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The optimum shape of a hydrofoil giving maximum lift in steady two-dimensional partial cavitating flow

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WE CONSIDER a two-dimensional hydrofoil at rest in the (xy)-plane embedded in a steady two-dimensional partial cavitating flow. In maximizing the lift we use standard variational calculus techniques and the problem of the optimum shape of the hydrofoil is shown to reduce to the solution of a pair of coupled singular integral equations from which we show that the shape of the mean chord of the hydrofoil has to satisfy a differential equation of second order.

Rozpatrzono dwuwymiarowy hydropłat spoczywający w płaszczyźnie (x, y) i zanurzony w dwuwymiarowym, częściowo kawitacyjnym przepływie. W procesie maksymalizacji siły nośnej zastosowano standardowe techniki rachunku wariacyjnego i wykazano, że problem optymalizacji kształtu sprowadza się do rozwiązania układu dwóch sprzężonych równań całkowych osobliwych. Z analizy wynika, że kształt średniej cięciwy płata spełniać musi równanie różniczkowe drugiego rzędu.

Рассмотрено двумерное подводное крыло, находящееся в плоскости (x, y), и погруженное в двумерном, частично кавитационном потоке. В процессе максимизации несущей силы применены стандартные техники вариационного исчисления и показано, что проблема оптимизации формы сводится к решению системы двух сопряженных сингулярных интегральных уравнений. Из анализа следует, что форма средней хорды крыла должна удовлетворять дифференциальному уравнению второго порядка.

1. Introduction

IT IS ASSUMED that in a flow of an incompressible liquid past a thin smooth hydrofoil C a constant-pressure cavity develops behind the hydrofoil, the length of the cavity being assumed less than the length of the hydrofoil (see Fig. 1).

The linearized theory is assumed; this problem has been studied by GUERST [1] and ACOSTA [2] in the case of a flat plate hydrofoil, GUERST [3] for the circular arc hydrofoil; WADE [4] for the ogival section and NISHIYAMA [5] for the partial cavitating hydrofoil with thickness, all these are based on either conformal mapping or Fourier expansion techniques.

The method of the lifting line theory has been used by DAVIES [6] to solve the cavity flow past an aerofoil and in the present paper we extend this method to the partially cavitating hydrofoil problem.

We use variational calculus techniques to obtain the optimum shape of the unknown curve C in order to maximize the lift, the curve C being assumed to be of a given length and prescribed mean curvature.

The problem is shown to reduce to the solution of a pair of coupled singular integral equations from which we show that the slope of the curve C has to satisfy a differential equation of the second order, which is solved numerically subject to boundary conditions at the end points.

2. Expression of the problem in integral equation form

ABD in Fig. 1 (p. 382) represents the mean chord $(0 < x < c)$ of the thin hydrofoil and it is assumed that a vapour-filled cavity AEB extends along a part AB $(0 < x < l < c)$ on the suction side of the hydrofoil as shown in the diagram, this being known as partially cavitating flow.

The speed of the liquid at infinity is U and the pressure there is P_∞. The pressure inside the cavity is uniform and equal to P_c $(< P_\infty)$. The problem will be solved on the basis of the lifting line theory and for this purpose we distribute singularities as follows along the x-axis:

a) sources of strength $m(\xi)$ per unit length in $0 < \xi < l$;

b) vortices of strength $\gamma_1(\xi)$ per unit length in $0 < \xi < l$;

c) vortices of strength $\gamma_2(\xi)$ per unit length in $l < \xi < c$, $(\gamma > 0$, clockwise). The total potential due to these singularities is given by

$$\phi(x,y) = \frac{1}{2\pi}\int_0^l \gamma_1(\xi)\tan^{-1}\left(\frac{y}{x-\xi}\right)d\xi + \frac{1}{2\pi}\int_l^c \gamma_2(\xi)\tan^{-1}\left(\frac{y}{x-\xi}\right)d\xi - \frac{1}{2\pi}\int_0^l m(\xi)\log r\, d\xi, \tag{2.1}$$

where

$$r = \{(x-\xi)^2 + y^2\}^{1/2}. \tag{2.2}$$

Differentiation of Eq. (2.1) with respect to x and y yields

$$\lim_{y\to 0\pm}\left(\frac{\partial\phi}{\partial x}\right) \equiv \begin{cases} -\dfrac{1}{2\pi}\displaystyle\oint_0^l \frac{m(\xi)d\xi}{x-\xi} \mp \frac{1}{2}\gamma_1(x) & (0 < x < l), \\ -\dfrac{1}{2\pi}\displaystyle\int_0^l \frac{m(\xi)d\xi}{x-\xi} \mp \frac{1}{2}\gamma_2(x) & (l < x < c). \end{cases} \tag{2.3}$$

$$\lim_{y\to 0\pm}\left(\frac{\partial\phi}{\partial y}\right) \equiv \begin{cases} \mp\dfrac{1}{2}m(x) + \dfrac{1}{2\pi}\displaystyle\oint_0^l \frac{\gamma_1(\xi)d\xi}{x-\xi} + \frac{1}{2\pi}\int_l^c \frac{\gamma_2(\xi)d\xi}{x-\xi} & (0 < x < l), \\ \dfrac{1}{2\pi}\displaystyle\int_0^l \frac{\gamma_1(\xi)d\xi}{x-\xi} + \frac{1}{2\pi}\oint_l^c \frac{\gamma_2(\xi)d\xi}{x-\xi} & (l < x < c). \end{cases} \tag{2.4}$$

It follows therefore that the boundary condition of zero normal velocity on the hydrofoil will be satisfied if m, γ_1 and γ_2 satisfy

$$\frac{1}{2\pi}\oint_0^l \frac{\gamma_1(\xi)d\xi}{x-\xi} + \frac{1}{2\pi}\int_l^c \frac{\gamma_2(\xi)d\xi}{x-\xi} + \frac{1}{2}m(x) = -Uy'(x) \quad (0 < x < l); \tag{2.5}$$

and

$$\frac{1}{2\pi}\int_0^l \frac{\gamma_1(\xi)d\xi}{x-\xi} + \frac{1}{2\pi}\oint_l^c \frac{\gamma_2(\xi)d\xi}{x-\xi} = -Uy'(x) \quad (l < x < c). \tag{2.6}$$

Applying Bernoulli's theorem in its linearized form, the condition of constant pressure on the surface of the cavity becomes

$$\phi_x = -\frac{1}{2}U\sigma, \tag{2.7}$$

where the cavitation number is defined by

$$\sigma = \frac{P_\infty - P_c}{\frac{1}{2}\varrho U^2}. \tag{2.8}$$

It will then follow from Eqs. (2.3) and (2.7) that the following equation must be satisfied:

$$-\frac{1}{2\pi}\int_0^l \frac{m(\xi)d\xi}{\xi-x} + \frac{1}{2}\gamma_1(x) = \frac{1}{2}U\sigma \quad (0 < x < l). \tag{2.9}$$

The three integral equations of the problem are thus Eqs. (2.5), (2.6) and (2.9).

3. Solution of the system integral equations

The problem is thus reduced to finding the solution of the coupled singular integral equations:

$$\oint_0^l \frac{\gamma_1(\xi)d\xi}{\xi-x} + \int_l^c \frac{\gamma_2(\xi)d\xi}{\xi-x} - \pi m(x) = 2\pi Uy'(x) \quad (0 < x < l); \tag{3.1}$$

$$\int_0^l \frac{\gamma_1(\xi)d\xi}{\xi-x} + \oint_l^c \frac{\gamma_2(\xi)d\xi}{\xi-x} = 2\pi Uy'(x) \quad (l < x < c); \tag{3.2}$$

$$\oint_0^l \frac{m(\xi)d\xi}{\xi-x} - \pi\gamma_1(x) = -\pi U\sigma \quad (0 < x < l), \tag{3.3}$$

for m, γ_1 and γ_2 with the Kutta condition of finite velocity at the trailing edge of the hydrofoil and with the closure condition (3.4) satisfied. We consider first Eq. (3.3); it may be verified that the inverse of Eq. (3.3) which satisfies the closure condition

$$\int_0^l m(x)dx = 0 \tag{3.4}$$

is

$$\sqrt{x(l-x)}\,m(x) = \frac{1}{2}U\sigma(l-2x) - \frac{1}{\pi}\oint_0^l \frac{\sqrt{\xi(l-\xi)}}{\xi-x}\gamma_1(\xi)d\xi \quad (0 < x < l). \tag{3.5}$$

We now solve Eq. (3.2) for γ_2; it is convenient to write Eq. (3.2) in the form

$$\oint_l^c \frac{\gamma_2(\xi)d\xi}{\xi-x} = 2\pi Uy'(x) - \int_0^l \frac{\gamma_1(\xi)d\xi}{\xi-x} \quad (l < x < c) \tag{3.6}$$

then the general solution for γ_2 is as follows:

$$\sqrt{(c-x)(x-l)}\,\gamma_2(x) = B - \frac{1}{\pi^2}\oint_l^c \frac{\sqrt{(c-\xi)(\xi-l)}}{\xi-x}\left[2\pi Uy'(x) - \int_0^l \frac{\gamma_1(\eta)d\eta}{\eta-\xi}\right]d\xi, \tag{3.7}$$

where B is an arbitrary constant. Using the Kutta condition which can be expressed in the form

$$\gamma_2(c) = 0, \tag{3.8}$$

we find that the arbitrary constant B is given by

$$B = \oint_l^c \frac{\sqrt{(c-\xi)(\xi-l)d\xi}}{\xi-c}\left\{2\pi Uy'(x) - \int_0^l \frac{\gamma_1(\eta)d\eta}{\eta-\xi}\right\}; \tag{3.9}$$

and Eq. (3.7) now reduces to

$$\gamma_2(x) = \frac{1}{\pi}\sqrt{\frac{c-x}{x-l}}\oint_l^c \sqrt{\frac{\xi-l}{c-\xi}}\frac{d\xi}{\xi-x}\left\{-2Uy'(\xi)d\xi + \frac{1}{\pi}\int_0^l \frac{\gamma_1(\eta)d\eta}{\eta-\xi}\right\}. \tag{3.10}$$

It is permissible to change the order of integration in the second term on the right-hand side of Eq. (3.10) and when this is done it becomes

$$\gamma_2(x) = \frac{1}{\pi}\sqrt{\frac{c-x}{x-l}}\left\{-2U\oint_l^c \sqrt{\frac{\xi-l}{c-\xi}}\frac{y'(\xi)d\xi}{\xi-x} + \int_0^l \sqrt{\frac{l-\eta}{c-\eta}}\frac{\gamma_1(\eta)d\eta}{\eta-x}\right\} \quad (l < x < c). \tag{3.11}$$

We require now to calculate the integral

$$I = \int_l^c \frac{\gamma_2(\xi)d\xi}{\xi-x}, \quad \text{where} \quad 0 < x < l. \tag{3.12}$$

Using Eq. (3.11) and making use of the Poincaré–Bertrand formula

$$(3.13)\qquad \int\limits_{c_1} \frac{\phi_1(\xi)d\xi}{\xi-x} \int\limits_{c_2} \frac{\phi_2(\eta)d\eta}{\eta-\xi} = -\pi^2\phi_1(x)\phi_2(x) + \int\limits_{c_2} \phi_2(\eta)d\eta \int\limits_{c_1} \frac{\phi_1(\xi)d\xi}{(\xi-x)(\eta-\xi)},$$

we obtain, after some reduction,

$$(3.14)\qquad \int\limits_l^c \frac{\gamma_2(\xi)d\xi}{\xi-x} = z\pi Uy'(x) - 2U\sqrt{\frac{c-x}{l-x}} \int\limits_l^c \sqrt{\frac{\eta-l}{c-\eta}} \frac{y'(\eta)d\eta}{\eta-x} + \oint\limits_0^l \left[\sqrt{\left(\frac{(l-\eta)(c-x)}{(c-\eta)(l-x)}\right)} - 1\right] \frac{\gamma_1(\eta)d\eta}{\eta-x}.$$

Substituting the expression for $m(x)$ from Eq. (3.5) in Eq. (3.1), we obtain

$$(3.15)\qquad \oint\limits_0^l \left[1 + \sqrt{\left(\frac{(l-\xi)}{x(l-x)}\right)}\right] \frac{\gamma_1(\xi)d\xi}{\xi-x} + \int\limits_l^c \frac{\gamma_2(\xi)d\xi}{\xi-x} = f_1(x),$$

where

$$(3.16)\qquad f_1(x) = 2\pi Uy'(x) + \frac{\pi U\sigma}{2\sqrt{x(l-x)}}(l-2x),$$

and elimination of γ_2 between Eqs. (3.14) and (3.15) gives

$$(3.17)\qquad \oint\limits_0^l \left[\sqrt{\frac{\xi}{x}} + \sqrt{\frac{c-x}{c-\xi}}\right] \frac{\sqrt{l-\xi}\,\gamma_1(\xi)d\xi}{\xi-x} = \sqrt{l-x}\,g(x),$$

where

$$(3.18)\qquad 2\sqrt{x(l-x)}\,g(x) = \pi U\sigma(l-2x) + 4U\sqrt{x(c-x)} \int\limits_l^c \sqrt{\frac{\eta-l}{c-\eta}} \frac{y'(\eta)d\eta}{\eta-x}.$$

For convenience we write

$$(3.19)\qquad \sqrt{l-\xi}\,\gamma_1(\xi) = \sqrt{(\xi)}, \qquad \sqrt{l-x}\,g(x) = G(x),$$

and, consequently, Eq. (3.17) can be expressed in the form

$$(3.20)\qquad \oint\limits_0^l \left[\sqrt{\frac{\xi}{x}} + \sqrt{\frac{c-x}{c-\xi}}\right] \frac{\Gamma(\xi)d\xi}{\xi-x} = G(x) \quad (0 < x < l).$$

In order to invert Eq. (3.20) we proceed as follows:

We make the transformation

$$(3.21)\qquad \xi = c\sin^2\theta, \quad x = c\sin^2\theta_0, \quad l = c\sin^2\alpha,$$

then we obtain the following integral equation with a cotangent kernel:

$$(3.22)\qquad \oint\limits_0^\alpha \psi(\theta)\cot(\theta-\theta_0)d\theta = f(\theta_0),$$

where

(3.23) $$f(\theta_0) = \sin\theta_0 G(c\sin^2\theta_0), \quad \psi(\theta) = 2\Gamma(c\sin^2\theta)\sin\theta.$$

We transform the integral equation (3.22) into standard Cauchy form to give

(3.24) $$\oint_0^b \frac{\Psi(t)dt}{t-t_0} = F(t_0),$$

where

(3.25) $$F(t_0) = \frac{f(\tan^{-1}t_0)}{t_0^2+1} - \frac{At_0}{t_0^2+1}, \quad \Psi(t) = \frac{\psi(\tan^{-1}t)}{t^2+1}, \quad A = \int_0^\alpha \psi(\theta)d\theta,$$

and

(3.26) $$t_0 = \tan\theta_0, \quad t = \tan\theta, \quad b = \tan\alpha.$$

The general solution of Eq. (3.24) is as follows:

(3.27) $$\sqrt{t(b-t)}\,\Psi(t) = D - \frac{1}{\pi^2}\oint_0^b \sqrt{t_0(b-t_0)}\,F(t_0)\frac{dt_0}{t_0-t},$$

where D is an arbitrary constant.

Using Eqs. (3.25), (3.26) and (3.27), we obtain

(3.28) $$\sin^{1/2}\theta\sin^{1/2}(\alpha-\theta)\psi(\theta) = \frac{D\sqrt{\cos\alpha}}{\cos\theta} - \frac{1}{\pi^2}\int_0^\alpha \sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,[f(\theta_0)-A\tan\theta_0]\frac{d\theta_0}{\sin(\theta_0-\theta)},$$

(3.29) $$F(\theta_0) = \frac{1}{2}\pi U\sigma\sqrt{c}\,(\sin^2\alpha-2\sin^2\theta_0) + 4\sqrt{c}\,U\sin\theta_0\cos\theta_0\int_\alpha^{\frac{\pi}{2}} \frac{\sqrt{\sin^2\phi-\sin^2\alpha}\,z(\phi)\sin\phi\,d\phi}{(\sin^2\phi-\sin^2\theta_0)}$$

and $z(\phi) = y'(\eta)|_{\eta=c\sin^2\phi}$ is the slope of the curve C at the point $c\sin^2\phi$.

From Eqs. (3.19), (3.21) and (3.23) we can write

(3.30) $$\nu(\theta) = \gamma_1(c\sin^2\theta) = \frac{\Gamma(c\sin^2\theta)}{\sqrt{l}\sqrt{\sin^2\alpha-\sin^2\theta}} = \frac{\psi(\theta)}{2\sqrt{l}\sin\theta\sqrt{\sin^2\alpha-\sin^2\theta}},$$

then Eq. (3.28) can be written in the form

(3.31) $$2\pi\{\sin^3\theta(\sin^2\alpha-\sin^2\theta)\sin(\alpha-\theta)\}^{1/2}\nu(\theta) = \frac{\pi D\cos^{1/2}\alpha}{c^{1/2}\cos\theta} - Ac^{-1/2}\left\{\frac{\cos^{1/2}\alpha}{\cos\theta} - \cos\left(\theta-\frac{1}{2}\alpha\right)\right\} - \frac{1}{2}U\sigma\oint_0^\alpha \sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,\frac{(\sin^2\alpha-2\sin^2\theta_0)}{\sin(\theta_0-\theta)}d\theta_0$$

$$+2U\int_{\alpha}^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha z(\phi)\sin\phi\, d\phi}\left\{\frac{\sqrt{\cos\alpha}\sqrt{\tan\phi}}{\cos\theta(\tan^2\phi-\tan^2\theta)}\left[\sqrt{\tan\phi-\tan\alpha}\,(\tan\phi\right.\right.$$

$$\left.\left.+\tan\theta)+\sqrt{\tan\phi+\tan\alpha}\,(\tan\phi-\tan\theta)\right]-2\cos\left(\theta-\frac{1}{2}\alpha\right)\right\} \quad (0<\theta<\alpha).$$

The constant A can be calculated from Eqs. (3.25), (3.30) and (3.31) to give

$$A = \pi D, \tag{3.32}$$

then we can write Eq. (3.31) as follows:

$$2\pi\{\sin^3\theta(\sin^2\alpha-\sin^2\theta)\sin(\alpha-\theta)\}^{1/2}\nu(\theta) = \frac{\pi D}{c^{1/2}}\cos\left(\theta-\frac{1}{2}\alpha\right) \tag{3.33}$$

$$-\frac{1}{2}U\sigma\oint_0^{\alpha}\sqrt{\sin\theta_0\sin(\alpha-\theta_0)}\,\frac{(\sin^2\alpha-\sin^2\theta)}{\sin(\theta_0-\theta)}\,d\theta_0$$

$$+2U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\; z(\phi)\sin\phi\, d\phi\left\{\frac{\sqrt{\cos\alpha}\sqrt{\tan\phi}}{\cos\theta(\tan^2\phi-\tan^2\theta)}\left[\sqrt{\tan\phi-\tan\theta}\,(\tan\phi\right.\right.$$

$$\left.\left.+\tan\theta)+\sqrt{\tan\phi+\tan\alpha}\,(\tan\phi-\tan\theta)\right]-2\cos\left(\theta-\frac{1}{2}\alpha\right)\right\} \quad (0<\theta<\alpha).$$

Using Eqs. (3.5), (3.21) and (3.33) we can write $m(c\sin^2\theta_0)$ after some simplification in the form

$$-\sin\theta_0\sqrt{\sin^2\alpha-\sin^2\theta_0}\sqrt{\sin\theta_0\sin(\alpha+\theta_0)}\, m(c\sin^2\theta_0) = \frac{\pi D}{2\sqrt{c}}\cos\left(\frac{1}{2}\alpha+\theta_0\right) \tag{3.34}$$

$$-\frac{\pi U\sigma}{4}\left[\sin\left(\frac{1}{2}\alpha+\theta_0\right)(\sin^2\alpha-2\sin^2\theta_0)-\sin^2\frac{1}{2}\alpha\sin\left(\frac{1}{2}\alpha-\theta_0\right)\right]$$

$$+U\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\; z(\phi)\sin\phi\, d\phi\left\{(\sin^2\phi-\sin^2\theta_0)^{-1}\left[\sqrt{\sin\phi\sin(\theta-\alpha)}\,\sin(\phi-\theta_0)\right.\right.$$

$$\left.\left.+\sqrt{\sin\phi\sin(\phi+\alpha)}\,\sin(\phi-\theta_0)\right]-2\cos\left(\frac{1}{2}\alpha+\theta_0\right)\right\} \quad (0<\theta<\alpha).$$

It follows from Eq. (3.34) that the function $m(c\sin^2\theta_0)$ for small values of θ_0 is of the form

$$m(x) = m(c\sin^2\theta_0) = \frac{A_0}{\theta_0^{3/2}}+\frac{A_1}{\theta_2^{1/2}}+O(1) = \frac{A_0}{\left(\frac{x}{c}\right)^{3/4}}+\frac{A_1}{\left(\frac{x}{c}\right)^{1/4}}+O(1), \tag{3.35}$$

where

$$A_0 = -\frac{1}{\sin^{3/2}\alpha}\left\{\frac{\pi D}{2\sqrt{c}}\cos\frac{1}{2}\alpha-\frac{\pi U\sigma}{4}\sin^3\frac{1}{2}\alpha(2\cos\alpha+1)\right.$$

$$+U\int_{\alpha}^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}\, z(\phi)\sin\phi\left[\sqrt{\frac{\sin(\phi-\alpha)}{\sin\phi}}+\sqrt{\frac{\sin(\phi+\alpha)}{\sin\phi}}-2\cos\frac{1}{2}\alpha\right]d\phi\Bigg\},$$

$$(3.36)\qquad A_1 = -\frac{1}{\sin^{3/2}\alpha}\Bigg\{-\frac{\pi D}{2\sqrt{c}}\sin\frac{1}{2}\alpha-\frac{\pi U\sigma}{4}\sin^2\frac{1}{2}\alpha\cos\frac{1}{2}\alpha(3+2\cos\alpha)$$

$$+U\int_{\alpha}^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}\, z(\phi)\,d\phi\left[\sqrt{\cos\phi}\left(\sqrt{\frac{\sin(\phi+\alpha)}{\sin\phi}}-\sqrt{\frac{\sin(\phi-\alpha)}{\sin\phi}}\right)\right.$$

$$\left.+2\sin\frac{1}{2}\alpha\sin\phi\right]d\phi\Bigg\}.$$

The behaviour of $m(x)$ for small x has been discussed in earlier papers in the cavity theory and the accepted behaviour of $m(x)$ for small values of x is $m(x) \propto x^{-1/4}$ [see, for example, DAVIES (6)] hence we choose A_0 to be zero in order to achieve the proper behaviour and this defines the unknown constant D;

$$(3.37)\qquad D = \frac{2\sqrt{c}\,U}{\pi\cos\frac{1}{2}\alpha}\Bigg\{\frac{\pi\sigma}{4}\sin^3\frac{1}{2}\alpha(2\cos\alpha+1)$$

$$-\int_{\alpha}^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}\, z(\phi)\sin\phi\left[\sqrt{\frac{\sin(\phi-\alpha)}{\sin\phi}}+\sqrt{\frac{\sin(\phi+\alpha)}{\sin\phi}}-2\cos\frac{1}{2}\alpha\right]d\phi\Bigg\}.$$

Substituting from Eq. (3.37) into Eq. (3.34) we obtain

$$(3.38)\qquad \frac{\pi}{U}\{\sin^3\theta\sin(\alpha-\theta)\,(\sin^2\alpha-2\sin^2\theta_0)\}^{1/2}\nu(\theta) = \frac{\pi\sigma}{2\cos\frac{1}{2}\alpha}\Bigg\{\sin^2\frac{1}{2}\alpha$$

$$\times\cos\left(\frac{1}{2}\alpha\sin\left(\frac{1}{2}\alpha-\theta\right)+\sin^3\frac{1}{2}\alpha\cos\left(\theta-\frac{1}{2}\alpha\right)(2\cos\alpha+1)-\sin\left(\frac{1}{2}\alpha+\theta_0\right)\right.$$

$$\left.\times\cos\frac{1}{2}\alpha(\sin^2\alpha-2\sin^2\theta_0)\right\}+\int_{\alpha}^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}\, z(\phi)\sin\phi\, d\phi\left\{-\frac{\cos\left(\theta-\frac{1}{2}\alpha\right)}{\cos\frac{1}{2}\alpha}\right.$$

$$\times\left[\sqrt{\frac{\sin(\phi+\alpha)}{\sin\phi}}+\sqrt{\frac{\sin(\phi-\alpha)}{\sin\phi}}\right]+(\sin^2\phi-\sin^2\theta)^{-1}\Big[\sqrt{\sin\phi\sin(\theta-\alpha)}\,\sin(\phi+\alpha)$$

$$\left.+\sqrt{\sin\phi\sin(\phi+\alpha)}\,\sin(\phi-\theta)\Big]\right\}\qquad (0<\theta<\alpha).$$

An alternative formula for the constant D can now be determined by using the closure condition (3.5), (3.21) and (3.34) and we obtain

$$(3.39)\quad D = \frac{2\sqrt{c}\,U}{\pi\cos\frac{1}{2}\alpha}\left\{\frac{1}{4}\pi\sigma\sin^3\frac{1}{2}\alpha\right.$$

$$\left.-\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,z(\phi)\sin\phi\,d\phi\left[\frac{\sin\phi\sin(\phi+\alpha)}{\sin(\phi-\alpha)}+\frac{\sqrt{\sin\phi\sin(\phi-\alpha)}}{\sin(\phi+\alpha)}-2\cos\frac{1}{2}\alpha\right]\right\}.$$

If we eliminate D between Eqs. (3.37) and (3.39), we obtain the relation between the cavity number σ and the integral of the slope of the foil

$$(3.40)\quad \pi\sigma\sin^2\frac{1}{2}\alpha\cos\alpha = -4\int_{\alpha}^{\frac{\pi}{2}}\left\{\sqrt{\frac{\sin\phi}{\sin(\phi-\alpha)}}\sin(\phi+\alpha)\cos\left(\phi-\frac{1}{2}\alpha\right)\right.$$

$$\left.-\sqrt{\frac{\sin\phi}{\sin(\phi+\alpha)}}\sin(\phi-\alpha)\cos\left(\phi+\frac{1}{2}\alpha\right)\right\}z(\phi)\,d\phi.$$

We now determine the formula for the lift L on the foil as follows:

$$(3.41)\qquad L = \int_0^l \{P|_{y=0^-}-P_c\}dx+\int_l^c\{P|_{y=0^-}-P|_{y=0^+}\}dx,$$

where

$$(3.42)\qquad P|_{y=0\pm} = P_\infty+\varrho U\phi_x|_{y=0\pm}.$$

Using Eqs. (2.3), (2.7), (2.8) and (3.42) we can write Eq. (3.41) as follows:

$$(3.43)\qquad L = \varrho U\int_0^l \gamma_1(x)dx+\varrho U\int_l^c\gamma_2(x)dx.$$

We substitute for $\gamma_2(x)$ Eq. (3.11) and change the order of integration; using Eq. (3.21), Eq. (3.43) reduces to

$$(3.44)\quad L = 2\varrho Uc\int_0^{\alpha}\sqrt{\sin^2\alpha-\sin^2\theta}\,\nu(\theta)\sin\theta\,d\theta-4\varrho U^2c\int_{\alpha}^{\frac{\pi}{2}}\sqrt{\sin^2\phi-\sin^2\alpha}\,z(\phi)\sin\phi\,d\phi.$$

4. The variational treatment

We now pose the problem of finding the optimum shape of the hydrofoil curve C so that the lift Eq. (3.44) is maximized, the curve C being of a given length S and prescribed mean curvature K; S and K are related to z as follows:

$$(4.1)\qquad S = 2c\int_0^{\frac{\pi}{2}}\sqrt{(1+z^2(\theta))}\sin\theta\cos\theta\,d\theta,$$

and

(4.2) $$K = \frac{1}{2c}\int\limits_0^{\frac{\pi}{2}} z'^2(\theta)\sec\theta\operatorname{cosec}\theta.$$

This problem is equivalent to minimizing the following functional I:

(4.3) $$I[\nu(\theta), z(\theta), z'(\theta), \theta] = -L+\lambda_1 S+\lambda_2 K = \int\limits_0^{\frac{\pi}{2}} F[\nu(\theta), z(\theta), z'(\theta), \theta; \lambda_1, \lambda_2]d\theta,$$

with the function F given by

(4.4) $$F \equiv \begin{cases} 2\lambda_1 c\sqrt{(1+z^2(\theta))}\sin\theta\cos\theta \\ \qquad +\dfrac{\lambda_2}{2c} z'^2(\theta)\sec\theta\operatorname{cosec}\theta - 2\varrho U c\sqrt{(\sin^2\alpha-\sin^2\theta)}\,\nu(\theta)\sin\theta & (0<\theta<\alpha), \\ 2\lambda_1 c\sqrt{(1+z^2(\theta))}\sin\theta\cos\theta \\ \quad +\dfrac{\lambda_2}{2c} z'^2(\theta)\sec\theta\operatorname{cosec}\theta + 4\varrho U^2 c\sqrt{(\sin^2\alpha-\sin^2\theta)}z(\theta)\sin\theta & \left(\alpha<\theta<\dfrac{\pi}{2}\right), \end{cases}$$

where $\nu(\theta)$, $z(\theta)$ are related by Eq. (3.38) and λ_1, λ_2 are determined Lagrange multipliers.

Let $\mu(\theta)$, $z(\theta)$ denote the required optimum functions and $\varepsilon\xi$, $\varepsilon\eta$ the respective variations from the optimum; then we can write the following relation between $\xi(\theta)$ and $\eta(\theta)$:

(4.5) $$\pi\{\sin^3\theta\sin(\alpha-\theta)(\sin^2\alpha-\sin^2\theta)\}^{1/2}\xi(\theta) = U\int\limits_\alpha^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}\,z(\phi)\sin\phi\, d\phi$$
$$\times\left\{-\frac{\cos\left(\theta-\frac{1}{2}\alpha\right)}{\cos\frac{1}{2}\alpha}\left[\sqrt{\frac{\sin(\phi-\alpha)}{\sin\phi}}+\sqrt{\frac{\sin(\phi+\alpha)}{\sin\phi}}\right]+\frac{1}{(\sin^2\phi-\sin^2\theta)}\right.$$
$$\left.\times\left[\sqrt{\sin\phi\sin(\phi-\alpha)}\sin(\phi+\theta)+\sqrt{\sin\phi\sin(\theta+\alpha)}\sin(\phi-\theta)\right]\right\} \quad (0<\theta<\alpha).$$

The variation of the functional I in Eq. (4.3) due to the variations $\xi(\theta)$ and $\eta(\theta)$ is

(4.6) $$\Delta I = \int\limits_0^{\frac{\pi}{2}} F[\nu+\varepsilon\xi, z+\varepsilon\eta, z'+\varepsilon\eta', \theta)]d\theta - \int\limits_0^{\alpha} F[\nu, z, z', \theta]d\theta;$$

for sufficiently small ε, we can write

(4.7) $$\Delta I = \varepsilon\delta I+\frac{\varepsilon^2}{2!}\delta^2 I+\dots\dots,$$

where the first and second variations ξI and $\xi^2 I$ are defined by

(4.8) $$\delta I = \int_0^{\frac{\pi}{2}} \{\xi(\theta)F_\nu(\nu, z, z', \theta)+\eta(\theta)F_z(\nu, z, z', \theta)+\eta'(\theta)F_{z'}(\nu, z, z', \theta)\}\, d\theta,$$

and

(4.9) $$\delta^2 I = \int_0^{\frac{\pi}{2}} \{\xi^2 F_{\nu\nu}+\eta^2 F_{zz}+\eta'^2 F_{z'z'}+2\xi\theta F_{\nu z}+2\xi\eta' F_{\nu z'}+2\eta\eta' F_{zz'}\}\, d\theta,$$

in which the sub-indices denote partial derivatives.

As $\xi(\theta)$ and $\eta(\theta)$ are related by Eq. (4.5), substitution of Eq. (4.5) in Eq. (4.8) and integration by parts yields

(4.10) $$\delta I = \int_0^{\frac{\pi}{2}} \Bigg\{ \frac{U \operatorname{cosec}\theta F_\nu(\nu, z, z', \theta)}{\pi\sqrt{\sin^2\alpha-\sin^2\theta}\,\sqrt{\sin\theta\sin(\alpha-\theta)}} \int_\alpha^{\frac{\pi}{2}} \sqrt{\sin^2\phi-\sin^2\alpha}\,\eta(\phi)\sin\phi\, d\phi \times \Bigg[-\frac{\cos\left(\theta-\frac{1}{2}\alpha\right)}{\cos\frac{1}{2}\alpha}\left(\sqrt{\frac{\sin(\theta-\alpha)}{\sin\phi}}+\sqrt{\frac{\sin(\theta+\alpha)}{\sin\phi}}\right)+\frac{1}{(\sin^2\phi-\sin^2\theta)} \times\left(\sqrt{\sin\phi\sin(\phi-\alpha)}\,\sin(\phi+\theta)+\sqrt{\sin\phi\sin(\phi+\alpha)}\,\sin(\phi-\theta)\right)\Bigg] + \eta(\theta)\left[F_z-\frac{d}{d\theta}F_{z'}\right]\Bigg\} d\theta + [\eta(\theta)F_{z'}]_0^{\frac{\pi}{2}}.$$

We change the order of integration in Eq. (4.10), use Eq. (4.4), and we obtain

(4.11) $$\delta I = \frac{\lambda_2}{c}\left[\eta(\theta)\frac{z'(\theta)}{\sin\theta\cos\theta}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \left\{\frac{2\lambda_1 c z(\theta)\sin\theta\cos\theta}{\sqrt{1+z^2(\theta)}} - \frac{\lambda_2}{c}\frac{d}{d\theta}\left(\frac{z'(\theta)}{\sin\theta\cos\theta}\right)\right\} d\theta + \frac{2pU^2c}{\cos\frac{1}{2}\alpha}\int_0^{\frac{\pi}{2}} \sqrt{\sin^2\theta-\sin^2\alpha}\left\{\sqrt{\frac{\sin(\theta-\alpha)}{\sin\theta}}+\sqrt{\frac{\sin(\theta+\alpha)}{\sin\theta}}\right\}\eta(\theta)\sin\theta\, d\theta.$$

For $I[z, \eta]$ to be a minimum it is necessary that

(4.12) $$\delta I[z, \eta] = 0,$$

and since this must be satisfied for all admissible $\eta(\theta)$ it follows that

$$(4.13)\qquad \lambda_2 \frac{d}{d\theta}\left(\frac{z'(\theta)}{\sin\theta\cos\theta}\right) - \frac{2\lambda_1 c^2 z(\theta)\sin\theta\cos\theta}{\sqrt{1+z^2(\theta)}} \equiv \begin{cases} 0 & (0 < \theta < \alpha), \\ \varrho U^2 f(\theta) & \left(\alpha < \theta < \dfrac{\pi}{2}\right), \end{cases}$$

where

$$(4.14)\qquad f(\theta) = \frac{2c^2\sqrt{\sin^2\theta - \sin^2\alpha\sin\theta}}{\cos\dfrac{1}{2}\alpha}\left[\sqrt{\frac{\sin(\theta-\alpha)}{\sin\theta}} + \sqrt{\frac{\sin(\theta+\alpha)}{\sin\theta}}\right].$$

This is a differential equation of the second order for $z(\theta)$ and it is necessary to postulate either the slope or the slope derivative of the hydrofoil at the end points; here we solve the problem subject to the following boundary conditions:

$$(4.15)\qquad z(0) = A, \quad z(c) = B.$$

The square bracket of Eq. (4.11) can be written in the form $[z'(x)\eta(\theta)]_0^{\frac{\pi}{2}}$, $x = c\sin^2\theta$ and we note that this is identically zero here since $\eta(\theta)$ vanishes at each end.

We consider the solution of Eq. (4.13) for the slope $z(\theta)$ in the case of a small slope (this is consistent with the linearization hypothesis), and we approximate Eq. (4.13) as follows:

$$(4.16)\qquad \frac{1}{2c\sin\theta\cos\theta}\frac{d}{d\theta}\frac{1}{2c\sin\theta\cos\theta}z'(\theta)\Big] - nz(\theta) \equiv \begin{cases} 0 & (0 < \theta < \alpha), \\ Ef(\theta), \left(n = \dfrac{\lambda_1}{2\lambda_2}, E = \dfrac{\varrho U^2}{\lambda_2}\right) & \left(\alpha < \theta < \dfrac{\pi}{2}\right), \end{cases}$$

where $f(\theta)$ is defined in (4.14).

Using the transformation (3.21) we obtain

$$(4.17)\qquad z''(x) - nz(x) \equiv \begin{cases} 0 & (0 < x < l), \\ EF(x) & (l < x < c), \end{cases}$$

where $F(x)$ is defined by

$$(4.18)\qquad 2^{1/2}x^{1/4}\{c^{1/2} + (c-l)^{1/2}\}^{1/2}(c-x)^{1/2}F(x) = (x-l)^{1/2} \times \left\{\left[\sqrt{x(c-l)} - \sqrt{l(c-x)}\right]^{1/2} + \left[\sqrt{x(c-l)} + \sqrt{l(c-x)}\right]^{1/2}\right\}.$$

Equation (4.17) is a nonlinear differential equation for $z(x)$; we solve it later subject to the boundary conditions (4.15) and the constraints (4.1) and (4.2).

In the special case when the noncavity case $l = 0$, $\alpha = 0$, and $\sigma = 0$ this gives

$$(4.19)\qquad L = 2\varrho U^2 \int_0^c \sqrt{\frac{\xi}{c-\xi}}\, y'(\xi)d\xi, \quad z''(x) - nz(x) = E\sqrt{\frac{x}{c-x}},$$

c is the hydrofoil chord, this is consistent with ESSAWY [7].

A sufficient condition for the extremum to be a minimum is

$$(4.20)\qquad \delta^2 I[\nu, z, z', \theta] \geqslant 0.$$

From Eq. (4.4), Eq. (4.9) can be written as follows:

$$(4.21)\qquad \delta^2 I = \int\limits_0^{\frac{\pi}{2}} \left\{2\lambda_1 c\sin\theta\cos\theta[1+z^2(\theta)]^{-3/2} + \frac{\lambda_2}{c}\,\eta'^2(\theta)\sec\theta\,\mathrm{cosec}\,\theta\right\}d\theta.$$

In the case of a small slope $z(\theta)$, we approximate Eq. (4.21) as follows:

$$(4.22)\qquad \delta^2 I = \int\limits_0^{\frac{\pi}{2}} \left\{2\lambda_1 c\sin\theta\cos\theta\eta'^2(\theta) + \frac{\lambda_2}{c}\,\eta'^2(\theta)\sec\theta\,\mathrm{cosec}\,\theta\right\}d\theta.$$

Using the transformation (3.21) we obtain

$$(4.23)\qquad \delta^2 I = \int\limits_0^c \{\lambda_1\phi^2(x)+2\lambda_2\phi'^2(x)\}dx,$$

where

$$(4.24)\qquad \phi(x) = \eta\left(\sin^{-1}\sqrt{\frac{x}{c}}\right),\quad \phi'(x) = \frac{1}{2}\,\eta'\left(\sin^{-1}\sqrt{\frac{x}{c}}\right)[x(c-x)]^{-1/2}.$$

Equation (4.23) has the same structure as in Ref. [7], Eq. (3.30); using the results of that paper we can write

$$(4.25)\qquad \lambda_1 + \frac{2\pi^2}{c^2}\,\lambda_2 > 0.$$

5. Optimum solution in the case when $z(0)$ and $z(c)$ are prescribed

We use the numerical method of solution to solve Eq. (4.17), namely

$$(5.1)\qquad z''(x)-nz(x) \equiv \begin{cases} 0 & (0 < x < l), \\ EF(x) & (l < x < c), \end{cases}$$

where $F(x)$ is defined by Eq. (4.18), and when $z(x)$ is subject to the boundary conditions

$$(5.2)\qquad z(0) = 0,\quad z(c) = B = \tan 12°,$$

in the case

$$(5.3)\qquad S = 4.02\text{ ft},\quad l \equiv \binom{3}{1}\text{ ft},\quad c = 4\text{ ft},\quad k = 0.0148\text{ ft}^{-1},$$

with

$$(5.4)\qquad \varrho = 62.4\text{ lb/ft}^3,\quad U = 40\text{ m.p.h.}$$

The problem has been solved numerically for tho different values of the cavity length.

Equation (5.1) has the same structure as in Ref. [7], therefore we solve Eq. (5.1) numerically by the same method stated in Ref. [7] and subject to the boundary condition (5.2) and constraints (4.1) and (4.2).

The values of n and E which satisfy the sufficient condition (4.25) are

(5.5)
$$n = 0.5852812, \qquad E = 0.188548 \; \forall\, l = 3\ \text{ft},$$
$$n = -0.6154366, \qquad E = 0.0856397 \; \forall\, l = 1\ \text{ft}.$$

The graphs of $y(x)$, the optimum shape, are shown in Fig. 2. Also included in Fig. 2 are the optimum shapes of foil for two cases: noncavity and full cavity.

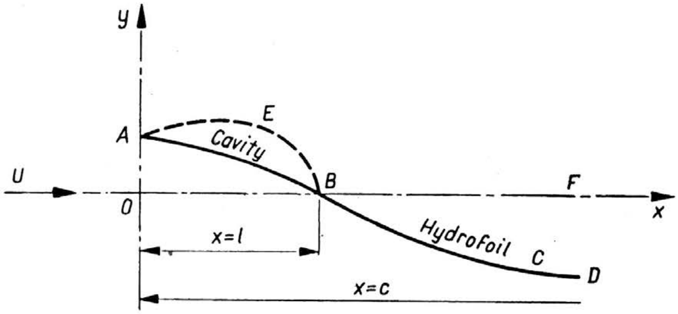

FIG. 1. The physical plane.

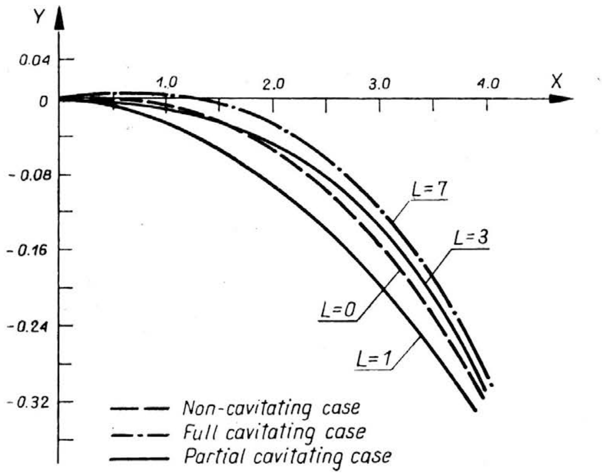

FIG. 2. The optimum hydrofoil shape in the case $z(0) = 0$, $z(c) = 3$.

The value of maximum lift

As seen from Eq. (3.44), the value of the lift L can be calculated by substituting $\nu(\theta)$, Eq. (3.38) in the lift formula (3.44) to give

(5.6)
$$L = \frac{2\pi U^2 c\sigma \sin^3\frac{1}{2}\alpha(2\cos\alpha+1)}{\cos\frac{1}{2}\alpha} - \frac{2\varrho U^2 c}{\cos\frac{1}{2}\alpha}\int\limits_{\alpha}^{\frac{\pi}{2}} \sqrt{\sin^2\phi - \sin^2\alpha}\, z(\phi)\sin\phi$$

$$\times\left[\sqrt{\frac{\sin(\phi+\alpha)}{\sin\phi}}+\sqrt{\frac{\sin(\phi-\alpha)}{\sin\phi}}\right]d\phi,$$

where σ is defined in Eq. (3.40).

Using the numerical results we find that the value of maximum lift is

$$L = 94835.727 \text{ lbf}, \quad l = 3 \text{ ft},$$
$$L = 30810.46 \text{ lbf}, \quad l = 1 \text{ ft}. \tag{5.7}$$

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