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NOTE ON AN EXPRESSION FOR THE RESULTANT OF TWO BINARY CUBICS.

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Mr Warren, in his paper "Illustrations of the Theory of Critical Functions," *Quarterly Mathematical Journal*, t. VI. pp. 231—237, (1864), has given for the Resultant of two binary cubic functions, an expression which is in effect as follows; viz. considering the cubic

$$(a, b, c, d)(x, y)^3,$$

its Hessian

$$(\mathrm{a}, \mathrm{b}, \mathrm{c})(x, y)^2, =(ac-b^2,\ ad-bc,\ bd-c^2)(x, y)^2,$$

and the cubicovariant

$$(A, B, C, D)(x, y)^3, =\left\{\begin{array}{l} \quad a^2d\ -3abc+2b^3\ , \\ \ 3abd-6ac^2+3b^2c, \\ -3acd+6b^2d\ -3bc^2, \\ -\ \ ad^2+3bcd-2c^3\ , \end{array}\right\}(x, y)^3;$$

and in like manner the cubic

$$(a', b', c', d')(x, y)^3,$$

its Hessian

$$(\mathrm{a}', \mathrm{b}', \mathrm{c}')(x, y)^2,$$

and the cubicovariant

$$(A', B', C', D')(x, y)^3;$$

and writing

$$\begin{aligned} \mathfrak{A} &= ad'\ -3bc'\ -3b'c\ -a'd\ , \\ \mathfrak{B} &= \mathrm{ac}'\ +\mathrm{a}'\mathrm{c}\ \ -2\mathrm{bb}'\qquad\quad , \\ \mathfrak{C} &= AD'-3BC'+3B'C-A'D, \end{aligned}$$

then the Resultant is

$$= -2\,\mathfrak{A}^3 + 27\,\mathfrak{A}\mathfrak{B} + 27\,\mathfrak{C},$$

that is, the Resultant is

$$\begin{aligned}= &- 2\,(ad' - a'd - 3bc' + 3b'c)^3 \\ &+ 27\,(ad' - a'd - 3bc' + 3b'c) \times \\ &\{(ac - b^2)(b'c' - d'^2) - \tfrac{1}{2}(ad - bc)(a'd' - b'c') + (bd - c^2)(a'c' - b'^2)\} \\ &+ 27\,\{\ (a^2d - 3abc + 2b^3)(-a'd'^2 + 3b'c'd' - 2c'^3) \\ &\quad - 3\,(abd - 2ac^2 + b^2c)(-a'c'd' + 2b'^2d' - b'c'^2) \\ &\quad + 3\,(-acd + 2b^2d - bc^2)(a'b'd' - 2a'c'^2 + b'^2c) \\ &\quad - (-ad^2 + 3bcd - 2c^3)(a'^2d' - 3a'b'c' + 2b'^3)\}.\end{aligned}$$

In particular assume

$$(a', b', c', d' \between x, y)^3 = x^3 + y^3,$$

so that

$$(\mathrm{a}', \mathrm{b}', \mathrm{c}' \between x, y)^2 = xy,$$
$$(A', B', C', D' \between x, y)^3 = x^3 - y^3,$$

and thus

$$a' = d' = 1, \quad b' = c' = 0,$$
$$\mathrm{a}' = \mathrm{c}' = 0, \quad \mathrm{b}' = \tfrac{1}{2},$$
$$A' = -D' = 1, \quad B' = C' = 0.$$

$$\begin{aligned}\mathfrak{A} &= a - d, \\ \mathfrak{B} &= -\mathrm{b} = bc - ad, \\ \mathfrak{C} &= A + D = a^2d - ad^2 - 3abc + 3bcd + 2b^3 - 2c^3,\end{aligned}$$

or, putting for shortness,

$$a - d = \theta, \text{ and therefore } a = d + \theta,$$

we have

$$\begin{aligned}\mathfrak{A} &= \theta, \\ \mathfrak{B} &= bc - d\theta - d^2, \\ \mathfrak{C} &= 2\,(b^3 - c^3) - 3bc\theta + d^2\theta + d\theta^2,\end{aligned}$$

and Resultant is

$$\begin{aligned}&- 2\theta^3 \\ &+ 27\theta\,(bc - d^2 - d\theta) \\ &+ 27\,\{2\,(b^3 - c^3) - 3bc\theta + d^2\theta + d\theta^2\},\end{aligned}$$

which is

$$= - 2\theta^3 + 54b^3 - 54c^3 - 54bc\theta,$$

or rejecting the factor -2, it is

$$= \theta^3 - 27b^3 + 27c^3 + 27cb\theta.$$

But the two equations are

$$(a,\ b,\ c,\ d\between x,\ y)^3 = 0,$$

$$x^3 + y^3 \qquad\qquad = 0,$$

the last of which gives $y = -x$, $y = -\omega x$, $y = -\omega^2 x$, if ω be an imaginary cube root of unity, and hence the Resultant is

$$= (a - 3b + 3c - d)(a - 3b\omega + 3c\omega^2 - d)(a - 3b\omega^2 + 3c\omega - d),$$

which is

$$= (\theta - 3b + 3c)(\theta - 3b\omega + 3c\omega^2)(\theta - 3b\omega^2 + 3c\omega),$$

or finally is

$$= \theta^3 - 27b^3 + 27c^3 + 27bc\theta,$$

and the formula is thus verified.

If the two cubics are taken to be

$$(a,\ b,\ c,\ d\between x,\ y)^3 = 0,$$

$$(b,\ c,\ d,\ e\between x,\ y)^3 = 0,$$

then the formula gives for the Discriminant of the quartic function $(a,\ b,\ c,\ d,\ e\between x,\ y)^4$ a new expression, which however does not appear to be an elegant one.