points lying, one of them on the line $y + z = 0$, and the other two on the conic $x(x + y + z) - 4yz = 0$.

27. Assume that θ has its twofold value $=-\frac{1}{\alpha\beta\gamma}$, the equations

$$x : y : z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu},$$

substituting also therein for λ, μ, ν the values $\alpha^{-3}, \beta^{-3}, \gamma^{-3}$, give for the coordinates of the twofold centre

$$x : y : z = \frac{\alpha^3\beta\gamma}{\beta\gamma-\alpha^2} : \frac{\beta^3\gamma\alpha}{\gamma\alpha-\beta^2} : \frac{\alpha\beta\gamma^3}{\alpha\beta-\gamma^2},$$

but in virtue of the relation $\alpha+\beta+\gamma=0$ we have

$$\beta\gamma-\alpha^2=\beta\gamma+\gamma\alpha+\alpha\beta=\gamma\alpha-\beta^2=\alpha\beta-\gamma^2;$$

or the values are $x : y : z = \alpha^2 : \beta^2 : \gamma^2$. Hence also we have as the equation of the locus of the twofold centre,

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0,$$

or, what is the same thing,

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

which is a conic touching the lines $x=0$, $y=0$, $z=0$ at their intersections with the lines $y-z=0$, $z-x=0$, $x-y=0$ respectively, or, what is the same thing, in the points (0, 1, 1), (1, 0, 1), (1, 1, 0) respectively.

28. Similarly, if θ has its one-with-twofold value $=\frac{2}{\alpha\beta\gamma}$, the equations

$$x : y : z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu},$$

substituting also therein for λ, μ, ν the values $\alpha^{-3}, \beta^{-3}, \gamma^{-3}$, give for the coordinates of the one-with-twofold centre

$$x : y : z = \frac{\alpha^3\beta\gamma}{2\alpha^2+\beta\gamma} : \frac{\beta^3\gamma\alpha}{2\beta^2+\gamma\alpha} : \frac{\gamma^3\alpha\beta}{2\gamma^2+\alpha\beta};$$

but in virtue of $\alpha+\beta+\gamma=0$ we have

$$2\alpha^2+\beta\gamma=\alpha^2-\alpha(\beta+\gamma)+\beta\gamma=(\alpha-\beta)(\alpha-\gamma)=-(\gamma-\alpha)(\alpha-\beta),$$

and similarly

$$2\beta^2+\gamma\alpha=-(\alpha-\beta)(\beta-\gamma), \quad 2\gamma^2+\alpha\beta=-(\beta-\gamma)(\gamma-\alpha);$$

and thence these values are

$$x : y : z = \alpha^2(\beta-\gamma) : \beta^2(\gamma-\alpha) : \gamma^2(\alpha-\beta),$$

for the coordinates of the one-with-twofold centre.

We thence deduce

$$y+z=\beta^2\gamma-\beta^2\alpha+\gamma^2\alpha-\gamma^2\beta=(\beta\gamma-\alpha\beta-\alpha\gamma)(\beta-\gamma)=(\beta\gamma+\alpha^2)(\beta-\gamma),$$

and consequently

$$-x+y+z=\beta\gamma(\beta-\gamma),$$

that is

$$\begin{array}{cccccc} x & : \ y & : \ z & : -x+y+z & : x-y+z & : x+y-z \\ =\alpha^2(\beta-\gamma) & : \beta^2(\gamma-\alpha) & : \gamma^2(\alpha-\beta) & : \beta\gamma(\beta-\gamma) & : \gamma\alpha(\gamma-\alpha) & : \alpha\beta(\alpha-\beta), \end{array}$$

and these give

$$-(-x+y+z)(x-y+z)(x+y-z)+xyz=0,$$

or, what is the same thing,

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz=0,$$

as the equation of the locus of the one-with-twofold centre, which locus is thus a cubic curve.

29. The equation

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz=0,$$

of the one-with-twofold centre locus may be transformed as follows, viz. writing for a moment $x+y+z=-w$, we have

$$\begin{aligned}
&(9x+4w)(9y+4w)(9z+4w)-w^3 \\
&=729\,xyz+324\,w(yz+zx+xy)-144\,w^3+64w^3-w^3, \\
&=81\ \{9xyz+4w(yz+zx+xy)-w^3\}, \\
&=81\ \{9xyz-12xyz-4(yz^2+\&c.)+(x^3+y^3+z^3)+(3yz^2+\&c.)+6xyz\}, \\
&=81\ \{x^3+y^3+z^3-(yz^2+\&c.)+3xyz\},
\end{aligned}$$

so that the equation may be written

$$(9x+4w)(9y+4w)(9z+4w)-w^3=0,$$

or, what is the same thing,

$$(5x-4y-4z)(-4x+5y-4z)(-4x-4y+5z)+(x+y+z)^3=0,$$

which shows that the intersections of the line $x+y+z=0$, with the sides $x=0$, $y=0$, $z=0$ of the triangle are inflexions on the curve; and that the tangents at these points are respectively

$$5x-4y-4z=0,\quad -4x+5y-4z=0,\quad -4x-4y+5z=0.$$

30. The curve passes through the point (1, 1, 1), which is the harmonic of $x+y+z=0$ in regard to the triangle; and this point is moreover a node on the curve; in fact if the equation be represented by $W=0$, then we have

$$d_xW=3x^2-2x(y+z)-y^2+3yz-z^2,$$

$=0$ for the point in question; and similarly $d_yW=0$, and $d_zW=0$.

31. The equation for the twofold centre conic may also be obtained as follows: viz. the equations

$$\frac{-x+y+z}{\lambda x} = \frac{x-y+z}{\mu y} = \frac{x+y-z}{\nu z} = \frac{2}{\theta},$$

give

$$\frac{(-x+y+z)(x-y+z)(x+y-z)}{xyz} = \frac{8}{\theta^3}\lambda\mu\nu = \frac{8}{\theta^2(\alpha\beta\gamma)^3},$$

which substituting for θ the twofold value $= -\dfrac{1}{\alpha\beta\gamma}$,

gives

$$(-x+y+z)(x-y+z)(x+y-z) + 8xyx = 0,$$

an equation which may be written

$$(x+y+z)^2(x^2+y^2+z^2-2yz-2zx-2xy) = 0,$$

which is the former result affected by the extraneous factor $x+y+z$.

If, instead, we substitute for θ the onefold value $= \dfrac{2}{\alpha\beta\gamma}$, we find

$$-(-x+y+z)(x-y+z)(x+y-z) + xyz = 0,$$

or, what is the same thing,

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz = 0,$$

which is the one-with-twofold centre cubic.

32. Recollecting that

$$k = \frac{-\frac{1}{4}\theta^2}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)},$$

we deduce for the twofold value of k

$$k_1 = k_2 = \frac{-\frac{1}{4}\alpha^3\beta^3\gamma^3}{(\beta\gamma-\alpha^2)(\gamma\alpha-\beta^2)(\alpha\beta-\gamma^2)},$$

$$= \frac{-\frac{1}{4}\alpha^3\beta^3\gamma^3}{(\beta\gamma+\gamma\alpha+\alpha\beta)^3},$$

$$= \frac{2\alpha^3\beta^3\gamma^3}{(\alpha^2+\beta^2+\gamma^2)^3},$$

and for the one-with-twofold value,

$$k_3 = \frac{-\frac{1}{4}\alpha^3\beta^3\gamma^3}{(2\alpha^2+\beta\gamma)(2\beta^2+\gamma\alpha)(2\gamma^2+\alpha\beta)}$$

$$= \frac{\frac{1}{4}\alpha^3\beta^3\gamma^3}{(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2}.$$

Article Nos. 33 to 38, *relating to the Tangents at a Node or Critic Centre.*

33. I proceed to investigate the equation of the tangents at the node of the curve

$$xyz + k(x+y+z)^2(\lambda x + \mu y + \nu z) = 0;$$

it will be recollected that if x, y, z are the coordinates of the node, then we have

$$x : y : z : x+y+z : \lambda x + \mu y + \nu z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu} : \frac{2}{\theta} : 1.$$

Representing for a moment the equation of the curve by $U=0$, then the second derived functions of U are

$$k\,.\,2(\lambda x + \mu y + \nu z) + 4k(x+y+z)\lambda,$$
$$k\,.\,2(\lambda x + \mu y + \nu z) + 4k(x+y+z)\mu,$$
$$k\,.\,2(\lambda x + \mu y + \nu z) + 4k(x+y+z)\nu,$$

$$x + k\,.\,2(\lambda x + \mu y + \nu z) + 2k(x+y+z)(\mu+\nu),$$
$$y + k\,.\,2(\lambda x + \mu y + \nu z) + 2k(x+y+z)(\nu+\lambda),$$
$$z + k\,.\,2(\lambda x + \mu y + \nu z) + 2k(x+y+z)(\lambda+\mu),$$

or calling these (a, b, c, f, g, h) respectively, and substituting the values $x = \dfrac{1}{\theta+1}$, &c., we find

$$a = 2k + \frac{8k\lambda}{\theta}, \quad = \frac{2k}{\theta}(\theta + 4\lambda),$$

with the like values for b, c; and

$$f = \frac{1}{\theta+\lambda} + 2k + \frac{4k}{\theta}(\mu+\nu) = \frac{2k}{\theta}\left(\frac{\theta}{\theta+\lambda}\,\frac{1}{2k} + \theta + 2\mu + 2\nu\right),$$

where the term in ( ) is

$$= \frac{\theta}{\theta+\lambda}\cdot\frac{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}{-\frac{1}{2}\theta^2} + \theta + 2\mu + 2\nu,$$

$$= \frac{-2(\theta+\mu)(\theta+\nu)}{\theta} + \theta + 2\mu + 2\nu,$$

$$= -2\theta - 2\mu - 2\nu - \frac{2\mu\nu}{\theta} + \theta + 2\mu + 2\nu,$$

$$= -\theta - \frac{2\mu\nu}{\theta},$$

that is

$$f = \frac{2k}{\theta}\left(-\theta - \frac{2\mu\nu}{\theta}\right),$$

with the like values for g and h; or omitting the common factor $\frac{2k}{\theta}$, we have

$$(a,\ b,\ c,\ f,\ g,\ h) = \left(\theta + 4\lambda,\ \theta + 4\mu,\ \theta + 4\nu,\ -\theta - \frac{2\mu\nu}{\theta},\ -\theta - \frac{2\nu\lambda}{\theta},\ -\theta - \frac{2\lambda\mu}{\theta}\right),$$

and thence, taking now x, y, z as current coordinates, the equation of the tangents at the node is

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2 = 0.$$

34. Substituting for λ, μ, ν the values $\alpha^{-3}, \beta^{-3}, \gamma^{-3}$, and for θ the twofold value $-\frac{1}{\alpha\beta\gamma}$, the equation of the tangents at the twofold centre becomes

$$\left(\beta^2\gamma^2(4\beta\gamma - \alpha^2), \ldots,\ \alpha^2\beta\gamma(\beta\gamma + 2\alpha^2), \ldots\right)(x,\ y,\ z)^2 = 0,$$

which is at once reduced to

$$\left(\beta^2\gamma^2(\beta - \gamma)^2, \ldots,\ \alpha^2\beta\gamma(\gamma - \alpha)(\alpha - \beta), \ldots\right)(x,\ y,\ z)^2 = 0,$$

or, what is the same thing,

$$\{\beta\gamma(\beta - \gamma)x + \gamma\alpha(\gamma - \alpha)y + \alpha\beta(\alpha - \beta)z\}^2 = 0,$$

which shows that the twofold centre is a cusp, and that the tangent is

$$\beta\gamma(\beta - \gamma)x + \gamma\alpha(\gamma - \alpha)y + \alpha\beta(\alpha - \beta)z,$$

or, what is the same thing,

$$(\beta - \gamma)\frac{x}{\alpha} + \quad (\gamma - \alpha)\frac{y}{\beta} + \quad (\alpha - \beta)\frac{z}{\gamma} = 0.$$

35. Writing in like manner $\lambda, \mu, \nu = \alpha^{-3}, \beta^{-3}, \gamma^{-3}$, and θ for the one-with-twofold value $= \frac{2}{\alpha\beta\gamma}$, we find for the equation of the tangents at the one-with-twofold centre

$$\left(2\beta^2\gamma^2(2\beta\gamma + \alpha^2), \ldots, \qquad - \alpha^2\beta\gamma(2\beta\gamma + \alpha^2), \ldots\right)(x,\ y,\ z)^2 = 0,$$

which may be reduced to

$$\left(2\beta^2\gamma^2(2\beta\gamma + \alpha^2), \ldots,\ \alpha^2\beta\gamma(2\gamma\alpha + \beta^2 + 2\alpha\beta + \gamma^2), \ldots\right)(x,\ y,\ z)^2 = 0,$$

or, what is the same thing,

$$(\beta\gamma x + \gamma\alpha y + \alpha\beta z)\{(2\beta\gamma + \alpha^2)\beta\gamma x + (2\gamma\alpha + \beta^2)\gamma\alpha y + (2\alpha\beta + \gamma^2)\alpha\beta z\} = 0.$$

Hence at the one-with-twofold centre the equation of one of the tangents is

$$(2\beta\gamma + \alpha^2)\beta\gamma x + (2\gamma\alpha + \beta^2)\gamma\alpha y + (2\alpha\beta + \gamma^2)\alpha\beta z = 0,$$

or, as this may otherwise be written,

$$(2\beta\gamma + \alpha^2)\frac{x}{\alpha} + (2\gamma\alpha + \beta^2)\frac{y}{\beta} + (2\alpha\beta + \gamma^2)\frac{z}{\gamma} = 0.$$

36. The equation of the other tangent is

$$\beta\gamma x + \gamma\alpha y + \alpha\beta z = 0,$$

or, what is the same thing,

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0.$$

This is in fact equivalent to

$$\begin{vmatrix} x, & y, & z \\ \alpha^2, & \beta^2, & \gamma^2 \\ \alpha^2(\beta-\gamma), & \beta^2(\gamma-\alpha), & \gamma^2(\alpha-\beta) \end{vmatrix} = 0;$$

for, developing the determinant, we find

$$x\,.\,\beta^2\gamma^2(2\alpha-\beta-\gamma) + y\,.\,\gamma^2\alpha^2(2\beta-\gamma-\alpha) + z\,.\,\alpha^2\beta^2(2\gamma-\alpha-\beta) = 0,$$

or, what is the same thing,

$$\alpha^2\beta^2\gamma^2\left(\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma}\right) = 0;$$

hence the line

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0,$$

which is one of the tangents at the one-with-twofold centre, is also the line joining this point with the twofold centre.

37. The equation of the tangents at a critic centre or node may be obtained in a different form, involving, instead of the parameter θ, the coordinates (x, y, z) of the node. We have

$$(\theta+\lambda)\,x = \frac{2}{\theta}(\ \ x+y+z),$$

or, what is the same thing,

$$\lambda x = \frac{2}{\theta}(-x+y+z),$$

and similarly

$$\mu y = \frac{2}{\theta}(\ \ x-y+z),$$

$$\nu z = \frac{2}{\theta}(\ \ x+y-z),$$

thence also

$$(\theta+4\lambda)\,x = \theta\,(x-2x+2y+2z) = \theta\,(-x+2y+2z),$$

and

$$\left(\theta + \frac{2\mu\nu}{\theta}\right) yz = \theta\,\{yz + \tfrac{1}{2}(x-y+z)(x+y-z)\},$$

$$= \tfrac{1}{2}\,\theta\,\{2yz + x^2 - (y-z)^2\},$$

$$= \tfrac{1}{2}\,\theta\,(x^2-y^2-z^2+4yz)$$

from which we obtain

$$\begin{aligned} a : b : c : f : g : h \\ = \ & 2yz\,(-\ x+2y+2z) \\ : \ & 2zx\,(\ 2x-\ y+\ z) \\ : \ & 2xy\,(\ 2x+2y-\ z) \\ : \ & -x\,(\ x^2-\ y^2-\ z^2+4yz) \\ : \ & -y\,(-x^2+\ y^2-\ z^2+4zx) \\ : \ & -z\,(-x^2-\ y^2+\ z^2+4xy), \end{aligned}$$

which are the required new forms.

38. We have

$$bc-f^2=4x^2yz\,(2x-y+z)\,(2x+2y-z)-x^2\,(x^2-y^2-z^2+4yz)^2$$

$$=(x+y+z)^2\,(x^2+y^2+z^2-2yz-2zx-2xy),$$

which is $=0$, if $x+y+z=0$, or if $x^2-2x\,(y+z)+(y-z)^2=0$. In the former case, viz. if $x+y+z=0$, we find $a=b=c=f=g=h=-6xyz$, and therefore

$$(a,\ b,\ c,\ f,\ g,\ h \between x',\ y',\ z')^2=-6xyz\,(x'+y'+z')^2,$$

but this corresponds merely to the value $k=\infty$, for which the cubic is

$$(x+y+z)^2\,(\lambda x+\mu y+\nu z)=0,$$

which is not a proper cuspidal curve. In the latter case, or where

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

or, what is the same thing, $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$, we have a proper cuspidal curve.

Article Nos. 39 to 43, *relating to the Triangle of the Critic Centres.*

39. The equation

$$\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}=0,$$

is satisfied by substituting therein

$$x : y : z=\frac{1}{\theta_2+\lambda} : \frac{1}{\theta_2+\mu} : \frac{1}{\theta_2+\nu},\ \text{ or } x : y : z=\frac{1}{\theta_3+\lambda} : \frac{1}{\theta_3+\mu} : \frac{1}{\theta_3+\nu};$$

in fact, for the first set of values the equation becomes

$$\frac{\lambda}{(\theta_1+\lambda)\,(\theta_2+\lambda)}+\frac{\mu}{(\theta_1+\mu)\,(\theta_2+\mu)}+\frac{\nu}{(\theta_1+\nu)\,(\theta_2+\nu)}=0,$$

or as this may be written

$$\left(\frac{\lambda}{\theta_1+\lambda}+\frac{\mu}{\theta_1+\mu}+\frac{\nu}{\theta_1+\nu}\right)-\left(\frac{\lambda}{\theta_2+\lambda}+\frac{\mu}{\theta_2+\mu}+\frac{\nu}{\theta_2+\nu}\right)=0,$$

that is, $1-1=0$, and similarly for the second set of values. Hence the equation in question is that of the line joining the critic centres corresponding to the roots θ_2 and θ_3. Hence

$$\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}=0,$$

$$\frac{\lambda x}{\theta_2+\lambda}+\frac{\mu y}{\theta_2+\mu}+\frac{\nu z}{\theta_2+\nu}=0,$$

$$\frac{\lambda x}{\theta_3+\lambda}+\frac{\mu y}{\theta_3+\mu}+\frac{\nu z}{\theta_3+\nu}=0,$$

are the equations of the sides of the triangle formed by the three critic centres.

40. It is to be remarked that the line

$$\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}=0,$$

is the polar of the critic centre $\left(\frac{1}{\theta_1+\lambda},\ \frac{1}{\theta_1+\mu},\ \frac{1}{\theta_1+\nu}\right)$ in regard to the twofold centre conic

$$x^2+y^2+z^2-2yz-2zx-2xy=0:$$

in fact, forming the equation of the polar in question, this is,

$$\left(-\frac{1}{\theta_1+\lambda}+\frac{1}{\theta_1+\mu}+\frac{1}{\theta_1+\nu}\right)x+\left(\frac{1}{\theta_1+\lambda}-\frac{1}{\theta_1+\mu}+\frac{1}{\theta_1+\nu}\right)y+\left(\frac{1}{\theta_1+\lambda}+\frac{1}{\theta_1+\mu}-\frac{1}{\theta_1+\nu}\right)z=0;$$

but from the equation in θ,

$$-\frac{1}{\theta_1+\lambda}+\frac{1}{\theta_1+\mu}+\frac{1}{\theta_1+\nu}=\frac{2}{\theta_1}-\frac{2}{\theta_1+\lambda}=\frac{2\lambda}{\theta_1+\lambda},$$

and the like for the coefficients of y and z; this proves the theorem, and it thus appears that the critic centres are conjugate poles in regard to the twofold centre conic.

Article Nos. 41 to 50. *Transformation of the Equation of the Nodal Tangents; the Three-Centre Conic.*

41. Writing as above,

$$l_1=\theta_2-\theta_3,\quad l_2=\theta_3-\theta_1,\quad l_3=\theta_1-\theta_2,$$

$$\Theta_1=(\theta_1+\lambda)(\theta_1+\mu)(\theta_1+\nu),$$

$$\Theta_2=(\theta_2+\lambda)(\theta_2+\mu)(\theta_2+\nu),$$

$$\Theta_3=(\theta_3+\lambda)(\theta_3+\mu)(\theta_3+\nu).$$

I put for greater convenience

$$X = -\frac{\Theta_1}{l_2 l_3}\left(\frac{\lambda x}{\theta_1+\lambda} + \frac{\mu y}{\theta_1+\mu} + \frac{\nu z}{\theta_1+\nu}\right),$$

$$Y = -\frac{\Theta_2}{l_3 l_1}\left(\frac{\lambda x}{\theta_2+\lambda} + \frac{\mu y}{\theta_2+\mu} + \frac{\nu z}{\theta_2+\nu}\right),$$

$$Z = -\frac{\Theta_3}{l_1 l_2}\left(\frac{\lambda x}{\theta_3+\lambda} + \frac{\mu y}{\theta_3+\mu} + \frac{\nu z}{\theta_3+\nu}\right),$$

so that $X=0$, $Y=0$, $Z=0$ are the equations of the sides of the triangle formed by the critic centres.

42. Then X, Y, Z may if we please be considered as new coordinates replacing the original coordinates x, y, z; the relation between the two sets being given by the equations last written down; the values of x, y, z in terms of X, Y, Z are given by the converse system

$$x = \frac{1}{\theta_1+\lambda}X + \frac{1}{\theta_2+\lambda}Y + \frac{1}{\theta_3+\lambda}Z,$$

$$y = \frac{1}{\theta_1+\mu}X + \frac{1}{\theta_2+\mu}Y + \frac{1}{\theta_3+\mu}Z,$$

$$z = \frac{1}{\theta_1+\nu}X + \frac{1}{\theta_2+\nu}Y + \frac{1}{\theta_3+\nu}Z.$$

43. To show the identity of the two systems, I start from the last-mentioned one; this gives

$$\left|\begin{matrix} x, & \dfrac{1}{\theta_2+\lambda}, & \dfrac{1}{\theta_3+\lambda} \\ y, & \dfrac{1}{\theta_2+\mu}, & \dfrac{1}{\theta_3+\mu} \\ z, & \dfrac{1}{\theta_2+\nu}, & \dfrac{1}{\theta_3+\nu} \end{matrix}\right| = X \left|\begin{matrix} \dfrac{1}{\theta_1+\lambda}, & \dfrac{1}{\theta_2+\lambda}, & \dfrac{1}{\theta_3+\lambda} \\ \dfrac{1}{\theta_1+\mu}, & \dfrac{1}{\theta_2+\mu}, & \dfrac{1}{\theta_3+\mu} \\ \dfrac{1}{\theta_1+\nu}, & \dfrac{1}{\theta_2+\nu}, & \dfrac{1}{\theta_3+\nu} \end{matrix}\right|,$$

where the coefficient of X is

$$= \frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)(\theta_2-\theta_3)(\theta_3-\theta_1)(\theta_1-\theta_3)}{(\theta_1+\lambda)(\theta_1+\mu)(\theta_1+\nu)(\theta_2+\lambda)(\theta_2+\mu)(\theta_2+\nu)(\theta_3+\lambda)(\theta_3+\mu)(\theta_3+\nu)},$$

or, what is the same thing,

$$= \frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)\, l_1 l_2 l_3}{\Theta_1\Theta_2\Theta_3}.$$

The first side is a linear function of x, y, z which vanishes for

$$x : y : z = \frac{1}{\theta_2+\lambda} : \frac{1}{\theta_2+\mu} : \frac{1}{\theta_2+\nu},$$

and for

$$x : y : z = \frac{1}{\theta_3+\lambda} : \frac{1}{\theta_3+\mu} : \frac{1}{\theta_3+\nu};$$

and hence it is of the form

$$K\left(\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}\right),$$

and by comparing the coefficients of x we have

$$K\frac{\lambda}{\theta_1+\lambda} = \frac{1}{(\theta_2+\mu)(\theta_3+\nu)} - \frac{1}{(\theta_2+\nu)(\theta_3+\mu)} = \frac{(\theta_2-\theta_3)(\mu-\nu)}{(\theta_2+\mu)(\theta_2+\nu)(\theta_3+\mu)(\theta_3+\nu)},$$

$$= \frac{l_1(\mu-\nu)(\theta_2+\lambda)(\theta_3+\lambda)}{\Theta_2\Theta_3},$$

that is

$$K = \frac{l_1(\mu-\nu)(\theta_1+\lambda)(\theta_2+\lambda)(\theta_3+\lambda)}{\lambda\Theta_2\Theta_3},$$

and it is easy to see that

$$(\theta_1+\lambda)(\theta_2+\lambda)(\theta_3+\lambda) = -\lambda(\nu-\lambda)(\lambda-\mu),$$

so that

$$K = \frac{-l_1(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{\Theta_2\Theta_3},$$

and the equation becomes

$$X = \frac{-\Theta_1}{l_2 l_3}\left(\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}\right),$$

which is right; and similarly for the values of Y and Z.

44. The equation of the tangents at the node corresponding to the root θ_1 is

$$\left(\theta_1+4\lambda,\ \theta_1+4\mu,\ \theta_1+4\nu,\ -\theta_1-\frac{2\mu\nu}{\theta_1},\ -\theta_1-\frac{2\nu\lambda}{\theta_1},\ -\theta_1-\frac{2\lambda\mu}{\theta_1}\right)(x,\ y,\ z)^2 = 0;$$

and substituting for x, y, z their values in terms of X, Y, Z, it appears in the first place that the coefficients of X^2, XY, XZ, YZ, all of them vanish.

45. In fact

$$\text{coeff. } X^2 = \left(\theta_1+4\lambda, \ldots, -\theta_1-\frac{2\mu\nu}{\theta_1}, \ldots\right)\left(\frac{1}{\theta_1+\lambda}, \frac{1}{\theta_1+\mu}, \frac{1}{\theta_1+\nu}\right)^2,$$

$$= \Sigma\frac{\theta_1+4\lambda}{(\theta_1+\lambda)^2} - 2\Sigma\frac{\theta_1+\dfrac{2\mu\nu}{\theta_1}}{(\theta_1+\mu)(\theta_1+\nu)}.$$

First term is

$$= \Sigma\frac{1}{\theta_1+\lambda} + 3\Sigma\frac{\lambda}{(\theta_1+\lambda)^2},$$

$$= \frac{2}{\theta_1} + \frac{3}{\Theta_1}\{3\theta_1^2 - (\mu\nu+\nu\lambda+\lambda\mu)\},$$

where the value in question for $\Sigma \dfrac{\lambda}{(\theta_1+\lambda)^2}$ is most readily found from the identical equation

$$\frac{\lambda}{\theta+\lambda}+\frac{\mu}{\theta+\mu}+\frac{\nu}{\theta+\nu}-1=-\frac{\theta^3-\theta(\mu\nu+\nu\lambda+\lambda\mu)-\lambda\mu\nu}{\Theta}$$

by differentiating and then writing $\theta=\theta_1$.

Second term is

$$=-\frac{2}{\Theta_1}\Sigma(\theta_1+\lambda)\left(\theta_1+\frac{2\mu\nu}{\theta_1}\right),$$

$$=-\frac{2}{\Theta_1}\Sigma\left(\theta_1^2+\lambda\theta_1+2\mu\nu+\frac{2\lambda\mu\nu}{\theta_1}\right),$$

$$=-\frac{2}{\Theta_1}\left\{3\theta_1^2+(\lambda+\mu+\nu)\,\theta_1+2(\mu\nu+\nu\lambda+\lambda\mu)+\frac{6\lambda\mu\nu}{\theta_1}\right\}.$$

Whole is $=\dfrac{1}{\Theta_1}$ multiplied into

$$\frac{2}{\theta_1}\Theta_1+3\{3\theta_1^2-(\mu\nu+\nu\lambda+\lambda\mu)\}$$

$$-2\left\{3\theta_1^2(\lambda+\mu+\nu)\,\theta_1+2(\mu\nu+\nu\lambda+\lambda\mu)+\frac{6\lambda\mu\nu}{\theta_1}\right\},$$

$$=\frac{5}{\theta_1}\{\theta_1^3-(\mu\nu+\nu\lambda+\lambda\mu)\,\theta_1-2\lambda\mu\nu\},$$

which is $=0$.

46. We have next

$$\text{coeff. } XY=\left(\theta_1+4\lambda,\ldots,-\theta_1-\frac{2\mu\nu}{\theta_1},\ldots\right)\left(\frac{1}{\theta_1+\lambda},\frac{1}{\theta_1+\mu},\frac{1}{\theta_1+\nu}\right)\left(\frac{1}{\theta_2+\lambda},\frac{1}{\theta_2+\mu},\frac{1}{\theta_2+\nu}\right)$$

$$=\Sigma\frac{\theta_1+4\lambda}{(\theta_1+\lambda)(\theta_2+\lambda)}-\Sigma\left(\theta_1+\frac{2\mu\nu}{\theta_1}\right)\left\{\frac{1}{(\theta_1+\mu)(\theta_2+\nu)}+\frac{1}{(\theta_1+\nu)(\theta_2+\lambda)}\right\}.$$

First term is

$$=\Sigma\frac{1}{\theta_2+\lambda}+3\Sigma\frac{\lambda}{(\theta_1+\lambda)(\theta_2+\lambda)},$$

$$=\Sigma\frac{1}{\theta_2+\lambda}+\frac{3}{\theta_1-\theta_2}\Sigma\left(\frac{\lambda}{\theta_1+\lambda}-\frac{\lambda}{\theta_2+\lambda}\right),$$

$$=\frac{2}{\theta_2},$$

$\left(\text{since } \Sigma\dfrac{\lambda}{\theta_1+\lambda}=1=\Sigma\dfrac{\lambda}{\theta_2+\lambda}\right).$

Second term, writing it out in full and collecting the terms which contain $\frac{1}{\theta_1+\lambda}$, is

$$=-\Sigma\,\frac{\theta_1}{\theta_1+\lambda}\left(\frac{1}{\theta_2+\mu}+\frac{1}{\theta_2+\nu}\right)-2\Sigma\,\frac{\lambda}{\theta_1(\theta_1+\lambda)}\left(\frac{\mu}{\theta_2+\mu}+\frac{\nu}{\theta_2+\nu}\right),$$

whereof the first part is

$$=-\theta_1\Sigma\,\frac{1}{\theta_1+\lambda}\,\Sigma\,\frac{1}{\theta_2+\lambda}+\theta_1\Sigma\,\frac{1}{\theta_1+\lambda}\,\frac{1}{\theta_2+\lambda},$$

$$=-\theta_1\Sigma\,\frac{1}{\theta_1+\lambda}\Sigma\,\frac{1}{\theta_2+\lambda}+\frac{\theta_1}{\theta_1-\theta_2}\,\Sigma\left(\frac{1}{\theta_2+\lambda}-\frac{1}{\theta_1+\lambda}\right),$$

$$=-\theta_1\,.\,\frac{2}{\theta_1}\,.\,\frac{2}{\theta_2}+\frac{\theta_1}{\theta_1-\theta_2}\left(\frac{2}{\theta_2}-\frac{2}{\theta_1}\right),\;=-\frac{4}{\theta_2}+\frac{2}{\theta_2},$$

$$=-\frac{2}{\theta_2}:$$

and the second part is

$$=-\frac{2}{\theta_1}\left\{\Sigma\,\frac{\lambda}{\theta_1+\lambda}\,\Sigma\,\frac{\lambda}{\theta_2+\lambda}-\Sigma\,\frac{\lambda^2}{(\theta_1+\lambda)(\theta_2+\lambda)}\right\},$$

$$=-\frac{2}{\theta_1}\left\{\Sigma\,\frac{\lambda}{\theta_1+\lambda}\,\Sigma\,\frac{\lambda}{\theta_2+\lambda}-\frac{1}{\theta_1-\theta_2}\,\Sigma\left(\frac{\lambda^2}{\theta_2+\lambda}-\frac{\lambda^2}{\theta_1+\lambda}\right)\right\};$$

and observing that

$$\Sigma\,\frac{\lambda^2}{\theta_1+\lambda}=\Sigma\,\frac{\{(\theta_1+\lambda)-\theta_1\}^2}{\theta_1+\lambda}=\Sigma\,(\theta_1+\lambda)-2\theta_1\Sigma_1+\theta_1^2\Sigma\,\frac{1}{\theta_1+\lambda},$$

$$=3\theta_1+(\lambda+\mu+\nu)-6\theta_1+\theta_1^2\,.\,\frac{2}{\theta_1},$$

$$=-\theta_1+\lambda+\mu+\nu,$$

with the like value for $\Sigma\,\frac{\lambda^2}{\theta_2+\lambda}$, the second part is

$$=-\frac{2}{\theta_1}\left\{1-\frac{1}{\theta_1-\theta_2}\,(-\theta_2+\theta_1)\right\},\;=-\frac{2}{\theta_1}\,(1-1),\;=0.$$

47. Hence the whole second term is $=-\frac{2}{\theta_2}$, and combining the two terms we have

$$\text{coeff. } XY=\frac{2}{\theta_2}-\frac{2}{\theta_2}=0.$$

In the same manner precisely it appears that

$$\text{coeff. } XZ=\qquad 0.$$

48. Next,

$$\text{coeff. } YZ = \left\{\theta_1 + 4\lambda, \ldots, -\left(\theta_1 + \frac{2\mu\nu}{\theta_1}\right), \ldots\right\}\left(\frac{1}{\theta_2+\lambda}, \frac{1}{\theta_2+\mu}, \frac{1}{\theta_2+\nu}\right)\left(\frac{1}{\theta_3+\lambda}, \frac{1}{\theta_3+\mu}, \frac{1}{\theta_3+\nu}\right).$$

First term is

$$= \frac{\theta_1}{\theta_2-\theta_3}\Sigma\left(\frac{1}{\theta_3+\lambda} - \frac{1}{\theta_2+\lambda}\right)$$

$$+ \frac{4}{\theta_2-\theta_3}\Sigma\left(\frac{\lambda}{\theta_3+\lambda} - \frac{\lambda}{\theta_2+\lambda}\right),$$

which is

$$= \frac{\theta_1}{\theta_2-\theta_3}\left(\frac{2}{\theta_3} - \frac{2}{\theta_2}\right) + \frac{4}{\theta_2-\theta_3}(1-1), \ = \frac{2\theta_1}{\theta_2\theta_3}.$$

Second term, writing it out at full length and rearranging the parts, is easily seen to be

$$= -\theta_1\left\{\Sigma\frac{1}{\theta_2+\lambda}\Sigma\frac{1}{\theta_3+\lambda} - \Sigma\frac{1}{(\theta_2+\lambda)(\theta_3+\lambda)}\right\}$$

$$-\frac{2}{\theta_1}\left\{\Sigma\frac{\lambda}{\theta_2+\lambda}\Sigma\frac{\lambda}{\theta_3+\lambda} - \Sigma\frac{\lambda^2}{(\theta_2+\lambda)(\theta_3+\lambda)}\right\},$$

where the first part is

$$= -\theta_1\left\{\Sigma\frac{1}{\theta_2+\lambda}\Sigma\frac{1}{\theta_3+\lambda} - \frac{1}{\theta_2-\theta_3}\Sigma\left(\frac{1}{\theta_3+\lambda} - \frac{1}{\theta_2+\lambda}\right)\right\},$$

$$= -\theta_1\left\{\frac{2}{\theta_2}\cdot\frac{2}{\theta_3} - \frac{1}{\theta_2-\theta_3}\left(\frac{2}{\theta_3} - \frac{2}{\theta_2}\right)\right\}, \ = -\theta_1\left(\frac{4}{\theta_2\theta_3} - \frac{2}{\theta_2\theta_3}\right),$$

$$= -\frac{2\theta_1}{\theta_2\theta_3},$$

and the second part is

$$= -\frac{2}{\theta_1}\left\{\Sigma\frac{\lambda}{\theta_2+\lambda}\Sigma\frac{\lambda}{\theta_3+\lambda} - \frac{1}{\theta_2-\theta_3}\Sigma\left(\frac{\lambda^2}{\theta_3+\lambda} - \frac{\lambda^2}{\theta_2+\lambda}\right)\right\},$$

$$= -\frac{2}{\theta_1}\left\{1 - \frac{1}{\theta_2-\theta_3}(\theta_2-\theta_3)\right\}, \ = 0,$$

so that the whole second term is

$$= -\frac{2\theta_1}{\theta_2\theta_3},$$

whence combining the two terms we have

$$\text{coeff. } YZ = \frac{2\theta_1}{\theta_2\theta_3} - \frac{2\theta_1}{\theta_2\theta_3} = 0.$$

49. We have now to find the coefficients of Y^2 and Z^2.

$$\text{coeff. } Y^2 = \left(\theta_1 + 4\lambda, \ldots, -\theta_1 - \frac{2\mu\nu}{\theta_1}, \ldots\right)\left(\frac{1}{\theta_2+\lambda}, \frac{1}{\theta_2+\mu}, \frac{1}{\theta_2+\nu}\right)^2,$$

$$= \Sigma \frac{\theta_1 + 4\lambda}{(\theta_2+\lambda)^2} - 2\Sigma \frac{\theta_1 + \dfrac{2\mu\nu}{\theta_1}}{(\theta_2+\mu)(\theta_2+\nu)},$$

$$= \Sigma \frac{\theta_1 - \theta_2 + \theta_2 + 4\lambda}{(\theta_2+\lambda)^2} - 2\Sigma \frac{\theta_1 - \theta_2 + \dfrac{2\mu\nu}{\theta_1} - \dfrac{2\mu\nu}{\theta_2} + \theta_2 + \dfrac{2\mu\nu}{\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)},$$

and observing that the terms

$$\Sigma \frac{\theta_2 + 4\lambda}{(\theta_2+\lambda)^2} - 2\Sigma \frac{\theta_2 + \dfrac{2\mu\nu}{\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)}$$

only differ from those of coeff. X^2 by having θ_2 in the place of θ_1 and are therefore $=0$, we have

$$\text{coeff. } Y^2 = \Sigma \frac{\theta_1 - \theta_2}{(\theta_2+\lambda)^2} - 2\Sigma \frac{\theta_1 - \theta_2 + \dfrac{2\mu\nu}{\theta_1} - \dfrac{2\mu\nu}{\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)}$$

$$= (\theta_1 - \theta_2)\left\{\Sigma \frac{1}{(\theta_2+\lambda)^2} - 2\Sigma \frac{1 - \dfrac{2\mu\nu}{\theta_1\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)}\right\}.$$

Here

$$\Sigma \frac{1}{(\theta_2+\lambda)^2} = \Sigma\left\{\frac{-1}{(\theta_2+\lambda)(\theta_2+\mu)} - \frac{1}{(\theta_2+\lambda)(\theta_2+\nu)} + \frac{2}{\theta_2(\theta_2+\lambda)}\right\}.$$

$$= \frac{1}{\Theta_2}\Sigma\left\{-(\theta_2+\nu) - (\theta_2+\mu) + \frac{2}{\theta_2}(\theta_2+\mu)(\theta_2+\nu)\right\},$$

$$= \frac{1}{\Theta_2}\Sigma\left(\mu + \nu + \frac{2\mu\nu}{\theta_2}\right),$$

$$= \frac{1}{\Theta_2}\left\{2(\lambda+\mu+\nu) + \frac{2(\mu\nu + \nu\lambda + \lambda\mu)}{\theta_2}\right\},$$

and

$$-2\Sigma \frac{1 - \dfrac{2\mu\nu}{\theta_1\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)} = -\frac{2}{\Theta_2}\Sigma\left(1 - \frac{2\mu\nu}{\theta_1\theta_2}\right)(\theta_2+\lambda),$$

$$= -\frac{2}{\Theta_2}\Sigma\left(\theta_2 + \lambda - \frac{2\mu\nu}{\theta_1} - \frac{2\lambda\mu\nu}{\theta_1\theta_2}\right),$$

$$= -\frac{2}{\Theta_2}\left\{3\theta_2 + \lambda + \mu + \nu - \frac{2(\mu\nu + \nu\lambda + \lambda\mu)}{\theta_1} - \frac{6\lambda\mu\nu}{\theta_1\theta_2}\right\},$$

and hence, substituting for $\theta_1 - \theta_2$ its value $= l_3$,

$$\text{coeff. } Y^2 = \frac{2l_3}{\Theta_2}\left\{-3\theta_2 + \frac{\mu\nu+\nu\lambda+\lambda\mu}{\theta_2} + \frac{2(\mu\nu+\nu\lambda+\lambda\mu)}{\theta_1} + \frac{6\lambda\mu\nu}{\theta_1\theta_2}\right\},$$

$$= \frac{2l_3}{\Theta_2\theta_1\theta_2}\{-3\theta_1\theta_2^2 + (\mu\nu+\nu\lambda+\lambda\mu)(\theta_1+2\theta_2) + 6\lambda\mu\nu\};$$

but we have $\theta_1+\theta_2+\theta_3=0$, whence $\theta_1+2\theta_2=\theta_2-\theta_3=l_1$; $\mu\nu+\nu\lambda+\lambda\mu=-(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)$, $2\lambda\mu\nu=\theta_1\theta_2\theta_3$, and therefore $-3\theta_1\theta_2^2+6\lambda\mu\nu=-3\theta_1\theta_2^2+3\theta_1\theta_2\theta_3=-3\theta_1\theta_2(\theta_2-\theta_3)=-3l_1\theta_1\theta_2$; and hence

$$\text{coeff. } Y^2 = \frac{2l_1l_3}{\Theta_2\theta_1\theta_2}\{-3\theta_1\theta_2 - \theta_1\theta_2 - \theta_3(\theta_1+\theta_2)\},$$

$$= \frac{2l_1l_3}{\Theta_2\theta_1\theta_2}\{-4\theta_1\theta_2 + (\theta_1+\theta_2)^2\},$$

$$= \frac{2l_1l_3}{\Theta_2\theta_1\theta_2}(\theta_1-\theta_2)^2;$$

that is

$$\text{coeff. } Y^2 = \frac{2l_1l_3^3}{\Theta_2\theta_1\theta_2},$$

and, by merely interchanging θ_2 and θ_3,

$$\text{coeff. } Z^2 = \frac{2l_1l_2^3}{\Theta_3\theta_1\theta_3}.$$

50. Hence the equation of the tangents is

$$\frac{2l_1l_3^3}{\Theta_2\theta_1\theta_2}Y^2 + \frac{2l_1l_2^3}{\Theta_3\theta_1\theta_3}Z^2 = 0,$$

or, what is the same thing,

$$\frac{1}{l_2^3\theta_2\Theta_2}Y^2 + \frac{1}{l_3^3\theta_3\Theta_3}Z^2 = 0,$$

or putting

$$A = \frac{1}{l_1^3\theta_1\Theta_1},\ B = \frac{1}{l_2^3\theta_2\Theta_2},\ C = \frac{1}{l_3^3\theta_3\Theta_3},$$

the equation of the tangents at the node corresponding to θ_1 is $BY^2+CZ^2=0$. And hence the equations of the tangents at the three nodes respectively are

$$\begin{aligned} BY^2 + CZ^2 &= 0,\\ AX^2 \quad . \quad + CZ^2 &= 0,\\ AX^2 + BY^2 \quad . \quad &= 0; \end{aligned}$$

that is, the nodes or critic centres are conjugate poles in regard to a conic

$$AX^2 + BY^2 + CZ^2 = 0,$$

which is the three-centre conic; and the tangents at each node are the tangents from such node to the conic in question.

Article Nos. 51 and 52. *Special Case of the Three-Centre Conic.*

51. Write for a moment

$$X' = \frac{\lambda x}{\theta_1+\lambda} + \frac{\mu y}{\theta_1+\mu} + \frac{\nu z}{\theta_1+\nu},$$

$$Y' = \frac{\lambda x}{\theta_2+\lambda} + \frac{\mu y}{\theta_2+\mu} + \frac{\nu z}{\theta_2+\nu},$$

$$Z' = \frac{\lambda x}{\theta_3+\lambda} + \frac{\mu y}{\theta_3+\mu} + \frac{\nu z}{\theta_3+\nu},$$

so that

$$X = -\frac{\Theta_1}{l_2 l_3} X',\ Y = -\frac{\Theta_2}{l_3 l_1} Y',\ Z = -\frac{\Theta_3}{l_2 l_1} Z',$$

the equation of the three-centre conic expressed in terms of (X', Y', Z') is

$$\frac{\Theta_1}{l_1\theta_1} X'^2 + \frac{\Theta_2}{l_2\theta_2} Y'^2 + \frac{\Theta_3}{l_3\theta_3} Z'^2 = 0,$$

say

$$A'X'^2 + B'Y'^2 + C'Z'^2 = 0.$$

When $\theta_1 = \theta_2$, we have $C' = \infty$, $A' = -B'$, $X' = Y'$; by writing the equation in the form

$$(A'+B')X'^2 + B'(Y'^2 - X'^2) + C'Z'^2 = 0,$$

and observing that in the limit $Y'^2 - X'^2 = 2X'(Y' - X')$, we see that the equation will thus assume the form

$$(\theta_1-\theta_2) X'S + \frac{1}{\theta_1-\theta_2}\frac{\Theta_3}{\theta_3} Z'^2 = 0,$$

where

$$S, = \frac{A'+B'}{\theta_1-\theta_2} X' + 2B' \frac{Y'-X'}{\theta_1-\theta_2},$$

is a finite function; $X' = 0$ is the line joining the twofold centre and the one-with-twofold centre, $S = 0$ is the other tangent at the one-with-twofold centre, $Z' = 0$ the tangent at the twofold centre or cusp; the form $X'S + \infty\, Z'^2 = 0$ shows that the three-centre conic reduces itself to a pair of points, viz. the twofold centre or cusp, and the point where the tangent at the cusp is met by the other tangent (that is the tangent not passing through the cusp) at the one-with-twofold centre.

52. To verify the value of S I proceed as follows:

$$\frac{A'+B'}{\theta_1-\theta_2} = \frac{1}{\theta_1-\theta_2}\left\{\frac{\Theta_1}{\theta_1(\theta_2-\theta_3)} + \frac{\Theta_2}{\theta_2(\theta_3-\theta_1)}\right\},$$

$$= \frac{1}{(\theta_2-\theta_3)(\theta_3-\theta_1)} \cdot \frac{1}{\theta_1-\theta_2}\left\{\theta_3\left(\frac{\Theta_1}{\theta_1} - \frac{\Theta}{\theta_2}\right) - (\Theta_1 - \Theta_2)\right\},$$

$$= \frac{1}{9\theta_1^2(\theta_1-\theta_2)}\left\{2\theta_1\left(\frac{\Theta_1}{\theta_1} - \frac{\Theta_2}{\theta_2}\right) + (\Theta_1 - \Theta_2)\right\}:$$

$$\frac{\Theta_1}{\theta_1}-\frac{\Theta_2}{\theta_2}=\theta_1^2-\theta_2^2+(\lambda+\mu+\nu)(\theta_1-\theta_2)+\lambda\mu\nu\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right),$$

$$=(\theta_1-\theta_2)\left(\theta_1+\theta_2+\lambda+\mu+\nu-\frac{\lambda\mu\nu}{\theta_1\theta_2}\right),$$

$$=(\theta_1-\theta_2)(\lambda+\mu+\nu+3\theta_1);$$

$$\Theta_1-\Theta_2=\theta_1^3-\theta_2^3+(\lambda+\mu+\nu)(\theta_1^2-\theta_2^2)+(\mu\nu+\nu\lambda+\lambda\mu)(\theta_1-\theta_2),$$

$$=(\theta_1-\theta_2)\{\theta_1^2+\theta_1\theta_2+\theta_2^2+(\lambda+\mu+\nu)(\theta_1+\theta_2)+\mu\nu+\nu\lambda+\lambda\mu\},$$

$$=(\theta_1-\theta_2)\,.\,\{6\theta_1^2+2(\lambda+\mu+\nu)\,\theta_1\}:$$

and thence

$$\frac{A'+B'}{\theta_1-\theta_2}=\frac{4}{9\theta_1}(3\theta_1+\lambda+\mu+\nu).$$

Moreover

$$Y'-X'=(\theta_1-\theta_2)\left\{\frac{\lambda x}{(\theta_1+\lambda)(\theta_2+\lambda)}+\frac{\mu y}{(\theta_1+\mu)(\theta_2+\mu)}+\frac{\nu z}{(\theta_1+\nu)(\theta_2+\nu)}\right\},$$

$$=(\theta_1-\theta_2)\left\{\frac{\lambda x}{(\theta_1+\lambda)^2}+\frac{\mu y}{(\theta_1+\mu)^2}+\frac{\nu z}{(\theta_1+\nu)^2}\right\},$$

and hence

$$S=\frac{4}{9\theta_1}(3\theta_1+\lambda+\mu+\nu)\left(\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}\right)$$

$$-\frac{2\Theta_1}{3\theta_1^2}\left\{\frac{\lambda x}{(\theta_1+\lambda)^2}+\frac{\mu y}{(\theta_1+\mu)^2}+\frac{\nu z}{(\theta_1+\nu)^2}\right\},$$

in which we have only now to substitute $(\lambda, \mu, \nu)=(\alpha^{-3}, \beta^{-3}, \gamma^{-3})$ and $\theta_1=\dfrac{-1}{\alpha\beta\gamma}$. We have

$$\theta_1+\lambda=\frac{M}{\alpha^2},\quad \theta_1+\mu=\frac{M}{\beta^2},\quad \theta_1+\nu=\frac{M}{\gamma^2},$$

where $M=\dfrac{1}{\alpha\beta\gamma}(\beta\gamma-\alpha^2)=\dfrac{1}{\alpha\beta\gamma}(\beta\gamma+\gamma\alpha+\alpha\beta)$, and then observing that

$$3\theta_1+\lambda+\mu+\nu=M\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)=M\frac{(\beta\gamma+\gamma\alpha+\alpha\beta)^2}{\alpha^2\beta^2\gamma^2}=M^3,$$

the equation $S=0$ becomes

$$2(\beta\gamma+\gamma\alpha+\alpha\beta)\left(\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}\right)+3(\alpha x+\beta y+\gamma z)=0,$$

or, what is the same thing,

$$(2\beta\gamma+\alpha^2)\frac{x}{\alpha}+(2\gamma\alpha+\beta^2)\frac{y}{\beta}+(2\alpha\beta+\gamma^2)\frac{z}{\gamma}=0,$$

which agrees with a former result.

Article Nos. 53 to 55. *Transformation of the Equation of the Cubic.*

53. Let it be required to express the cubic

$$xyz + k(x+y+z)^2(\lambda x + \mu y + \nu z) = 0$$

in terms of the coordinates X, Y, Z. We have

$$x + y + z = 2\left(\frac{1}{\theta_1}X + \frac{1}{\theta_2}Y + \frac{1}{\theta_3}Z\right),$$

$$\lambda x + \mu y + \nu z = X + Y + Z,$$

and the equation therefore is

$$\Pi\left(\frac{X}{\theta_1+\lambda} + \frac{Y}{\theta_2+\mu} + \frac{Z}{\theta_2+\nu}\right) + 4k\left(\frac{X}{\theta_1} + \frac{Y}{\theta_2} + \frac{Z}{\theta_3}\right)^2 (X+Y+Z) = 0,$$

where Π denotes the product of the three factors obtained by writing λ, μ, ν successively in the place of λ.

For one of the nodal cubics we have

$$k = k_1 = \frac{-\frac{1}{4}\theta_1^2}{\Theta_1},$$

and the equation multiplied by Θ_1 is

$$\Pi\left\{X + \frac{Y(\theta_1+\lambda)}{\theta_2+\lambda} + \frac{Z(\theta_1+\lambda)}{\theta_3+\lambda}\right\} - \left(X + Y\frac{\theta_1}{\theta_2} + Z\frac{\theta_1}{\theta_3}\right)^2 (X+Y+Z) = 0,$$

which it is clear *à priori* must be of the form

$$KX\left(\frac{Y^2}{l_2^3\theta_2\Theta_2} + \frac{Z^2}{l_3^3\theta_3\Theta_3}\right) + \Theta_1\Pi\left(\frac{Y}{\theta_2+\lambda} + \frac{Z}{\theta_3+\lambda}\right) - \theta_1^2\left(\frac{Y}{\theta_2} + \frac{Z}{\theta_3}\right)^2 (Y+Z) = 0,$$

and there is in fact no difficulty in verifying that the coefficients of X^3, X^2Y, X^2Z, XYZ all of them vanish. To find K, comparing the coefficients of XY^2 we have

$$K \cdot \frac{1}{l_2^3\theta_2\Theta_2} = \Sigma\frac{(\theta_1+\lambda)(\theta_1+\mu)}{(\theta_2+\lambda)(\theta_2+\mu)} - \frac{\theta_1^2}{\theta_2^2} - 2\frac{\theta_1}{\theta_2},$$

that is

$$K \cdot \frac{1}{l_2^3\theta_2} = \Sigma(\theta_1+\lambda)(\theta_1+\mu)(\theta_2+\nu) - \frac{\theta_1}{\theta_2^2}(\theta_1+2\theta_2)\Theta_2,$$

$$= \Sigma(\theta_1+\lambda)(\theta_1+\mu)(\theta_2-\theta_1+\theta_1+\nu) - \frac{\theta_1}{\theta_2^2}(\theta_1+2\theta_2)\Theta_2,$$

$$= \left\{(\theta_2-\theta_1)\frac{2}{\theta_1}+3\right\}\Theta_1 - \frac{\theta_1}{\theta_2^2}(\theta_1+2\theta_2)\Theta_2,$$

$$= \frac{1}{\theta_1}(\theta_1+2\theta_2)\Theta_1 - \frac{\theta_1}{\theta_2^2}(\theta_1+2\theta_2)\Theta_3,$$

$$= \frac{l_1}{\theta_1}\Theta_1 - \frac{l_1\theta_1}{\theta_2^2}\Theta_2,$$

$$= \frac{l_1}{\theta_1\theta_2^2}(\theta_2^2\Theta_1 - \theta_1^2\Theta_2):$$

and

$$\theta_2^2\Theta_1-\theta_1^2\Theta_2=\theta_2^2\theta_1^3-\theta_1^2\theta_2^3+(\theta_2^2\theta_1-\theta_1^2\theta_2)(\mu\nu+\nu\lambda+\lambda\mu)+(\theta_2^2-\theta_1^2)\lambda\mu\nu,$$

$$=l_3\{\theta_1^2\theta_2^2-\theta_1\theta_2(\mu\nu+\nu\lambda+\lambda\mu)-(\theta_1+\theta_2)\lambda\mu\nu\},$$

$$=l_3\{\theta_1^2\theta_2^2+\theta_1\theta_2(\theta_1\theta_2+\theta_3\overline{\theta_1+\theta_2})-\tfrac{1}{2}(\theta_1+\theta_2)\theta_1\theta_2\theta_3\},$$

$$=l_3\theta_1\theta_2\{2\theta_1\theta_2+\tfrac{1}{2}\theta_3(\theta_1+\theta_2)\},$$

$$=\tfrac{1}{2}l_3\theta_1\theta_2\{4\theta_1\theta_2-(\theta_1+\theta_2)^2\},$$

$$=-\tfrac{1}{2}\theta_1\theta_2l_3^3;$$

so that we have

$$K\cdot\frac{1}{l_2^3\theta_2}=\frac{l_1}{\theta_1\theta_2^2}\cdot-\tfrac{1}{2}\theta_1\theta_2l_3^3=-\tfrac{1}{2}\frac{l_1l_3^2}{\theta_2},$$

that is

$$K=-\tfrac{1}{2}l_1l_2^3l_3^3,$$

and the equation of the nodal cubic is

$$-\tfrac{1}{2}l_1l_2^3l_3^3X\left(\frac{Y^2}{l_2^3\theta_2\Theta_3}+\frac{Z^2}{l_3^3\theta_3\Theta_3)}\right)+\Theta_1\Pi\left(\frac{Y}{\theta_2+\lambda}+\frac{Z}{\theta_3+\lambda}\right)-\theta_1^2\left(\frac{Y}{\theta_2}+\frac{Z}{\theta_3}\right)^2(Y+Z)=0.$$

54. To complete the reduction we have

$$\text{coeff. } Y^3=\frac{\Theta_1}{\Theta_2}-\frac{\theta_1^2}{\theta_2^2}=\frac{1}{\theta_2^2\Theta_2}(\theta_2^2\Theta_1-\theta_1^2\Theta_2);$$

$$\text{coeff. } Y^2Z=\Theta_1\Sigma\frac{1}{(\theta_2+\lambda)(\theta_2+\mu)(\theta_3+\nu)}-\frac{\theta_1^2}{\theta_2^2}-\frac{2\theta_1^2}{\theta_2\theta_3},$$

$$=\frac{\Theta_1}{\Theta_2}\Sigma\frac{\theta_2+\lambda}{\theta_3+\lambda}-\frac{\theta_1^2}{\theta_2^2\theta_3}(\theta_3+2\theta_2),$$

$$=\frac{\Theta_1}{\Theta_2}\frac{\theta_3+2\theta_2}{\theta_2}-\frac{\theta_1^2}{\theta_2^2\theta_3}(\theta_3+2\theta_2),$$

$$=-\frac{l_3}{\theta_3}\frac{\Theta_1}{\Theta_2}+\frac{l_3\theta_1^2}{\theta_2^2\theta_3},$$

$$=-\frac{l_3}{\theta_2^2\theta_3\Theta_2}(\theta_2^2\Theta_1-\theta_1^2\Theta_2),$$

so that substituting for $\theta_2^2\Theta_1-\theta_1^2\Theta_2$ its value $=-\frac{1}{2}\theta_1\theta_2l_3^3$, the terms in Y^3 and Y^2Z are

$$=-\tfrac{1}{2}\theta_1l_3^3\frac{1}{\theta_2\Theta_2}\left(Y^3-\frac{l_3}{\theta_2}Y^2Z\right),$$

and in like manner the terms in YZ^2 and Z^3 are

$$=+\tfrac{1}{2}\theta_1l_2^3\frac{1}{\theta_3\Theta_3}\left(\frac{l_2}{\theta_2}YZ^2+Z^3\right),$$

so that the terms in $(Y, Z)^3$ are

$$= -\tfrac{1}{2}\,\theta_1 l_2^3 l_3^3 \left[\frac{1}{l_2^3\theta_2\Theta_3}\left(Y^3 - \frac{l_3}{\theta_3}\,Y^2Z\right) - \frac{1}{l_3^3\theta_3\Theta_3}\left(\frac{l_2}{\theta_2}\,YZ^2 + Z^3\right)\right],$$

and the equation, omitting the factor $-\frac{1}{2}\,l_1^3 l_2^3$, is

$$l_1 X\left(\frac{Y^2}{l_2^3\theta_2\Theta_2} + \frac{Z^2}{l_3^3\theta_3\Theta_3}\right) + \theta_1\left[\frac{1}{l_2^3\theta_2\Theta_2}\left(Y^3 - \frac{l_3}{\theta_3}\,Y^2Z\right) - \frac{1}{l_3^3\theta_3\Theta_3}\left(\frac{l_2}{\theta_2}\,YZ^2 + Z^3\right)\right] = 0.$$

55. But the term in [ ] is

$$(Y - Z)\left(\frac{Y^2}{l_2^3\theta_2\Theta_2} + \frac{Z^2}{l_3^3\theta_3\Theta_3}\right) + \frac{1}{l_2^3\theta_2\Theta_2}\left(1 - \frac{l_3}{\theta_3}\right)Y^2Z + \frac{1}{l_3^3\theta_3\Theta_3}\left(-1 - \frac{l_2}{\theta_2}\right)YZ^2,$$

which is

$$= (Y - Z)\left(\frac{Y^2}{l_2^3\theta_2\Theta_3} + \frac{Z^2}{l_3^3\theta_3\Theta_3}\right) - \frac{2\theta_1}{\theta_2\theta_3}\,YZ\left(\frac{1}{l_2^3\Theta_2}\,Y - \frac{1}{l_3^3\Theta_3}\,Z\right),$$

and the equation of the nodal cubic is finally

$$\{l_1 X + \theta_1 (Y - Z)\}\left(\frac{Y^2}{l_2^3\theta_2\Theta_2} + \frac{Z^2}{l_3^3\theta_3\Theta_3}\right) - \frac{2\theta_1^{\,2}}{\theta_2\theta_3}\,YZ\left(\frac{Y}{l_2^3\Theta_2} - \frac{Z}{l_3^3\Theta_3}\right) = 0.$$

The lines $Y = 0$, $Z = 0$, $\dfrac{Y}{l_2^3\Theta_2} - \dfrac{Z}{l_3^3\Theta_3} = 0$ each pass through the node and meet the cubic in a third point; the three points of intersection lie in the line $l_1 X + \theta_1 (Y - Z) = 0$.

Article Nos. 56 to 66. *The Cubic Locus, Harmoconics and Harmonic Conic.*

56. Suppose that the line $\lambda x + \mu y + \nu z = 0$ passes through a given point (a, b, c), then we have

$$\lambda a + \mu b + \nu c = 0;$$

and observing that $\theta + \lambda$, $\theta + \mu$, $\theta + \nu$, θ are proportional to

$$\frac{1}{x},\ \frac{1}{y},\ \frac{1}{z},\ \frac{2}{x + y + z} \text{ respectively,}$$

we find

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} - \frac{2(a + b + c)}{x + y + z} = 0$$

the equation of a cubic curve, the locus of the critic centres corresponding to the several lines $\lambda x + \mu y + \nu z = 0$ which pass through the point (a, b, c). The cubic curve passes, it is clear, through the six points which are the angles of the quadrilateral

$$x = 0,\ y = 0,\ z = 0,\ x + y + z = 0.$$

57. If we take for (a, b, c) the coordinates of the point of intersection of the line $\lambda x+\mu y+\nu z=0$ with any one of the lines $x=0$, $y=0$, $z=0$, $x+y+z=0$, then in each case the cubic breaks up into the same line and a conic, viz. we have the conics

$$\frac{\nu}{y}-\frac{\mu}{z}-\frac{2(\nu-\mu)}{x+y+z}=0,$$

$$\frac{\lambda}{z}-\frac{\nu}{x}-\frac{2(\lambda-\nu)}{x+y+z}=0,$$

$$\frac{\mu}{x}-\frac{\lambda}{y}-\frac{2(\mu-\lambda)}{x+y+z}=0,$$

$$\frac{\mu-\nu}{x}+\frac{\nu-\lambda}{y}+\frac{\lambda-\mu}{z}=0;$$

and it is to be noticed that in each case the critic centres all of them lie on the conic. In fact, since the point $(0, \nu, -\mu)$ is an arbitrary point on the line $x=0$, a line $\lambda x+\mu y+\nu z=0$ passing through the point in question is an absolutely arbitrary line, and the corresponding critic centres therefore do not lie on the line $x=0$; that is, they lie on the conic

$$\frac{\nu}{y}-\frac{\mu}{z}-\frac{2(\nu-\mu)}{x+y+z}=0;$$

and it may also be remarked that the elimination of λ, θ, from the system

$$\theta+\lambda : \theta+\mu : \theta+\nu : \theta=\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{2}{x+y+z},$$

or, what is the same thing, the elimination of θ from the system

$$\theta+\mu : \theta+\nu : \theta= \frac{1}{y} : \frac{1}{z} : \frac{2}{x+y+z},$$

gives the last-mentioned equation, unencumbered by the factor $x=0$.

We have thus four conics, each of them passing through the three critic centres which correspond to the line $\lambda x+\mu y+\nu z=0$; as to the signification of the first three of these conics, I remark as follows.

58. The 'harmoconic' of a point A as to the line T in respect of the conic Θ, may be defined as follows; viz. considering the pencil of lines through A, the locus of the fourth harmonic of the point in which a line of the pencil meets T, in regard to the two points in which the same line meets the conic Θ, is a conic which is the harmoconic in question. (In particular, if the line T pass through the point A the harmoconic breaks up into the line T and into the polar of A.) The conic Θ may of course be a pair of lines.

Consider any three lines x, y, z, a line S, and the line T; then the harmoconics being all as to the same line T, we have the theorem

Harmoconic of intersection of x, S in regard to pair of lines y, z,

Ditto " " of y, S " " z, x,

Ditto " " of z, S " " x, y,

all pass through the same three points.

And taking $x=0$, $y=0$, $z=0$ for the equations of the lines x, y, z; $\lambda x+\mu y+\nu z=0$ for the equation of the line S; and $x+y+z=0$ for the equation of the line T, the harmoconics just spoken of are the above-mentioned three conics respectively.

59. In fact, considering the harmoconic of intersection of x, S in regard to the pair y, z; and taking x', y', z' as the coordinates of a point P of the harmoconic, then the equation of the line AP is

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ 0, & \nu, & -\mu \end{vmatrix}=0,$$

that is

$$x+(\mu y'+\nu z')-x'(\mu y+\nu z)=0,$$

and at the point of intersection with the line T or $x+y+z=0$, we have

$$(y+z)(\mu y'+\nu z')+x'(\mu y+\nu z)=0,$$

or, what is the same thing,

$$y(\mu x'+\mu y'+\nu z')+z(\nu x'+\mu y'+\nu z')=0,$$

which is the line through the last-mentioned point and the point $(y=0,\ z=0)$.

The line from the point A to the point $(y=0,\ z=0)$ is

$$yz'-zy'=0.$$

60. By the definition of the harmoconic, the last-mentioned two lines are harmonics in regard to the lines $y=0$, $z=0$; that is, we have for the equation of the harmoconic in question

$$-y'(\mu x'+\mu y'+\nu z')+z'(\nu x'+\mu y'+\nu z')=0;$$

this equation may also be written

$$(\nu z'-\mu y')(x'+y'+z')-2(\nu-\mu)y'z'=0,$$

or, what is the same thing,

$$\frac{\nu}{y'}-\frac{\mu}{z'}-\frac{2(\nu-\mu)}{x'+y'+z'}=0,$$

whence writing x, y, z in place of x', y', z', we see that this harmoconic is in fact the first of the above-mentioned three conics.

61. The fourth conic through the critic centres is the conic

$$\frac{\mu-\nu}{x}+\frac{\nu-\lambda}{y}+\frac{\lambda-\mu}{z}=0,$$

which it will be observed passes through the vertices of the triangle $x=0$, $y=0$, $z=0$, and also through the point (1, 1, 1) which is the harmonic of the line $x+y+z=0$ in regard to the triangle: I call it the 'harmonic conic.' Representing the equation by

$$\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0,$$

or, what is the same thing,

$$2fyz+2gzx+2hxy=0,$$

we have $f=\mu-\nu$, $g=\nu-\lambda$, $h=\lambda-\mu$, and therefore $f+g+h=0$.

62. It is easy to show that the coordinates of the pole of the line $x+y+z=0$ in regard to the harmonic conic are $x:y:z=f^2:g^2:h^2$; these values satisfy the condition $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$, that is, the pole in question lies on the twofold centre conic.

63. The equation of the tangents to the harmonic conic at its intersection with the line $x+y+z=0$ (which tangents meet of course in the last-mentioned pole, that is in a point of the twofold centre conic) is found to be

$$2fgh\,(x+y+z)^2+\square\,(2fyz+2gzx+2hxy)=0\,;$$

if for shortness

$$\square=\quad f^2+g^2+h^2-2gh-2hf-2fg,$$

or what is the same thing

$$\square=-4\,(gh+hf+fg),\ =2\,(f^2+g^2+h^2).$$

64. We have identically

$$-6fgh\,(x^2+y^2+z^2-2yz-2zx-2xy)$$
$$=2fgh\,(x+y+z)^2+\square\,(2fyz+2gzx+2hxy)-8\,(fx+gy+hz)\,(ghx+hfy+fgz),$$

so that the tangents in question meet the twofold centre conic

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

at its intersections with the lines $fx+gy+hz=0$, and $ghx+hfy+fgz=0$: the latter of these is in fact the tangent of the conic at the point $(f^2,\ g^2,\ h^2)$ of intersection of the two tangents. Hence the two tangents meet at the point $(f^2,\ g^2,\ h^2)$ of the twofold centre conic and they besides meet the conic at its points of intersection with the line $fx+gy+hz=0$.

65. The line $\lambda x+\mu y+\nu z=0$ may be expressed in the form,

$$\frac{x}{\alpha^3}+\frac{y}{\beta^3}+\frac{z}{\gamma^3}+h(x+y+z)=0,$$

(where, *ut suprà*, $\alpha+\beta+\gamma=0$). The corresponding values of f, g, h are

$$f:g:h=\alpha^3(\beta^3-\gamma^3):\beta^3(\gamma^3-\alpha^3):\gamma^3(\alpha^3-\beta^3),$$

or, what is the same thing,

$$f:g:h=\alpha^3(\beta-\gamma):\beta^3(\gamma-\alpha):\gamma^3(\alpha-\beta),$$

or, again,

$$f:g:h=\alpha^2(\beta^2-\gamma^2):\beta^2(\gamma^2-\alpha^2):\gamma^2(\alpha^2-\beta^2).$$

The equation $fx+gy+hz=0$ may be written

$$\begin{vmatrix} x, & y, & z \\ 1, & 1, & 1 \\ \frac{1}{\alpha^2}, & \frac{1}{\beta^2}, & \frac{1}{\gamma^2} \end{vmatrix}=0,$$

that is, the line in question is the line joining the harmonic point (1, 1, 1) with the point

$$\left(\frac{1}{\alpha^2},\ \frac{1}{\beta^2},\ \frac{1}{\gamma^2}\right),$$

the inverse of the point $(\alpha^2, \beta^2, \gamma^2)$, which is (*ante*, No. 27) the point of contact of the line

$$\frac{x}{\alpha^3}+\frac{y}{\beta^3}+\frac{z}{\gamma^3}=0$$

with the envelope.

66. The harmonic conic passes through the vertices of the triangle $x=0$, $y=0$, $z=0$, through the harmonic point (1, 1, 1), and through the critic centres. Hence if one of the critic centres be given, the harmonic conic passes through five given points and is thus completely determined. But a critic centre being given, the line joining the other two critic centres is the polar of the given centre in regard to the twofold centre conic (*ante*, No. 40), and it is thus completely determined; and the other two critic centres are of course the intersections of this line with the harmonic conic.

Article Nos. 67 to 87. *Miscellaneous Investigations.*

67. I demonstrate by means of the last-mentioned formulæ a theorem already in effect demonstrated by the investigation which led to the three centre conic, viz. that the tangents at a node or critic centre, and the lines drawn to the other two critic centres, form a harmonic pencil.

In fact the tangents at the node or critic centre are given by the equation

$$\left(\theta+4\lambda, \ldots, -\theta-\frac{2\mu\nu}{\theta}, \ldots\right)\left(x, y, z\right)^2=0,$$

the other two critic centres are given as the intersection of the line

$$\frac{\lambda x}{\theta+\lambda}+\frac{\mu y}{\theta+\mu}+\frac{\nu z}{\theta+\nu}=0,$$

with the conic

$$\frac{\mu-\nu}{x}+\frac{\nu-\lambda}{y}+\frac{\lambda-\mu}{z}=0,$$

the theorem will be true if the pair of tangents and the last-mentioned conic are cu harmonically by the last-mentioned line. Now in general the condition in order tha the line $\xi x+\eta y+\zeta z=0$, may cut harmonically the conics $(a, b, c, f, g, h)(x, y, z)^2$ and $(a', b', c', f', g', h')(x, y, z)^2=0$ is

$$(bc'+b'c-2ff', \ldots, gh'+g'h-af'-a'f, \ldots)(\xi, \eta, \zeta)^2=0,$$

and if $a'=b'=c'=0$, then the condition is

$$(-2ff', \ldots gh'+g'h-af', \ldots)(\xi, \eta, \zeta)^2=0.$$

68. In the present case the equations of the two conics may be written

$$\left(\theta+4\lambda, \ldots -\theta-\frac{2\mu\nu}{\theta}, \ldots\right)(x, y, z)^2=0,$$

$$(0, \ldots, \mu-\nu, \ldots)(x, y, z)^2=0,$$

and we have

$$-2ff'=-2(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right),$$

$$\begin{aligned} gh'+g'h-af' &= -\left(\theta+\frac{2\nu\lambda}{\theta}\right)(\lambda-\mu)-\left(\theta+\frac{2\lambda\mu}{\theta}\right)(\nu-\lambda)-(\mu-\nu)(\theta+4\lambda), \\ &= -\theta(\lambda-\mu+\mu-\nu+\nu-\lambda) \\ &\quad +\frac{2}{\theta}(-\nu\lambda^2+\nu\lambda\mu-\lambda\mu\nu-\lambda^2\mu)+4\lambda(\mu-\nu), \\ &= (\mu-\nu)\left(\frac{2\lambda^2}{\theta}-4\lambda\right), \end{aligned}$$

and the condition is

$$\left\{(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right), \ldots (\mu-\nu)\left(\frac{2\lambda^2}{\theta}-4\lambda\right), \ldots\right\}\left(\frac{\lambda}{\theta+\lambda}, \frac{\mu}{\theta+\mu}, \frac{\nu}{\theta+\nu}\right)^2=0.$$

69. Writing this in the form

$$\Sigma(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right)\left(\frac{\lambda}{\theta+\lambda}\right)^2+\Sigma\, 2(\mu-\nu)\left(\frac{\lambda^2}{\theta}-2\lambda\right)\frac{\mu\nu}{(\theta+\mu)(\theta+\nu)}=0,$$

then observing that

$$\frac{\lambda}{\theta+\lambda}+\frac{\mu}{\theta+\mu}+\frac{\nu}{\theta+\nu}=1,$$

the first part is

$$=\Sigma(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right)\left(\frac{-\lambda\mu}{(\theta+\lambda)(\theta+\mu)}-\frac{\lambda\nu}{(\theta+\lambda)(\theta+\nu)}+\frac{\lambda}{\theta+\lambda}\right),$$

which is

$$=\frac{1}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}\Sigma(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right)\left\{-\lambda\mu(\theta+\nu)-\lambda\nu(\theta+\mu)+\lambda(\theta+\mu)(\theta+\nu)\right\},$$

$$=\frac{1}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}\Sigma(\mu+\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right)(\lambda\theta^2-\lambda\mu\nu),$$

and observing that the sum is

$$=\Sigma\lambda(\mu-\nu)(\theta^3+2\mu\nu\theta)-\lambda\mu\nu\,\Sigma(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right),$$

$$=-\frac{2\lambda\mu\nu}{\theta}\Sigma\,\mu\nu(\mu-\nu)=\frac{\theta}{2\lambda\mu\nu}(\mu-\nu)(\nu-\lambda)(\lambda-\mu),$$

the first part is

$$=\frac{2\lambda\mu\nu}{\theta}\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}.$$

The second part is

$$=\frac{2\lambda\mu\nu}{\theta(\theta+\lambda)(\lambda+\mu)(\theta+\nu)}\Sigma(\mu-\nu)(\lambda-2\theta)(\lambda+\theta),$$

in which the sum is

$$=\Sigma(\mu-\nu)(\lambda^2-\lambda\theta-2\theta^2)=\Sigma\lambda^2(\mu-\nu)=-(\mu-\nu)(\nu-\lambda)(\lambda-\mu),$$

so that the second part is

$$=-\frac{2\lambda\mu\nu}{\theta}\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)},$$

and the sum of the two parts is $=0$, which proves the theorem.

70. Let x_1, y_1, z_1 be the coordinates of a critic centre, then the equation of the polar in regard to the twofold centre conic is

$$(-x_1+y_1+z_1)x+(x_1-y_1+z_1)y+(x_1+y_1-z_1)z=0,$$

and the equation of the conic through the five points is

$$\frac{x_1(y_1-z_1)}{x}+\frac{y_1(z_1-x_1)}{y}+\frac{z_1(x_1-y_1)}{z}=0,$$

and these equations together determine the remaining two critic centres.

71. I remark in passing that the equation of the one-with-twofold centre locus may also be obtained by means of the equations

$$\frac{x_1(y_1-z_1)}{x}+\frac{y_1(z_1-x_1)}{y}+\frac{z_1(x_1-y_1)}{z}=0,$$

$$(-x_1+y_1+z_1)x+(x_1-y_1+z_1)y+(x_1+y_1-z_1)z=0,$$

which determine the remaining two critic centres corresponding to a given critic centre (x_1, y_1, z_1); in fact, in order that the centre (x_1, y_1, z_1) may be accompanied by a twofold centre the line must touch the conic; and the analytical condition, substituting therein (x, y, z) in the place of (x_1, y_1, z_1), is found to be

$$xyz\left\{x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz\right\}=0,$$

the three lines $xyz=0$ are not properly part of the locus, but their appearance may be accounted for without difficulty.

72. Assume that the line $\lambda x+\mu y+\nu z=0$ passes successively through the points

$$(x=0,\ y-z=0)\quad (y=0,\ z-x=0),\quad (z=0,\ x-y=0),$$

or, what is the same thing, the points (0, 1, 1), (1, 0, 1), (1, 1, 0): then (*ante*, No. 56) the critic centres are in all these cases respectively on the conics.

$$\frac{1}{y}+\frac{1}{z}-\frac{4}{x+y+z}=0,$$

$$\frac{1}{z}+\frac{1}{x}-\frac{4}{x+y+z}=0,$$

$$\frac{1}{x}+\frac{1}{y}-\frac{4}{x+y+z}=0;$$

or, as these may be written,

$$(y-z)^2+x(y+z)=0,$$

$$(z-x)^2+y(z+x)=0,$$

$$(x-y)^2+z(x+y)=0,$$

the first of which is a conic touching the lines $x=0$, $y+z=0$ at the points of intersection with the line $y-z=0$; and similarly for the other two conics.

73. Suppose that the line $\lambda x+\mu y+\nu z=0$ passes through the point $(4, -1, -1)$, or let $4\lambda-\mu-\nu=0$; we have $(\alpha, \beta, \gamma)=(4, -1, -1)$; and the critic centres lie on the curve

$$\frac{4}{x}-\frac{1}{y}-\frac{1}{z}-\frac{4}{x+y+z}=0,$$

that is

$$\frac{4(y+z)}{x(x+y+z)}-\frac{y+z}{yz}=0,$$

or, as this may be written,

$$(y+z)\left\{x(x+y+x)-4yz\right\}=0,$$

so that the cubic locus breaks up into the line $y+z=0$ and into the conic

$$x(x+y+z)-4yz=0.$$

74. I say that the critic centres lie, one of them on the line, and the other two on the conic.

In fact, putting $\lambda=\frac{1}{4}(\mu+\nu)$ the equation in θ is

$$\theta^3-\theta\left(\mu\nu+\tfrac{1}{4}(\mu+\nu)^2\right)-\tfrac{1}{2}\mu\nu(\mu+\nu)=0,$$

that is

$$\left\{\theta+\tfrac{1}{2}(\mu+\nu)\right\}\left\{\theta^2-\tfrac{1}{2}(\mu+\nu)\theta-\mu\nu\right\}=0,$$

and we have

$$x:y:z=\frac{1}{\theta+\frac{1}{4}(\mu+\nu)}:\frac{1}{\theta+\mu}:\frac{1}{\theta+\nu}.$$

75. Hence if $\theta+\frac{1}{2}(\mu+\nu)=0$, we obtain

$$x:y:z=\frac{1}{-\frac{1}{2}(\mu+\nu)}:\frac{1}{\frac{1}{2}(\mu-\nu)}:\frac{-1}{\frac{1}{2}(\mu-\nu)}$$

$$=\frac{\mu-\nu}{\mu+\nu}:-1:1;$$

whence also

$$(\mu+\nu)x+(\mu-\nu)y=0,$$
$$(\mu+\nu)x-(\mu-\nu)z=0,$$
$$y+z=0,$$

so that the corresponding critic centre lies on the line $y+z=0$; the last-mentioned equations, restoring the value 4λ in place of $\mu+\nu$, may also be written

$$4\lambda x+(\mu-\nu)y=0,$$
$$4\lambda x-(\mu-\nu)z=0,$$
$$y+z=0.$$

76. If on the other hand

$$\theta^2-\tfrac{1}{2}(\mu+\nu)\theta-\mu\nu=0,$$

or, as this equation may be written,

$$5\theta^2-(\theta+2\mu)(\theta+2\nu)=0,$$

then observing that in general, in virtue of the equation

$$\frac{1}{\theta+\lambda}+\frac{1}{\theta+\mu}+\frac{1}{\theta+\nu}-\frac{2}{\theta}=0,$$

we have

$$y : z : x+y : x+z = \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu} : \frac{2}{\theta} - \frac{1}{\theta+\nu} : \frac{2}{\theta} - \frac{1}{\theta+\mu},$$

$$= \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu} : \frac{\theta+2\nu}{\theta(\theta+\nu)} : \frac{\theta+2\mu}{\theta(\theta+\mu)},$$

and consequently

$$y : x+z = \theta : \theta+2\mu ; \quad z : x+y = \theta : \theta+2\nu,$$

the foregoing equation

$$5\theta^2 - (\theta+2\mu)(\theta+2\nu) = 0$$

gives

$$5yz - (x+z)(x+y) = 0,$$

that is

$$x(x+y+z) - 4yz = 0;$$

or the critic centres corresponding to the two values of θ lie on the conic. The line joining them is the polar of the point $\left(\frac{\mu-\nu}{\mu+\nu}, -1, 1\right)$ in regard to the twofold centre conic; the equation therefore is

$$(\mu-\nu)x - (3\mu+\nu)y + (\mu+3\nu)z = 0.$$

77. Starting with a critic centre on the line $y+z=0$, the other two critic centres lie on the conic $x(x+y+z)-4yz=0$, and they are the intersections of the conic by the polar of the first centre in regard to the twofold centre conic.

78. Starting with a critic centre on the conic $x(x+y+z)-4yz=0$, the other two critic centres lie one on the conic, and the other on the line $y+z=0$; viz. the polar of the first centre in regard to the twofold centre conic meets the line in one point, and the conic in two points; of these one is the harmonic of the point on the line in regard to the twofold centre conic; this point on the conic, and the point on the line, are the other two centres.

79. The point $(4, -1, -1)$ is of course one of a system of three points; viz. these are $(4, -1, -1)$ $(-1, 4, -1)$, $(-1, -1, 4)$; and the corresponding loci of the critic centres are

$$(y+z)\left\{x(x+y+z) - 4xy\right\} = 0,$$

$$(z+x)\left\{y(x+y+z) - 4zx\right\} = 0,$$

$$(x+y)\left\{z(x+y+z) - 4xy\right\} = 0,$$

the three points in question are (*ante*, No. 24) shown to be nodes of the twofold centre envelope.

80. The line $3x+y+z=0$ is the line through the points $(-1, 4, -1)$, $(-1, -1, 4)$, and as such the corresponding critic centres lie

one on the line $z+x=0$, two on the conic $x(x+y+z)-4yz=0$,

one on the line $x+y=0$, two on the conic $y(x+y+z)-4zx=0$.

The two lines meet in the point $(1, -1, -1)$.

The two conics meet in the points $(1, 0, 0)$, $(2, 3, 3)$; and touch at the point $(0, 1, -1)$, the common tangent being $5x+y+z=0$: this appears by writing the equations of the two conics in the forms

$$(y-z)(5x+y+z)+(y+z)(-3x+y+z)=0,$$
$$-(y-z)(5x+y+z)+(y+z)(-3x+y+z)=0,$$

for we have then the four points of intersection put in evidence; viz. these are

$$\begin{array}{llll} y-z=0, & y+z=0, & \text{that is} & (1,\ 0,\ 0), \\ y-z=0, & -3x+y+z=0, & \text{,,} & (2,\ 3,\ 3), \\ 5x+y+z=0, & y+z=0, & \text{,,} & (0,\ 1,\ -1), \\ 5x+y+z=0, & -3x+y+z=0, & \text{,,} & (0,\ 1,\ -1). \end{array}$$

The point of intersection $(1, 0, 0)$, which is an angle of the triangle, is not a critic centre; the three critic centres are the other point of intersection $(2, 3, 3)$; the point of contact $(0, 1, -1)$; and the point of intersection $(1, -1, -1)$ of the two lines.

81. To obtain in a different manner the last-mentioned result it may be remarked that for the line $3x+y+z=0$, for which $(\lambda, \mu, \nu)=(3, 1, 1)$, the equation in θ is

$$\theta^3-7\theta-6=(\theta+1)(\theta+2)(\theta-3)=0,$$

so that the values of $\theta+\lambda$, $\theta+\mu$, $\theta+\nu$ are

$$\begin{array}{llll} \text{for } \theta=-1, & 2, & 0, & 0, \\ \text{,, } \theta=-2, & 1, & -1, & -1, \\ \text{,, } \theta=\ \ 3, & 6, & 4, & 4, \end{array}$$

and the corresponding values of $x : y : z$ are

$$\begin{array}{ll} =\tfrac{1}{2} : \infty : \infty, & \text{that is, } (0,\ 1,\ -1), \\ =1 : -1 : -1, & \text{,, } (1,\ -1,\ -1), \\ =\tfrac{1}{6} : \tfrac{1}{4} : \tfrac{1}{4}, & \text{,, } (2,\ 3,\ 3), \end{array}$$

which points are therefore the critic centres for the line $3x+y+z=0$.

The last-mentioned line, it is clear, is one of the system of three lines

$$3x+y+z=0, \quad x+3y+z=0, \quad x+y+3z=0.$$

82. If $\lambda=0$, that is if the line $\lambda x+\mu y+\nu z=0$ pass through an angle $y=0$, $z=0$ of the triangle; then reverting to the original equations

$$\frac{-x+y+z}{\lambda x}=\frac{x-y+z}{\mu y}=\frac{x+y-z}{\nu z},$$

these give $(y=0,\ z=0)$ or else $(-x+y+z=0,\ \nu z^2-\mu y^2=0)$, that is, one of the three critic centres is the angle $(y=0,\ z=0)$ of the triangle; and the other two are the intersections of the line $-x+y+z=0$ with the pair of lines $\nu z^2-\mu y^2=0$.

It should be remarked that, given the critic centre $y=y_1=0,\ z=z_1=0$, the remaining two centres cannot be determined as the intersection of the polar $-x+y+z=0$ with the conic

$$\frac{x_1(y_1-z_1)}{x}+\frac{y_1(z_1-x_1)}{y}+\frac{z_1(x_1-y_1)}{z}=0,$$

inasmuch as the equation of this conic becomes the identity $0=0$.

83. The critic centres for the case in question, $\lambda=0$, may also be determined by means of the equation of the cubic through the three centres; in fact, since $\lambda=0$, the equation $\lambda\alpha+\mu\beta+\nu\gamma=0$ becomes $\mu\beta+\nu\gamma=0$, that is $\beta:\gamma=\nu:-\mu$; and the equation of the cubic therefore is

$$\alpha\left(\frac{1}{x}-\frac{2}{x+y+z}\right)+\frac{\beta}{\nu}\left(\frac{\nu}{y}-\frac{\mu}{z}-\frac{2(\nu-\mu)}{x+y+z}\right)=0,$$

and since the ratio $\alpha:\beta$ is arbitrary we have the two equations

$$\frac{1}{x}-\frac{2}{x+y+z}=0,\quad \frac{\nu}{y}-\frac{\mu}{z}-\frac{2(\nu-\mu)}{x+y+z}=0,$$

which resolve themselves into the above-mentioned two equations, $-x+y+z=0$, $\nu z^2-\mu y^2=0$.

84. Consider a critic centre the coordinates of which are $(0,\ y_1,\ z_1)$, that is, which is an arbitrary point on the side $x=0$ of the triangle: it is to be remarked that there is not any position of the line $\lambda x+\mu y+\nu z=0$, which properly gives rise to such a critic centre.

For writing $x_1=0$ the equations

$$\frac{-x_1+y_1+z_1}{\lambda x_1}=\frac{x_1-y_1+z_1}{\mu y_1}=\frac{x_1+y_1-z_1}{\nu z_1},$$

give $\mu=0,\ \nu=0$, that is, the line $\lambda x+\mu y+\nu z=0$ is found to be $x=0$; but in this case the cubic is $x\left(yz+k(x+y+z)^2\right)=0$, which irrespectively of the value of k has nodes at the points $x=0,\ yz+k(y+z)^2=0$, and which only for the value $k=0$ acquires a third node at the point $y=0,\ z=0$: the case is a singular and exceptional one.

85. If notwithstanding we assume a critic centre at the point $(0,\ y_1,\ z_1)$, then the other two critic centres are by the general theorem given as the intersection of the line

$$(y_1+z_1)x-(y_1-z_1)(y-z)=0$$

with the conic (pair of lines) $x(y-z)=0$, that is, we have a twofold centre $x=0,\ y-z=0$, or what is the same thing a twofold centre $(0,\ 1,\ 1)$.

86. If a critic centre lie on the line $y-z=0$, then of the other two critic centres, one lies on this same line and the other is the point $x=0$, $x+y+z=0$, or say the point $(0, 1, -1)$. And in this case the line $\lambda x+\mu y+\nu z=0$ passes through the last-mentioned point; that is, we have $\mu=\nu$. Conversely, starting from the equation $\mu=\nu$, so that the line $\lambda x+\mu y+\nu z=0$ is $\lambda x+\mu(y+z)=0$, a line through the intersection of the lines $x=0$, $x+y+z=0$, the equation in θ is

$$(\theta+\mu)(\theta^2-\mu\theta-2\lambda\mu)=0,$$

where the factor $\theta+\mu=0$ corresponds to the critic centre $x=0$, $x+y+z=0$, or $(0, 1, -1)$, (it will presently be shown that this is so), and the quadric equation $\theta^2-\mu\theta-2\lambda\mu=0$ corresponds to two critic centres on the line $y-z=0$. We have

$$x : y : z=\frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\mu},$$

and thence $y-z=0$; and $\theta(x-y)=-\lambda x+\mu y$, which substituted in the equation $\theta^2-\mu\theta-2\lambda\mu=0$ gives

$$(\lambda x-\mu y)\{(\lambda+\mu)x-2\mu y\}-2\lambda\mu(x-y)^2=0,$$

and the two critic centres are given as the intersections of this conic by the line $y-z=0$.

87. Consider for a moment the case $\nu=\mu+\epsilon$, where ϵ is ultimately $=0$, the equation in θ is

$$\frac{1}{\theta+\lambda}+\frac{1}{\theta+\mu}+\frac{1}{\theta+\mu+\epsilon}-\frac{2}{\theta}=0;$$

then if a root is $\theta=-\mu+A\epsilon$, we have

$$\frac{1}{A\epsilon+\lambda-\mu}+\frac{1}{A\epsilon}+\frac{1}{(A+1)\epsilon}-\frac{2}{A\epsilon-\mu}=0,$$

so that, ϵ being indefinitely small, we have

$$\frac{1}{A}+\frac{1}{A+1}=0, \text{ that is, } 2A+1=0 \text{ or } A=-\tfrac{1}{2},$$

and then

$$\theta=-\mu-\tfrac{1}{2}\epsilon,\quad \theta+\lambda=\lambda-\mu-\tfrac{1}{2}\epsilon;\quad \theta+\mu=-\tfrac{1}{2}\epsilon;\quad \theta+\nu=+\tfrac{1}{2}\epsilon,$$

which gives

$$x : y : z=\frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu}=\frac{1}{\lambda-\mu-\frac{1}{2}\epsilon} : -\frac{2}{\epsilon} : +\frac{2}{\epsilon},$$

or, ϵ being indefinitely small, $x : y : z=0 : 1 : -1$, so that the factor $\theta+\mu=0$ corresponds, as mentioned above, to the critic centre $(0, 1, -1)$.