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Behaviour of stationary singular points in one-dimensional materials with memory

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IT IS SHOWN that in one-dimensional materials with memory there can exist stationary singular points at which the second deformation gradients are discontinuous. An external force which is discontinuous at a material point gives rise to such a discontinuity. The variation of the discontinuity is governed by a Volterra's integral equation. The stress relaxation function can be determined by observing the behaviour of the discontinuity induced by a certain discontinuous external force.

Wykazano, że w jednowymiarowych materiałach z pamięcią mogą istnieć takie punkty osobliwe, w których drugie gradienty deformacji są nieciągłe. Obciążenie zewnętrzne nieciągłe w punkcie materialnym prowadzi do takiej nieciągłości. Zmianą takiej nieciągłości rządzi równanie całkowe Volterry. Funkcję relaksacji naprężeń można wyznaczyć drogą obserwacji zachowania się tej nieciągłości pod wpływem pewnego nieciągłego obciążenia zewnętrznego.

Показано, что в одномерных материалах с памятью могут существовать такие особые точки, в которых вторые градиенты деформаций имеют разрыв. Внешняя нагрузка разрывная в материальной точке приводит к такому разрыву. Изменением такого разрыва управляет интегральное уравнение Вольтерра. Функцию релаксации напряжений можно определить путем наблюдений поведения этого разрыва под влиянием некоторой разрывной внешней нагрузки.

1. Introduction

THERE HAVE BEEN many works on wave propagation in materials with memory and materials with internal variables. Shock waves in materials with memory were investigated in [1, 2], while acceleration waves were studied in [1, 3–8]. Exact solutions of steady flows, which admit both shock and acceleration waves, were obtained in [9, 10]. On the other hand, shock waves in materials with internal variables were investigated in [11, 12], while acceleration waves were studied in [11, 13–17]. All the waves considered above are defined by propagating singularities.

Recently, the author [18–20] showed that there may exist stationary singular points or surfaces in materials with internal variables. According to [21, 22], such materials with internal variables can be regarded as special materials with memory. Thus there is the possibility that stationary singular points or surfaces exist in familiar classes of materials with memory.

This paper considers a one-dimensional material with fading memory, and the constitutive assumptions are given in the next section. Across a stationary singular point in such a material, the spatial derivative of the history of the deformation gradient has a discontinuity. Since the history of the deformation gradient belongs to a functional space, the

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stationary singular point is also a discontinuity point with respect to a mapping whose range is the functional space. In the next section we obtain a kinematical compatibility condition for such a discontinuous mapping. Using the result, we prove in Sect. 3 that there may exist stationary singular points of the second order. In [18–20] it has not been discussed how the stationary singularities are generated: In this section we show that a stationary singular point is induced by an external force which is discontinuous across a fixed material point. In Sect. 4 it is shown that the amplitude of a stationary singular point, i.e. the jump of the second deformation gradient, is governed by a Volterra's integral equation. In Sect. 5, by applying the result to the material with an unknown response functional, we propose a method for determining the stress relaxation function by observing the behaviour of the stationary singular point.

2. Constitutive equation and generalized kinematical compatibility condition

Let us consider a one-dimensional material with memory defined as follows: Let u denote the deformation, X the coordinate in the reference configuration, v the velocity, and F the deformation gradient. Then F_r^t, the past history of the deformation gradient at time t, is a function on $(0, \infty)$ defined by

$$F_r^t(s) \equiv F(t-s), \qquad s \in (0, \infty). \tag{2.1}$$

Suppose that the space of all past histories of deformation gradients is a Hilbert space Y with the inner product: for $y, z \in Y$

$$(y, z) \equiv \int_0^\infty y(s)z(s)h^2(s)ds, \tag{2.2}$$

where h is a positive, monotone-decreasing, smooth function on $[0, \infty)$ decaying to zero fast enough. Suppose further that Y has a complete system of orthogonal smooth functions $\{l_i(s)\}$. The pair $(F(t), F_r^t)$ is called the history of the deformation gradient at time t.

The stress σ is given by a functional of the history of the deformation gradient:

$$\sigma(t) = \phi\big(F(t), F_r^t\big). \tag{2.3}$$

We assume that ϕ is continuously differentiable with respect to both arguments. Here it should be noted that the deformation gradient together with its past history depends on the coordinate also. Henceforth for simplicity we omit all the arguments of functions or functionals unless there is any danger of ambiguity. The past history of the deformation gradient F is denoted by F_r. Let y be a smooth function of $t \in (-\infty, \infty)$, and define a smooth curve in $(X-t)$ space by

$$\Sigma \equiv \bigcup_{t\in(-\infty,\infty)} \big(y(t), t\big). \tag{2.4}$$

We call a point $y(t)$ a second-order singular point if the following conditions are satisfied:

i) u, v, F, and F_r are continuous everywhere in $(X-t)$ space.

ii) v, F, and F_r are continuously differentiable with respect to X and t everywhere else except on Σ, where their first derivatives suffer finite jump discontinuities.

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Here the continuity and the differentiability of F_r with respect to X and t, respectively, are those of the mapping $(X, t) \to F^t(\cdot, X)$ from R^2 into Y.

The velocity of the singular point is given by

$$U = \frac{dy}{dt}. \tag{2.5}$$

If U vanishes identically, the singular point is called stationary.

Our conditions of the second-order singular point are weaker than those of the acceleration wave given by COLEMAN, GURTIN and HERRERA [1]. They assumed, in addition to i) and ii), that F_r is continuously differentiable everywhere with respect to X. It will be shown in the next section that under their assumption no stationary singular point of second order can exist.

Let us next derive a kinematic compatibility condition for a discontinuous mapping ψ from $(X-t)$ space into Y. Suppose that ψ is continuously differentiable everywhere except on Σ, and that ψ and its first derivatives suffer finite jump discontinuities across Σ. In view of the properties of Y, $\psi(X, t)$ for any point except on Σ can be expressed as

$$\psi(X, t) = c_i(X, t) l_i, \tag{2.6}$$

where

$$c_i(X, t) \equiv \big(\psi(X, t), l_i\big), \tag{2.7}$$

and where the summation is carried over all l_i. The mapping $\xi \to (\xi, l_i)$: $Y \to R$ is linear and continuous, and hence it is also smooth, cf. [23]. Since for every i c_i is the composition of the mappings ψ and $\xi \to (\xi, l_i)$, it is smooth everywhere except on Σ, and it as well as its first derivatives suffer finite jump discontinuities across Σ. It follows then that

$$[\psi] = [c_i] l_i, \ [\partial_X \psi] = [\partial_X c_i] l_i, \ [\partial_t \psi] = [\partial_t c_i] l_i, \tag{2.8}$$

where [ ] denotes the jump of a quantity within it across Σ, and ∂_X and ∂_t the partial differentiations with respect to X and t, respectively. The first-order kinematic compatibility condition for c_i is given by

$$\frac{\delta}{\delta t}[c_i] = [\partial_t c_i] + U[\partial_X c_i], \tag{2.9}$$

where $\delta/\delta t$ means differentiation with respect to t along the path $t \to (y(t), t)$ in $(X-t)$ space. Taking the inner product of Eq. (2.9) with l_i in Y for every i, and then adding the results, by use of Eq. (2.8) we obtain the kinematic compatibility condition for ψ:

$$\frac{\delta}{\delta t}[\psi] = [\partial_t \psi] + U[\partial_X \psi]. \tag{2.10}$$

3. Existence of stationary singular points of second order

The first-order kinematic compatibility conditions for v, F, and F_r across a second-order singular point are given by

$$[\partial_t v] + U[\partial_X v] = 0, \tag{3.1}$$

$$[\partial_t F] + U[\partial_X F] = 0, \tag{3.2}$$

$$[\partial_t F_r] + U[\partial_X F_r] = 0, \tag{3.3}$$

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where we have made use of Eq. (2.10) for Eq. (3.3). Note that $\partial_t F_r$ and $\partial_X F_r$ are the temporal and the spatial derivatives of the mapping $(X, t) \to F^t(\cdot, X)$ from R^2 into Y, and that they also belong to Y.

Substituting the constitutive equation into the balance law of linear momentum, we have

$$\partial_F \phi\, \partial_X F + \partial_{F_r}\phi\, \partial_X F_r = \varrho\, \partial_t v, \tag{3.4}$$

where the external forces are assumed to be absent. The derivative of ϕ with respect to $F_r \in Y$, $\partial_{F_r}\phi$, is a continuous linear functional on Y. The jump of Eq. (3.4) across Σ is

$$\partial_F \phi[\partial_X F] + \partial_{F_r}\phi[\partial_X F_r] = \varrho[\partial_t v]. \tag{3.5}$$

Eliminating $[\partial_t v]$ from Eq. (3.5) by use of Eqs. (3.1) and (3.2), we get

$$(\partial_F \phi - \varrho U^2)[\partial_X F] + \partial_{F_r}\phi[\partial_X F_r] = 0. \tag{3.6}$$

In order to show the existence of stationary singular points, we calculate at any point in $(X-t)$ space except on Σ,

$$(l_i, \partial_t F_r) = \int_0^\infty l_i(s)\, \partial_t F(t-s) h^2(s) ds = l_i(0) h^2(0) F(t) + \int_0^\infty F(t-s)\, \partial_s \{l_i(s) h^2(s)\} ds, \tag{3.7}$$

where $\partial_t F_r$ may be identified with the past history of $\partial_t F$ as an element of Y, and where we have assumed that $|\partial_s \{l_i(s) h^2(s)\}|$ is decaying to zero fast enough. We note furthermore that

$$\lim_{s\to\infty} \{l_i(s) F(t-s) h^2(s)\} = 0, \tag{3.8}$$

because $l_i(s)F(t-s)h^2(s)$ is integrable on $(0, \infty)$ with respect to s. Since F is continuous everywhere, so is the right hand side of Eq. (3.7). Hence, by taking the jump of Eq. (3.7) across Σ, we have

$$[(l_i, \partial_t F_r)] = (l_i, [\partial_t F_r]) = 0. \tag{3.9}$$

Since $\{l_i\}$ is a complete system on Y, Eq. (3.9) implies that

$$[\partial_t F_r] = 0, \tag{3.10}$$

which, when combined with Eq. (3.3), yields

$$U[\partial_X F_r] = 0. \tag{3.11}$$

Thus there are two possibilities: If $[\partial_X F_r] = 0$, the first derivatives of F_r are continuous across Σ. Then for $y(t)$ to be a second-order singular point, $[\partial_X F]$ should not vanish because otherwise the condition (ii) is not satisfied in view of Eqs. (3.1) and (3.2). Equation (3.6) implies then that

$$\varrho U^2 = \partial_F \phi. \tag{3.12}$$

Here and henceforth we assume that $\partial_F \phi > 0$, so that the singular point is a usual acceleration wave. Thus we see that if we assume a priori the continuity of $\partial_X F_r$ across Σ as done in [1], no stationary singular point can exist.

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In the second case when $U = 0$, $y(t)$ is a stationary singular point, and Eqs. (3.1), (3.2) and (3.6) become, respectively,

$$[\partial_t v] = [\partial_t F] = 0, \tag{3.13}$$

$$\partial_F \phi [\partial_X F] + \partial_{F_r} \phi [\partial_X F_r] = 0. \tag{3.14}$$

Henceforth we shall be concerned with stationary singular points only. So far we have assumed that external forces do not exist. If there is an external force which is continuous with respect to X, all the above jump relations remain valid. We next consider the case when the material is subject to a time-dependent external force f which is discontinuous across a fixed material point. We simply call such a force a discontinuous external force. The jump of the balance law of linear momentum across the fixed point is then given by

$$\partial_F \phi [\partial_X F] + \partial_{F_r} \phi [\partial_X F_r] + [f] = 0. \tag{3.15}$$

Since $[f] \neq 0$ in Eq. (3.15), either $[\partial_X F]$ or $[\partial_X F_r]$ or both do not vanish. Hence we conclude that a discontinuous external force induces a stationary singular point.

4. Integral equation for amplitude of stationary singular point

Let us call $[\partial_X F]$ the amplitude of the stationary singular point, and write

$$a \equiv [\partial_X F]. \tag{4.1}$$

We assume that F is a known function of time at the stationary singular point, so that F_r is also known in time. Then the derivatives of ϕ, $\partial_F \phi$ and $\partial_{F_r} \phi$, are also known in time at the singular point. We write

$$E(t) \equiv \partial_F \phi \left(F(t), F_r^t\right), \tag{4.2}$$

which is called the instantaneous tangent modulus. By the Riesz representation theorem for the inner product space Y, for the prescribed $F(t)$ and F_r^t the continuous linear functional $\partial_{F_r} \phi$ on Y may be expressed as

$$\partial_{F_r} \phi \left(F(t), F_r^t\right) p = \int_0^\infty K\left(s; F(t), F_r^t\right) p(s) h^2(s) ds = \int_0^\infty G'(s; t) p(s) ds \tag{4.3}$$

for every $p \in Y$, cf. [1]. By use of Eqs. (4.1), (4.2) and $(4.3)_2$, Eq. (3.14) may be written as

$$E(t) a(t) + \int_0^\infty G'(s; t) a(t-s) ds = 0. \tag{4.4}$$

Introducing the new variable of integration $\tau = t - s$ in place of s in Eq. (4.4) and then dividing the result by $E(t)$, we obtain

$$a(t) + \int_{-\infty}^{t} \frac{G'(t-\tau; t)}{E(t)} a(\tau) d\tau = 0, \quad \text{(I)} \tag{4.5}$$

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where recall that $E(t) > 0$. Thus we see that if no discontinuous external force has been applied in the entire past, the amplitude a is governed by a singular, homogeneous Volterra's integral equation of the second kind. When a discontinuous external force was applied to the material and is removed at time t_0, and when the value of a is prescribed prior to the time t_0, Eq. (4.5) leads to an inhomogeneous integral equation: for $t \geqslant t_0$

$$a(t)+g(t)+\int_{t}^{t_0} \frac{G'(t-\tau;t)}{E(t)}\, a(\tau)d\tau = 0, \qquad \text{(II)} \tag{4.6}$$

where g is a given function of $t \geqslant t_0$ defined by

$$g(t) \equiv \int_{-\infty}^{t_0} \frac{G'(t-\tau;t)}{E(t)}\, a(\tau)d\tau. \tag{4.7}$$

By a similar process, when a discontinuous external force is applied to the material, Eq. (3.15) yields another inhomogeneous integral equation:

$$a(t)+\frac{[f](t)}{E(t)}+\int_{-\infty}^{t} \frac{G'(t-\tau;t)}{E(t)}\, a(\tau)d\tau = 0. \qquad \text{(III)} \tag{4.8}$$

5. Application. Determination of stress relaxation function

In this section we present a method how to determine the stress relaxation function for an unknown response functional by use of the result in the previous section. The stress relaxation function for the history of the deformation gradient $(F(t), F_r^t)$ is defined by the solution of

$$\begin{aligned} &\frac{d}{ds} G(s;t) = K\big(s; F(t), F_r^t\big)h^2(s),\\ &G(0;t) = E(t) \equiv \partial_F\phi\big(F(t), F\big), \end{aligned} \tag{5.1}$$

where cf. Eqs. (4.2) and (4.3). The stress relaxation function is useful in the analysis of the mechanical behaviour of the material with memory, e.g. by using it the equation of motion (3.4) may be reduced to an integro-differential equation. If the response functional is linear, the stress relaxation function does not depend on the history of the deformation gradient, and the material may be regarded as a viscoelastic material of the integral type. In this case the stress relaxation function can be determined by the stress as a function of time which gives rise to the strain of the unit function of time. The method cannot be applied to the case when the response functional is nonlinear.

We consider a stationary singular point induced by a discontinuous external force, and we suppose that the history of the deformation gradient at the point is known in time. If we put

$$a(\tau) = H(\tau-t_0) \tag{5.2}$$

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in the corresponding integral equation (III), we have

$$G(t-t_0;t)+[f](t) = 0 \tag{5.3}$$

for $t \geqslant t_0$, where $H(\cdot)$ is the unit function and we have made use of Eq. $(5.1)_2$. Hence, to determine the stress relaxation function, we may measure the discontinuous external force which induces the discontinuity a given by Eq. (5.2) for each $t_0 \in (-\infty, t]$.

In pratice it may be difficult to control the external discontinuous force so as to obtain such a desired discontinuity a. We next present a method to determine the stress relaxation function by measuring a induced by a discontinuous external force. Equation (III) may be written as

$$a(t) = g(t)+ \int_{-\infty}^{t} L(t,\tau)a(\tau)d\tau, \tag{5.4}$$

where

$$g(t) \equiv -\frac{[f](t)}{E(t)}, \tag{5.5}$$

$$L(t,\tau) \equiv -\frac{\dot{G}'(t-\tau;t)}{E(t)}. \tag{5.6}$$

We suppose that $g(t)$ and $L(t, \tau)$ are continuous and bounded with respect to each argument, and that there exists a t_0 such that for every $t < t_0$

$$\int_{-\infty}^{t} |L(t,\tau)|d\tau < M < 1, \tag{5.7}$$

where M is a positive constant. Then Eq. (5.4) can be solved as

$$a(t) = g(t)+ \int_{-\infty}^{t} R(t,\tau)g(\tau)d\tau, \tag{5.8}$$

where $R(t, \tau)$ is the resolvent of the kernel $L(t, \tau)$. If we put

$$g(\tau) = H(\tau-t_0), \tag{5.9}$$

in Eq. (5.8), we have for $t \geqslant t_0$

$$a(t;t_0) = 1+ \int_{t_0}^{t} R(t,\tau)d\tau. \tag{5.10}$$

By differentiating Eq. (5.10) with respect to t_0, we get

$$\frac{\partial}{\partial t_0} a(t;t_0) = -R(t,t_0). \tag{5.11}$$

Thus the resolvent $R(t, \tau)$ can be determined by measuring the discontinuity a which is induced by the discontinuous external force given by Eqs. (5.5) and (5.9) for each $t_0 \in (-\infty, t]$. The function $L(t, \tau)$ can be calculated by use of $R(t, \tau)$. If we regard Eq.

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(5.8) as an integral equation for $g(t)$ and Eq. (5.4) as its solution, then $-L(t, \tau)$ becomes the resolvent of the kernel $-R(t, \tau)$. Thus it turns out that

$$
\begin{aligned}
L(t, \tau) &= -\sum_{n=1}^{\infty} S_n(t, \tau), \\
S_1(t, \tau) &= -R(t,\tau), \\
S_{n+1}(t, \tau) &= \int_{\tau}^{t} S_n(t, \xi) S_1(\xi, \tau) d\xi \qquad (n = 1, 2, \ldots),
\end{aligned} \tag{5.12}
$$

where we have assumed that $R(t, \tau)$ satisfies similar conditions to those for $L(t, \tau)$. From Eq. (5.6) we can obtain the stress relaxaton function by integrating $-E(t)L(t, t-s)$ with respect to s with Eq. $(5.1)_2$.

When the deformation gradient at the stationary singular point is constant in time, the stress relaxation function can be determined more easily than the above. In this case F_t at the point is also a constant element of Y, and hence $E(t)$ becomes a positive constant and $G(s; t)$ does not depend on t. In view of Eq. (5.6), $L(t, \tau)$ reduces then to a function $L(t-\tau)$ of the difference $t-\tau$. By substituting Eq. (5.9) with $t_0' = 0$ into Eq. (5.4) and by applying the Laplace transformation to the result, we obtain

$$
\bar{L}(s) = \frac{1}{\bar{a}(s)} \left\{ \bar{a}(s) - \frac{1}{s} \right\}, \tag{5.13}
$$

where $\bar{L}$ and $\bar{a}$ are the Laplace transformations of L and a, respectively. From Eq. (5.5) the discontinuous external force is then a scalar multiple of the unit function of time. To determine the kernel L, it is enough that we observe one discontinuity a induced by this discontinuous external force and apply the inverse transformation to the right hand side of Eq. (5.13).

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