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Propagation and damping of surface- and interface-type waves in viscoelastic fluids

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The problems of propagation and damping of small-amplitude harmonic waves in the neighbourhood of plane boundaries or plane interfaces in viscoelastic incompressible and compressible fluids are discussed in greater detail. Several examples of homogeneous and two-layer fluids, with free or rigid surfaces and the layers sliding or adhering at the interface, are presented. The conditions under which the layers may act like waveguides are also considered.

Przedyskutowano szczegółowo zagadnienia propagacji i tłumienia fal harmonicznych o małych amplitudach w sąsiedztwie płaskich powierzchni i powierzchni rozdziału faz w nieściśliwych i ściśliwych cieczach lepkosprężystych. Przedstawiono kilka przykładów cieczy jednorodnych i dwuwarstwowych, ze swobodnymi lub sztywnymi powierzchniami oraz z warstwami ślizgającymi się lub przylegającymi na powierzchni rozdziału. Rozważono również warunki, przy których warstwy mogą działać podobnie jak falowody.

Обсуждены подробно проблемы распространения и затухания гармонических волн малой амплитуды вблизи плоских поверхностей и поверхностей раздела фаз, в несжимаемых и сжимаемых вязкоупругих жидкостях. Представлено несколько примеров однородных и двухслоистых жидкостей, со свободными или жесткими поверхностями, а также со слоями скользящими или прилегающими на поверхности раздела. Рассмотрены тоже условия, при которых слои могут действовать аналогично как волноводы.

1. Introduction

Wave propagation problems in elastic solids have been extensively discussed in numerous monographs and books (cf. [1, 2, 3]). There exist also several papers (cf.e.g. [4, 5, 6, 7]) devoted to the problems of propagation, reflection and transmission of harmonic waves in viscoelastic, usually compressible, solids. In our previous papers [8, 9, 10, 11] some properties of small-amplitude harmonic waves in viscoelastic compressible and incompressible fluids were considered. It was shown, among other things, that certain properties of such waves differ essentially from those observed in elastic and viscoelastic compressible solids.

In the present paper we discuss the conditions of existence, propagation and damping for small-amplitude harmonic waves in the neighbourhood of plane boundaries or plane interfaces in viscoelastic incompressible and compressible fluids. We shall call such waves the surface- and interface-type waves, bearing in mind an analogy to elastic cases, although a full similarity does not exist. In general, these waves may propagate parallely to a surface or interface, being simultaneously damped in other directions (cf. [12]).

In Sect. 2 some notions and notations concerned with the governing equations for viscoelastic fluids and various boundary conditions are briefly discussed. In subsequent Sects. 3 and 4, several examples of waves propagating in incompressible (shear waves) and compressible (shear and dilatational waves) viscoelastic fluids are presented in greater detail. At the end, in Sect. 5, certain general remarks are made and possible simplifications as well as generalizations are discussed.

2. Wave equations and boundary conditions

It can be shown (cf. [8, 9]) that the constitutive equation of a simple viscoelastic fluid subjected to small-amplitude oscillatory flows takes the following form:

$$\mathbf{T} = (-p+\lambda^*\mathrm{tr}\,\mathbf{D})\mathbf{1}+2\eta^*\mathbf{D}, \quad \mathbf{D} = \frac{1}{2}(\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}), \tag{2.1}$$

where $\mathbf{T}$ is the stress tensor, p — the hydrostatic pressure, $\mathbf{D}$ — the strain-rate tensor, λ^* and η^* denote the frequency-dependent dynamic second and shear viscosities, respectively. For plane flows realized in the xz-plane, the above relations introduced into the dynamic equations of equilibrium lead to the governing equations

$$\left(\nabla^2-\frac{\varrho}{\lambda^*+2\eta^*}\frac{\partial}{\partial t}\right)\Phi_1 = 0, \quad \left(\nabla^2-\frac{\varrho}{\eta^*}\frac{\partial}{\partial t}\right)\Phi_2 = 0, \quad \nabla^2 p = 0, \tag{2.2}$$

where ϱ is the mass density of a fluid(¹), and Φ_i $(i=1,2)$ denote the potentials determining the velocity components, viz.

$$u = \frac{\partial\Phi_1}{\partial x}+\frac{\partial\Phi_2}{\partial z}, \quad v = 0, \quad w = \frac{\partial\Phi_1}{\partial z}-\frac{\partial\Phi_2}{\partial x}. \tag{2.3}$$

Instead of dynamic viscosities, we often use the following dynamic moduli:

$$G_1^* = G_1'+iG_1'' = i\omega(\lambda^*+2\eta^*), \quad G_2^* = G_2'+iG_2'' = i\omega\eta^*, \tag{2.4}$$

where primes and double-primes denote the real and imaginary parts, respectively. G_1^* may be called the bulk (dilatational) modulus and G_2^* — the shear (distorsional) modulus. For incompressible fluids only shearing flows are admissible; then $\mathrm{tr}\,\mathbf{D} = 0$ or $\mathrm{div}\,\mathrm{grad}\,\Phi_1 = 0$, and the two last equations of the set (2.2) are taken into account. Thus, for small-amplitude harmonic oscillations, Eqs. (2.2) describe the dilatational (longitudinal), shear (transverse) and pressure waves, respectively.

In what follows, being interested in the surface- or interface-type waves, we consider solutions of Eqs. (2.4) in the forms

$$\Phi_i = (A_i e^{\nu_i z}+B_i e^{-\nu_i z})\exp(\mu x+i\omega t), \quad i = 1,2, \tag{2.5}$$

$$p = p_0(z)\exp(\mu x+i\omega t), \tag{2.6}$$

where A_i, B_i $(i = 1,2)$ are integration constants, and ν_i $(i = 1,2)$, μ, viz.

$$\nu_i^2 = -k_{iz}^2 = -\left(\frac{\varrho\omega^2}{G_i^*}+\mu^2\right), \quad \mu^2 = -k_x^2, \tag{2.7}$$

are simply related to the components of the wave vectors $\mathbf{k}_i(k_x, k_{iz})$, $i = 1,2$.

(¹) It can be proved [9] that for small disturbances in compressible fluids, the density ϱ is independent of time.

It is seen from Eqs. (2.5) and (2.6) that the waves considered may propagate parallely to the x-axis, if $\mathrm{Re}\,\mu^2 < 0$, being simultaneously damped in the z-direction if also $\mathrm{Re}\,\nu_i^2 > 0$. Damping in that direction is considered to be full if, moreover, $\mathrm{Im}\,\nu_i^2 = 0$ for certain discrete values of the frequency ω.

The solutions (2.5) and (2.6) must satisfy the appropriate boundary (continuity) conditions determined at the surfaces or interfaces. Therefore, for homogeneous fluids contained in the lower half-space $z \leqslant 0$, we have (Fig. 1)

$$T^{13} = T^{33} = 0, \quad \text{or} \quad u = w = 0, \tag{2.8}$$

if the surface is free or rigid, respectively. Similarly, for two immiscible fluids in which a thin upper layer is superposed on a bulk fluid (hereafter we call it the two-layer fluid), we obtain (Fig. 2)

$$T^{33} = \overline{T}^{33}, \quad w = \overline{w}, \tag{2.9}$$

if the layers can slide freely at the interface and, moreover,

$$T^{13} = \overline{T}^{13}, \quad u = \overline{u}, \tag{2.10}$$

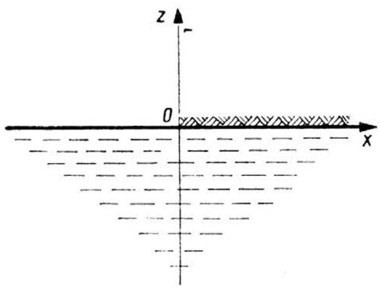

FIG. 1.

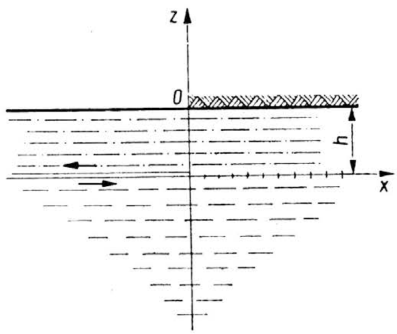

FIG. 2.

if the layers adhere at the interface. All the quantities concerned with the lower fluid are marked with overbars. We should also take into account that amplitudes of the waves considered vanish far away from the surface or interface. Therefore, for $z \to -\infty$, the waves must be damped entirely. This means that the constants B_i or $\overline{B}_i$ $(i = 1, 2)$ should be disregarded in Eqs. (2.5). Moreover, in such an approach to the problem we neglect possible effects of surface or interface tensions.

3. Propagation and damping in incompressible fluids

3.1. Shear waves in homogeneous fluids

For homogeneous incompressible fluids, the boundary conditions $(2.8)_1$ for $z = 0$ lead to [2]

([2]) In this Section we are interested only in shear (transverse) waves of the type considered. The Rayleigh surface waves in homogeneous incompressible viscoelastic media were discussed elsewhere (cf. [13, 14]).

$$\eta^*(\nu_2^2-\mu^2)A_2 = 0, \quad -p_0(0)-2\eta^*\mu\nu_2 A_2 = 0, \tag{3.1}$$

if the surface is free. The above system of equations has a solution only if

$$\mu^2 = \nu_2^2 = -\frac{\varrho\omega^2}{2G_2^*} = -\frac{\omega^2}{2C_2^2}\cos^2\delta_2(1-i\tan\delta_2), \tag{3.2}$$

where

$$C_i^2 = \frac{G_i'(\omega)}{\varrho}, \quad \tan\delta_i = \frac{G_i''(\omega)}{G_i'(\omega)}, \quad i = 1, 2, \tag{3.3}$$

denote the frequency-dependent real wave speeds in the fluids for which $\tan\delta_i = 0$, and the frequency-dependent real loss angles.

Thus the condition of propagation along the x-axis and the condition of damping in the z-direction, viz.

$$-\frac{\omega^2}{2C_2^2}\cos^2\delta_2 < 0, \quad \frac{\omega^2}{4C_2^2}\sin 2\delta_2 > 0 \tag{3.4}$$

are always satisfied for any frequency ω. The resulting shear waves propagate and are damped in both directions; they are never fully damped.

3.2. Shear waves in two-layer fluids

For two-layer incompressible fluids with the upper layer of thickness h (Fig. 2), the boundary conditions $(2.8)_1$ and (2.9) lead to

$$\begin{aligned} \eta^*(\nu_2^2-\mu^2)(A_2 e^{\nu_2 h}+B_2 e^{-\nu_2 h}) &= 0, \\ -p_0(h)-2\eta^*\mu\nu_2(A_2{}^{\nu_2 h}-B_2 e^{-\nu_2 h}) &= 0, \\ A_2+B_2-\bar{A}_2 &= 0, \\ -\big(p_0(0)-\bar{p}_0(0)\big)-2\eta^*\mu\nu_2(A_2-B_2)+2\bar{\eta}^*\mu\bar{\nu}_2\bar{A}_2 &= 0, \end{aligned} \tag{3.5}$$

if the surface is free and the layers can slide freely at the interface. Therefore we have four equations (3.5) and five quantities to be determined $\big(A_2, B_2, \bar{A}_2, p_0(h), p_0(0)-\bar{p}_0(0)\big)$. Two types of solutions are possible, on assuming either $p_0(h) = 0$ or $p_0(0)-\bar{p}_0(0) = 0$. Other assumptions do not lead to interesting cases; e.g. for $\bar{A}_2 = 0$, the waves are fully reflected in the upper layer (cf. Sect. 4.3).

In the first case ($p_0(h) = 0$), a solution exists if Eq. (3.2) is valid. Thus the condition of propagation along the x-axis and the condition of damping in the z-direction in the lower fluid are as follows:

$$-\frac{\omega^2}{2C_2^2}\cos^2\delta_2 < 0, \quad \frac{\bar{C}_2^2}{C_2^2} > 2\frac{\cos^2\bar{\delta}_2}{\cos^2\delta_2}. \tag{3.6}$$

Damping in the lower fluid may be full for certain discrete values of frequency ω for which

$$\frac{\bar{C}_2^2}{C_2^2} = 2\frac{\sin 2\bar{\delta}_2}{\sin 2\delta_2}. \tag{3.7}$$

The conditions $(3.6)_2$ and (3.7) may be satisfied simultaneously only for $\tan\bar{\delta}_2 > \tan\delta_2$, i.e. in the case in which the loss angle in the lower fluid is greater than that in the upper one.

In the second case ($p_0(0)-\bar{p}_0(0) = 0$), the condition of solution existence takes the form

$$\operatorname{th}\nu_2 h = -\frac{\nu_2 G_2^*}{\bar{\nu}_2 \bar{G}_2^*}. \tag{3.8}$$

This complex transcendental equation can easily be solved in the following limit cases: for $\operatorname{th}\nu_2 h \to 0$, i.e. for a very thin upper layer, and for $\operatorname{th}\nu_2 h \to 1$, i.e. for sufficiently high frequencies [3].

The conditions equivalent to Eq. (3.6) for $\operatorname{th}\nu_2 h = 1$ are as follows:

$$\frac{\bar{C}_2}{C_2} > \frac{\varrho}{\bar{\varrho}}, \quad \frac{\bar{C}_2}{C_2} < 1 \quad \left(\frac{\varrho}{\bar{\varrho}} < 1\right), \tag{3.9}$$

while for $\operatorname{th}\nu_2 h = 0$,

$$-\frac{\omega^2}{C_2^2}\cos^2\delta_2 < 0, \quad \frac{\bar{C}_2^2}{C_2^2} > \frac{\cos^2\bar{\delta}_2}{\cos^2\delta_2}. \tag{3.10}$$

In the latter case the condition of full damping, viz.

$$\frac{\bar{C}_2^2}{C_2^2} = \frac{\sin 2\bar{\delta}_2}{\sin 2\delta_2}, \tag{3.11}$$

can be satisfied only for $\tan\bar{\delta}_2 > \tan\delta_2$.

For two-layer incompressible fluids with rigid surfaces, the first two boundary conditions (3.5) change into

$$\begin{aligned} \nu_2(A_2 e^{\nu_2 h} - B_2 e^{-\nu_2 h}) &= 0, \\ \mu(A_2 e^{\nu_2 h} + B_2 e^{-\nu_2 h}) &= 0. \end{aligned} \tag{3.12}$$

Now, on assuming that $p_0(0)-\bar{p}_0(0) = 0$, we arrive at the following condition of existence:

$$\nu_2 = 0, \quad \mu^2 = -\frac{\varrho\omega^2}{G_2^*} = -\frac{\omega^2}{C_2^2}\cos^2\delta_2(1-i\tan\delta_2). \tag{3.13}$$

Thus the conditions of propagation and damping take exactly the form given by Eqs. (3.10) and (3.11).

Consider the case of two-layer incompressible fluids with the upper layer of thickness h (Fig. 2), and the layers adhering at the interface. The boundary conditions (3.5) should be completed with the following equations resulting from Eqs. (2.10):

$$\begin{aligned} \eta^*(\nu_2^2-\mu^2)(A_2+B_2) - \bar{\eta}^*(\bar{\nu}_2^2-\mu^2)\bar{A}_2 &= 0, \\ \nu_2(A_2-B_2) - \bar{\nu}_2\bar{A}_2 &= 0, \end{aligned} \tag{3.14}$$

if the surface is free. Therefore we have six equations (3.5) and (3.14) and five quantities

[3] This fact can be shown, expanding $\operatorname{th}\nu_2 h$ into real and imaginary parts, respectively.

to be determined $(A_2, B_2, \bar{A}_2, p_0(h), p_0(0)-\bar{p}_0(0))$. It is seen by inspection of Eqs. (3.5) and (3.14), that some particular solution can be obtained if

$$\eta^*(\nu_2^2-\mu^2) = \bar{\eta}^*(\bar{\nu}_2^2-\mu^2), \quad \mathrm{th}\nu_2 h = \frac{\nu_2}{\bar{\nu}_2}. \tag{3.15}$$

This solution gives

$$B_2 = -A_2 e^{2\nu_2 h}, \quad \bar{A}_2 = A_2(1-e^{2\nu_2 h}), \tag{3.16}$$

and the corresponding expressions for $p_0(h)$ and $p_0(0)-\bar{p}_0(0)$, resulting directly from Eqs. (3.5).

The conditions of propagation, damping and full damping, determined in a manner analogous to the previous cases, take the forms

$$\frac{\varrho}{\bar{\varrho}} < \frac{\bar{C}_2^2}{C_2^2} < \frac{2\varrho}{\varrho+\bar{\varrho}}, \quad \frac{\bar{C}_2^2}{C_2^2} = \frac{\varrho\tan\delta_2}{\bar{\varrho}\tan\bar{\delta}_2}, \tag{3.17}$$

for $\mathrm{th}\nu_2 h \to 1$, and

$$\frac{\varrho}{2\bar{\varrho}} < \frac{\bar{C}_2^2}{C_2^2} < \frac{\varrho}{\bar{\varrho}}, \quad \frac{\bar{C}_2^2}{C_2^2} = \frac{\varrho\tan\delta_2}{2\bar{\varrho}\tan\bar{\delta}_2}, \tag{3.18}$$

for $\mathrm{th}\nu_2 h \to 0$, respectively. The above conditions may be satisfied simultaneously only for certain discrete values of frequency ω for which

$$\tan\bar{\delta}_2 < \tan\delta_2 < \frac{2\bar{\varrho}}{\bar{\varrho}+\varrho}\tan\bar{\delta}_2, \tag{3.19}$$

and

$$\tan\bar{\delta}_2 < \tan\delta_2 < 2\tan\bar{\delta}_2, \tag{3.20}$$

respectively.

If in the above case of layers adhering at the interface the outer surface is rigid, there is no reasonable solution for the waves considered.

3.3. Waves of the Love type

It can be shown, like for elastic media, that in two-layer incompressible fluids with the layers adhering at the interface, there exist the surface-type waves whose amplitudes oscillate only in a direction perpendicular to the xz-plane. Such waves which may be called the Love-type waves do not occur in homogeneous fluids.

Now we assume the velocity components in the form

$$u = 0, \quad v = (A_2 e^{\nu_2 z} + B_2 e^{-\nu_2 z})\exp(\mu x + i\omega t), \quad w = 0, \tag{3.21}$$

and similar expressions with overbars. The boundary conditions are (Fig. 2)

$$T^{23} = 0 \quad \text{or} \quad v = 0, \quad T^{23} = \bar{T}^{23}, \quad v = \bar{v}, \tag{3.22}$$

if the surface is free or rigid, respectively. Nontrivial solutions of the problem exist if either

$$\mathrm{th}\nu_2 h = -\frac{\bar{G}_2^*\bar{\nu}_2}{G_2^*\nu_2} \quad \text{or} \quad \mathrm{th}\nu_2 h = -\frac{G_2^*\nu_2}{\bar{G}_2^*\bar{\nu}_2} \tag{3.23}$$

are satisfied.

In the limit case of very high frequencies, i.e. for $\operatorname{th}\nu_2 h \to 1$, we obtain results analogous to those described by Eqs. (3.9)

In the limit case of very thin upper layers, i.e. for $\operatorname{th}\nu_2 h \to 0$, Eqs. (3.23) give either $\bar{\nu}_2 \to 0$ or $\nu_2 \to 0$. Thus, for a free surface, these particular Love-type waves may be damped only in the upper layer, while for a rigid surface they may be damped only in the lower fluid.

The conditions of propagation along the x-axis, viz.

$$(3.24)\qquad -\frac{\omega^2}{\bar{C}_2^2}\cos^2\bar{\delta}_2 < 0 \quad \text{or} \quad -\frac{\omega^2}{C_2^2}\cos^2\delta_2 < 0,$$

are always satisfied for any frequency ω. The conditions of damping in the z-direction are as follows:

$$(3.25)\qquad \frac{\bar{C}_2^2}{C_2^2} < \frac{\cos^2\bar{\delta}_2}{\cos^2\delta_2} \quad \text{or} \quad \frac{\bar{C}_2^2}{C_2^2} > \frac{\cos^2\bar{\delta}_2}{\cos^2\delta_2}$$

for a free or rigid surface, respectively. In both cases the waves may be entirely damped in the z-direction if for certain discrete values of frequency ω Eq. (3.11) is true. The inequality $(3.25)_1$ and Eq. (3.11) require that $\tan\delta_2 > \tan\bar{\delta}_2$, while the inequality $(3.25)_2$ and Eq. (3.11) require that $\tan\bar{\delta}_2 > \tan\delta_2$.

4. Propagation and damping in compressible fluids

4.1. Waves in homogeneous fluids

For homogeneous compressible fluids, the boundary conditions $(2.8)_1$ for $z = 0$ lead to

$$(4.1)\qquad \begin{aligned} &\eta^*[2\mu\nu_1 A_1 + (\nu_2^2 - \mu^2)A_2] = 0,\\ &-p_0(0) + [\lambda^*(\nu_1^2+\mu^2) + 2\eta^*\nu_1^2]A_1 - 2\eta^*\mu\nu_2 A_2 = 0, \end{aligned}$$

if the surface is free. Therefore, we have two equations and three quantities to be determined $(A_1, A_2, p_0(0))$. Without any loss of generality we can put $p_0(0)=0$, and then a solution exists if

$$(4.2)\qquad 4\eta^*\mu^2\nu_1\nu_2 + [\lambda^*(\nu_1^2+\mu^2) + 2\eta^*\nu_1^2](\nu_2^2-\mu^2) = 0.$$

On introducing the notations

$$(4.3)\qquad \begin{aligned} n &= -\frac{\varrho\omega^2}{\mu^2 G_2^*},\\ \vartheta &= \frac{G_2^*}{G_1^*} = \frac{C_2^2}{C_1^2}\cos^2\delta_1[(1+\tan\delta_1\tan\delta_2) - i(\tan\delta_1 - \tan\delta_2)], \end{aligned}$$

we arrive at the following algebraic equation with complex coefficients:

$$(4.4)\qquad n^3 - 8n^2 + (24 - 16\vartheta)n - 16(1-\vartheta) = 0,$$

the form of which is similar to that describing the Rayleigh surface waves in elastic solids. Since for real viscoelastic fluids usually $\tan\delta_1 > \tan\delta_2$ and $C_1 > C_2$ (cf. [4]), we expect that also $\mathrm{Re}\,\vartheta < 1$ and $\mathrm{Im}\,\vartheta < 0$. Solutions of the above equation, such that $\mathrm{Re}\,n > 0$ or $\mathrm{Re}\,\mu^2 < 0$, essentially depend on the numerical values of complex ϑ.

The conditions of damping and full damping in the z-direction take the forms

$$\mathrm{Re}\,\mu^2 < -\frac{\omega^2}{C_2^2}\cos^2\delta_2, \quad \mathrm{Im}\,\mu^2 = \frac{\omega^2}{2C_2^2}\sin 2\delta_2. \tag{4.5}$$

They may be satisfied simultaneously only for $\tan\delta_2 < -\mathrm{Im}\,\mu^2/\mathrm{Re}\,\mu^2$.

If the surface of homogeneous compressible fluids is rigid, the boundary conditions $(2.8)_2$ for $z = 0$ lead to

$$\mu A_1 + \nu_2 A_2 = 0, \quad \nu_1 A_1 - \mu A_2 = 0. \tag{4.6}$$

The condition of solution existence requires that

$$\nu_1\nu_2 + \mu^2 = 0 \quad \text{or} \quad n = 1 + \frac{1}{\vartheta}, \tag{4.7}$$

where n, ϑ are defined by Eqs. (4.3).

On denoting

$$C_{12}^2 = \frac{G_1'(\omega)+G_2'(\omega)}{\varrho}, \quad \tan\delta_{12} = \frac{G_1''(\omega)+G_2''(\omega)}{G_1'(\omega)+G_2'(\omega)}, \tag{4.8}$$

we arrive at the following condition of propagation:

$$-\frac{\omega^2}{C_{12}^2}\cos^2\delta_{12} < 0, \tag{4.9}$$

which is satisfied for any ω. This means that in the case considered the Rayleigh-type surface waves always propagate along the x-axis. The conditions of damping and full damping in the z-direction are

$$\frac{C_{12}^2}{C_1^2} < \frac{\cos^2\delta_{12}}{\cos^2\delta_1}, \quad \frac{C_{12}^2}{C_1^2} = \frac{\sin 2\delta_{12}}{\sin 2\delta_1}, \tag{4.10}$$

for dilatational waves, and

$$\frac{C_{12}^2}{C_2^2} < \frac{\cos^2\delta_{12}}{\cos^2\delta_2}, \quad \frac{C_{12}^2}{C_2^2} = \frac{\sin 2\delta_{12}}{\sin 2\delta_2}, \tag{4.11}$$

for shear waves. The above conditions may be satisfied for certain discrete values of the frequency ω if $\tan\delta_1 > \tan\delta_2$ and $\tan\delta_1 < \tan\delta_2$, respectively. Thus, for real fluids only dilatational waves may be fully damped in the z-direction; shear waves always propagate in both directions.

4.2. Waves in two-layer fluids

For two-layer compressible fluids, with the upper layer of thickness h (Fig. 2), the boundary conditions $(2.8)_1$ and (2.9) lead to

$$2\mu\nu_1(A_1 e^{\nu_1 h} - B_1 e^{-\nu_1 h}) + (\nu_2^2 - \mu^2)(A_2 e^{\nu_2 h} - B_2 e^{-\nu_2 h}) = 0,$$
$$-p_0(h) + [\lambda^*(\nu_1^2+\mu^2) + 2\eta^*\nu_1^2](A_1 e^{\nu_1 h} + B_1 e^{-\nu_1 h}) - 2\eta^*\mu\nu_2(A_2 e^{\nu_2 h} - B_2 e^{-\nu_2 h}) = 0,$$

$$\nu_1(A_1-B_1)-\mu(A_2+B_2)-\bar{\nu}_1\bar{A}_1+\mu\bar{A}_2 = 0, \tag{4.12}$$
$$-\big(p_0(0)-\bar{p}_0(0)\big)+[\lambda^*(\nu_1^2+\mu^2)+2\eta^*\nu_1^2](A_1+B_1)-2\eta^*\mu\nu_2(A_2-B_2)$$
$$-[\bar{\lambda}^*(\bar{\nu}_1^2+\mu^2)+2\bar{\eta}^*\bar{\nu}_1^2]\bar{A}_1+2\bar{\eta}^*\mu\bar{\nu}_2\bar{A}_2 = 0,$$

if the surface is free, and the layers can slide freely at the interface. The above four equations involve eight quantities to be determined (A_i, B_i, $\bar{A}_i$, $i = 1, 2$, $p_0(h)$, $p_0(0)-\bar{p}_0(0)$). Instead of seeking various particular solutions of the whole problem, we consider the case in which the waves are only transmitted, without being reflected, from the upper to lower fluid, i.e. for $B_1 = B_2 = 0$ ([4]). Without any loss of generality we again assume that $p_0(h) = 0$, $p_0(0)-\bar{p}_0(0) = 0$. Then the condition of solution existence can be expressed in the form of two equations:

$$\begin{aligned}&[\lambda^*(\nu_1^2+\mu^2)+2\eta^*\nu_1^2](\nu_2^2-\mu^2)-4\eta^*\mu^2\nu_1\nu_2 = 0,\\ &\bar{\lambda}^*(\bar{\nu}_1^2+\mu^2)+2\bar{\eta}^*\nu_1^2-2\bar{\eta}^*\bar{\nu}_1\bar{\nu}_2 = 0.\end{aligned} \tag{4.13}$$

The first equation, involving material functions of the upper fluid, is equivalent to Eq. (4.2) or Eq. (4.4), describing the Rayleigh-type surface waves. The second equation, involving material functions of the lower fluid, leads to the following propagation condition along the x-axis:

$$\frac{\bar{C}_2^2}{\bar{C}_1^2} > \frac{\cos^2\bar{\delta}_2}{4}[(\tan\bar{\delta}_1-\tan\bar{\delta}_2)\tan\bar{\delta}_2+(1+\tan\bar{\delta}_1\tan\bar{\delta}_2)]. \tag{4.14}$$

There exists no damping in that direction if, moreover,

$$\frac{\bar{C}_2^2}{\bar{C}_1^2} = \frac{\cos^2\bar{\delta}_2}{4\tan\bar{\delta}_2}[(\tan\bar{\delta}_1-\tan\bar{\delta}_2)-\tan\bar{\delta}_2(1+\tan\bar{\delta}_1\tan\bar{\delta}_2)] \tag{4.15}$$

for certain discrete values of the frequency ω. Both conditions may be satisfied simultaneously only for $\tan\bar{\delta}_1 > \tan\bar{\delta}_2$ (what is realistic) and for $\tan\bar{\delta}_2 < 1$. The damping condition in the lower fluid can be discussed only for dilatational waves; shear waves are never damped if $\tan\bar{\delta}_1 > \tan\bar{\delta}_2$. Thus the waves may be damped in the z-direction if $\bar{C}_2^2/\bar{C}_1^2$ is contained between the values resulting from the relations

$$\left(\frac{\bar{C}_2^2}{\bar{C}_1^2}\right)_{1,2} = \frac{1+\tan\bar{\delta}_1\tan\bar{\delta}_2}{2\cos^2\bar{\delta}_1[(1+\tan\bar{\delta}_1\tan\delta_2)^2-(\tan\bar{\delta}_1-\tan\bar{\delta}_2)^2]}\left(1\pm\frac{\tan\bar{\delta}_1-\tan\bar{\delta}_2}{1+\tan\bar{\delta}_1\tan\bar{\delta}_2}\right).$$

This damping is full if

$$\frac{\bar{C}_2^2}{\bar{C}_1^2} = \frac{1}{2}\frac{\sin 2\bar{\delta}_2}{\sin 2\bar{\delta}_1}\left(1+\sqrt{1+\sin^2\bar{\delta}_1}\right) \tag{4.17}$$

for certain discrete values of the frequency ω.

Similar conditions can be formulated for waves in the upper layer. Instead of Eqs. (4.16) and (4.17), we obtain

([4]) The case in which the waves are fully reflected in the upper layer ($\bar{A}_1 = \bar{A}_2 = 0$) is discussed separately in Sect. 4.3.

$$(4.18)\qquad \frac{\overline{C}_1^2}{C_1^2} < \frac{4\Delta\cos^2\overline{\delta}_1(1+\tan\overline{\delta}_1\tan\overline{\delta}_2)-1}{4\Delta^2\cos^2\overline{\delta}_1\cos^2\delta_1[(1+\tan\delta_1\tan\overline{\delta}_2)\times(1+\tan\overline{\delta}_1\tan\overline{\delta}_2)-(\tan\overline{\delta}_1-\tan\overline{\delta}_2)(\tan\delta_1-\tan\overline{\delta}_2)]},$$

$$(4.19)\qquad \frac{\overline{C}_1^2}{C_1^2} = \sin 2\overline{\delta}_2\,\frac{\cos^2\overline{\delta}_2\tan\overline{\delta}_2\tan\overline{\delta}_1+4\Delta}{4\Delta^2\sin 2\delta_1},$$

for dilatational waves and

$$(4.20)\qquad \frac{\overline{C}_2^2}{C_2^2} < \frac{4\Delta\cos^2\overline{\delta}_1(1+\tan\overline{\delta}_1\tan\overline{\delta}_2)-1}{4\Delta^2\cos^2\overline{\delta}_1\cos^2\delta_2[(1+\tan\overline{\delta}_2\tan\delta_2)\times(1+\tan\overline{\delta}_1\tan\overline{\delta}_2)-(\tan\overline{\delta}_1-\tan\overline{\delta}_2)(\tan\overline{\delta}_2-\tan\delta_2)]},$$

$$(4.21)\qquad \frac{\overline{C}_2^2}{C_2^2} = \sin 2\overline{\delta}_2\,\frac{\cos^2\overline{\delta}_2\tan\overline{\delta}_2\tan\overline{\delta}_1+4\Delta}{4\Delta^2\sin 2\delta_2}$$

for shear waves, respectively. In the above formulae $\Delta = \overline{C}_2^2/\overline{C}_1^2$.

If the surface of two-layer compressible fluids is rigid, the first two boundary conditions (4.12) should be replaced by

$$(4.22)\qquad \begin{aligned} &\mu(A_1 e^{\nu_1 h}+B_1 e^{-\nu_1 h})+\nu_2(A_2 e^{\nu_2 h}-B_2 e^{-\nu_2 h}) = 0,\\ &\nu_1(A_1 e^{\nu_1 h}-B_1 e^{-\nu_1 h})-\mu(A_2 e^{\nu_2 h}+B_2 e^{-\nu_2 h}) = 0. \end{aligned}$$

In this case the assumption $B_1 = B_2 = 0$ together with $p_0(0)-\overline{p}_0(0) = 0$, leads to the following existence conditions:

$$(4.23)\qquad \begin{aligned} &\mu^2-\nu_1\nu_2 = 0,\\ &\overline{\lambda}^*(\overline{\nu}_1^2+\mu^2)+2\overline{\eta}^*\overline{\nu}_1^2-2\overline{\eta}^*\overline{\nu}_1\overline{\nu}_2 = 0, \end{aligned}$$

involving, like in the previous case, material functions of the upper and lower fluid, respectively. The first equation is equivalent to Eq.$(4.7)_1$, while the second one to Eq.$(4.13)_2$. Therefore, everything said after those equations remains true in the case of two-layer fluids with the rigid surfaces.

In a similar way other cases of surface- and interface-type waves in two-layer compressible fluids can be discussed in greater detail. By way of illustration, consider the case of a fluid with the free surface and the layers adhering at the interface, under the additional assumption that in the upper fluid (incompressible) only shear waves can exist. From the boundary conditions $(2.8)_1$, (2.9) and (2.10), we obtain the following system of equations:

$$(4.24)\qquad \begin{aligned} (\nu_2^2-\mu^2)(A_2 e^{\nu_2 h}+B_2 e^{-\nu_2 h}) &= 0,\\ -p_0(h)-2\eta^*\mu\nu_2(A_2 e^{\nu_2 h}-B_2 e^{-\nu_2 h}) &= 0,\\ \nu_2(A_2-B_2)-\mu\overline{A}_1-\overline{\nu}_2\overline{A}_2 &= 0,\\ \mu(A_2+B_2)+\overline{\nu}_1\overline{A}_1-\mu\overline{A}_2 &= 0,\\ \eta^*(\nu_2^2-\mu^2)(A_2+B_2)-2\overline{\eta}^*\mu\overline{\nu}_1\overline{A}_1-\overline{\eta}^*(\overline{\nu}_2^2-\mu^2)\overline{A}_2 &= 0,\\ -\big(p_0(0)-\overline{p}_0(0)\big)-2\eta^*\mu\nu_2(A_2-B_2)-[\overline{\lambda}^*(\overline{\nu}_1^2+\mu^2)+2\overline{\eta}^*\overline{\nu}_1^2]\overline{A}_1+2\overline{\eta}^*\mu\overline{\nu}_2\overline{A}_2 &= 0. \end{aligned}$$

Therefore we have six equations and six quantities to be determined $\big(A_2, B_2, \bar{A}_1, \bar{A}_2, p_0(h), p_0(0)-\bar{p}_0(0)\big)$. The solution of the above system exists if

$$\mu^2 = \nu_2^2 = -\frac{\omega^2}{2C_2^2}\cos^2\delta_2(1-i\tan\delta_2). \tag{4.25}$$

The condition of propagation along the x-axis is exactly the same as Eq. $(3.6)_1$, and the waves always propagate in that direction. On the other hand, the condition of damping and full damping in the z-direction in the lower fluid (compressible) are

$$\frac{\bar{C}_1^2}{C_1^2} > 2\frac{C_2^2}{C_1^2}\frac{\cos^2\bar{\delta}_1}{\cos^2\delta_2}, \quad \frac{\bar{C}_1^2}{C_1^2} = 2\frac{C_2^2}{C_1^2}\frac{\sin 2\bar{\delta}_1}{\sin 2\delta_2}, \tag{4.26}$$

for dilatational waves and

$$\frac{\bar{C}_2^2}{C_2^2} > 2\frac{\cos^2\bar{\delta}_2}{\cos^2\delta_2}, \quad \frac{\bar{C}_2^2}{C_2^2} = 2\frac{\sin 2\bar{\delta}_2}{\sin 2\delta_2}, \tag{4.27}$$

for shear waves, respectively. The above conditions may be satisfied for certain discrete values of the frequency ω if $\tan\bar{\delta}_1 > \tan\delta_2$ and $\tan\bar{\delta}_2 > \tan\delta_2$.

4.3. Surface layers as waveguides

In the previous examples we already had cases in which the waves considered propagated along a thin surface layer, being fully damped in perpendicular directions. Interesting cases arise when in two-layer compressible fluids with free or rigid surfaces the waves are fully reflected in the upper layer, leading to some sort of a waveguide.

To this end we assume that in the boundary conditions (4.12) $\bar{A}_1 = \bar{A}_2 = 0$, what means that no transmitted waves can occur in the lower fluid. Without any loss of generality we also take $p_0(h) = 0$ and $p_0(0)-\bar{p}_0(0) = 0$. A solution of the problem exists if either of the following equations is satisfied:

$$\begin{gathered}[\lambda^*(\nu_2^2+\mu^2)+2\eta^*\nu_1^2](\nu_2^2-\mu^2)+4\eta^*\mu^2\nu_1\nu_2 = 0,\\ \lambda^*(\nu_1^2+\mu^2)+2\eta^*\nu_1^2+2\eta^*\nu_1\nu_2 = 0.\end{gathered} \tag{4.28}$$

The case of full reflection in the upper layer is possible if, moreover,

$$\frac{\bar{C}_1^2}{C_1^2} > \frac{1+\tan^2\bar{\delta}_1}{1+\tan\delta_1\tan\bar{\delta}_1}, \quad \frac{\bar{C}_2^2}{C_2^2} > \frac{1+\tan^2\bar{\delta}_2}{1+\tan\delta_2\tan\bar{\delta}_2}. \tag{4.29}$$

The condition $(4.28)_1$ is equivalent to Eq. (4.2) or Eq. (4.4) describing the Rayleigh-type surface waves in homogeneous fluids, while the condition $(4.28)_2$ reminds that Eq. $(4.13)_2$, but with material parameters referred to the upper fluid. The condition of propagation along the upper layer, resulting from Eq. $(4.28)_2$, takes the form

$$\frac{C_2^2}{C_1^2} > \frac{\cos^2\delta_2}{4}[(\tan\delta_1-\tan\delta_2)\tan\delta_2+(1+\tan\delta_1\tan\delta_2)]. \tag{4.30}$$

There is no damping along the upper layer if also

$$\frac{C_2^2}{C_1^2} = \frac{\cos^2\delta_2}{4\tan\delta_2}[(\tan\delta_1-\tan\delta_2)-\tan\delta_2(1+\tan\delta_1\tan\delta_2)] \tag{4.31}$$

for certain discrete values of the frequency ω. The above conditions may be satisfied simultaneously only for $\tan\delta_1 > \tan\delta_2$ and $\tan\delta_2 < 1$. The conditions of damping and full damping for dilatational waves are satisfied if C_2^2/C_1^2 is contained between the following values:

$$\left(\frac{C_2^2}{C_1^2}\right)_{1,2} = \frac{1+\tan\delta_1\tan\delta_2}{2\cos^2\delta_1[(1+\tan\delta_1\tan\delta_2)^2-(\tan\delta_1-\tan\delta_2)^2]}\left(1\pm\frac{\tan\delta_1-\tan\delta_2}{1+\tan\delta_1\tan\delta_2}\right) \tag{4.32}$$

and

$$\frac{C_2^2}{C_1^2} = \frac{1}{2}\frac{\sin 2\delta_2}{\sin 2\delta_1}\left(1+\sqrt{1+\sin^2\delta_1}\right) \tag{4.33}$$

for certain discrete values of the frequency ω. Note that shear waves are never damped if $\tan\delta_1 > \tan\delta_2$.

If the surface of the upper layer is rigid, we obtain, instead of Eqs. (4.28), the following conditions of existence:

$$\begin{gathered} \mu^2-\nu_1\nu_2 = 0, \\ \lambda^*(\nu_1^2+\mu^2)+2\eta^*\nu_1^2-2\eta^*\nu_1\nu_2 = 0. \end{gathered} \tag{4.34}$$

The condition $(4.34)_1$ is equivalent to Eq. (4.7); thus the Rayleigh-type surface waves always propagate along the upper layer, being damped in the z-direction according to Eqs. (4.10) and (4.11). The condition $(4.34)_2$ is similar to Eq. (4.13) or Eq. (4.23) but with material parameters referred to the upper fluid. The conditions of propagation and damping are again expressed by Eqs. (4.30)–(4.33).

On the basis of the above results, it can be concluded that both types of waves can propagate along the upper layer and, under certain additional conditions, the propagation in that direction is full, i.e. is not accompanied by any damping. In such a case the layer of the upper fluid, sliding freely at the interface, acts like a waveguide.

5. Final remarks

When discussing previous examples of the surface- and interface-type waves in homogeneous and two-layer fluids, we often emphasized the essential differences in the solutions as compared with elastic and viscoelastic solids. These differences result from the fact that for compressible as well as for incompressible fluids, there exists one more function to be determined — the hydrostatic pressure. Thus a number of variables in the equations resulting from the appropriate boundary conditions is usually greater than in the case of solids. Frequently, without any loss of generality, it was possible to choose the values of hydrostatic pressure at the surface or interface in such a way that unique solutions of the problem could be found.

In the majority of cases the waves can propagate parallely to the surface or interface, being simultaneously damped in both directions. Under certain additional assumptions damping in the direction perpendicular to the interface may be full, that is without any propagation in that direction. Such cases are possible only for certain discrete values of the frequency ω. Moreover, under the conditions of full reflection, the upper layer of a compressible fluid, sliding freely at the interface, may act like a waveguide.

Some results obtained in the paper can be specified or simplified for purely elastic or elastic-like fluids (cf. [5, 6]). Elastic fluids are the limit cases of viscoelastic fluids if frequencies tend to infinity; then the loss angles $\tan\delta_i (i = 1, 2)$ tend to zero and the speeds of propagation $C_i (i = 1, 2)$ become constants. A less realistic notion of the elastic-like fluids corresponds to the case in which $\tan\delta_i = 0$ $(i = 1, 2)$ but $C_i(\omega) (i = 1, 2)$ depend on the frequency ω. Let us mention certain examples of the results valid for purely elastic and elastic-like fluids.

The condition $(3.23)_1$ describing the Love-type shear waves leads to the following transcendental equation:

$$\tan\left(\omega h \sqrt{\frac{1}{C_2^2} - \frac{1}{C_L^2}}\right) = \frac{\bar{G}_2' \sqrt{\frac{1}{C_L^2} - \frac{1}{\bar{C}_2^2}}}{G_2' \sqrt{\frac{1}{C_2^2} - \frac{1}{C_L^2}}}, \tag{5.1}$$

where $C_L^2 = -\omega^2/\mu^2$ denotes the frequency-dependent velocity of waves. This equation has a real root C_L such that $\bar{C}_2 > C_L > C_2$, if $\bar{C}_2/C_2 > 1$.

Similarly, Eq. (4.4) becomes the well-known equation of the Rayleigh surface waves in elastic fluids if $\vartheta = C_2/C_1 < 1$ and $n < 1$. For example, for $C_2/C_1 = 1/3$, we obtain $n = 2-2\sqrt{3} = 0.8453$ or $C_R = 0.981 C_2$, where $C_R^2 = -\omega^2/\mu^2 = C_2^2 n$ denotes the frequency-independent velocity of Rayleigh waves.

The conditions of propagation (4.14) and damping (4.16), for the waves in two-layer compressible fluids with free surfaces and the layers sliding freely at the interfaces lead to the inequalities

$$\frac{1}{2} < \frac{\bar{C}_2}{\bar{C}_1} < \frac{1}{\sqrt{2}}. \tag{5.2}$$

Here only dilatational waves can be damped in the lower fluid; shear waves are never damped. If the upper fluid is incompressible and the layer adheres at the interface, we have instead of Eq. $(4.27)_1$

$$\frac{\bar{C}_2}{C_2} > \sqrt{2} \quad \text{since} \quad \bar{C}_1 > \bar{C}_2. \tag{5.3}$$

In this case dilatational as well as shear waves can be damped in the lower fluid.

It is also worth noting that the velocity amplitudes of the waves considered may be equal to zero at certain critical distances from the interface. Therefore, in the case of two-layer fluids composed of the upper incompressible fluid and the lower compressible one, with free surfaces and the layers adhering at the interfaces, the corresponding components of the velocity amplitudes (for $\bar{u}$ and $\bar{w}$) are equal to zero if

$$z_{cr} = \frac{1}{\bar{\nu}_1 - \bar{\nu}_2} \ln \frac{2\bar{\nu}_1\bar{\nu}_2}{\nu_2^2 - \mu^2} \quad \text{and} \quad z_{cr} = \frac{1}{\bar{\nu}_1 - \bar{\nu}_2} \ln \frac{2\nu_2^2}{\mu^2 - \bar{\nu}_2^2}, \tag{5.4}$$

respectively. By inspection of the right-hand sides of Eqs. (5.4) it can be proved that for both components $z_{cr} < 0$, i.e. the critical values of depth are situated in the lower fluid if

$$\frac{\bar{C}_2^2}{C_2^2} > \frac{4\Delta - \cos^2\bar{\delta}_2(1+\tan\bar{\delta}_1\tan\bar{\delta}_2)}{2\Delta\cos^2\delta_2(1+\tan\delta_2\tan\bar{\delta}_2)}, \quad \Delta = \frac{\bar{C}_2^2}{\bar{C}_1^2} \tag{5.5}$$

and

$$\frac{\bar{C}_2^2}{C_2^2} > 2\,\frac{1+\tan^2\delta_2}{1+\tan\delta_2\tan\bar{\delta}_2}, \tag{5.6}$$

respectively. For elastic or elastic-like fluids the latter inequality is identical to the inequality (5.3).

Possible generalizations of the problems considered can be realized either by solving other more complex cases with the layers sliding or adhering at the interface or by taking into account the effects of surface and interface tensions.

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