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Stability of nonlinear thermal convection in a porous medium

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THE PROPERTIES of the approximate solutions of the Darcy–Boussinesq equations, obtained for small amplitudes by the perturbation analysis and for finite amplitudes by using the Galerkin method, were investigated. It was shown that two branches of three, which emanate at the Rayleigh number $Ra = 4\pi^2$ representing two-dimensional time-independent flows, are stable. The third branch of three-dimensional convection is unstable. However, the branch of three-dimensional flow starting at $Ra = 4.5\pi^2$ is stable. Galerkin's analysis showed that three of the mentioned stable branches lose stability at $Ra = 18\pi^2$ and $21\pi^2$, respectively, at the Hopf bifurcation points. The existence of many branches of different stable pattern flows could explain the difficulties in determining the unique, second critical Rayleigh number of the transition from laminar to fluctuating flow.

Badano własności przybliżonych rozwiązań równań Darcy–Boussinesqa, otrzymanych dla małych amplitud przy pomocy analizy perturbacyjnej i w przypadku skończonych amplitud przy użyciu metody Galerkina. Pokazano, że dwie gałęzie z trzech, które rozwidlają się dla $Ra = 4\pi^2$ reprezentujące dwuwymiarowe ustalone przepływy, są stabilne. Trzecia gałąź trójwymiarowej konwekcji jest niestabilna. Jednakże gałąź trójwymiarowego przepływu wychodząca z $Ra = 4.5\pi^2$ jest stabilna. Analiza Galerkina pokazała, że trzy wspomniane stabilne gałęzie tracą stabilność w punktach bifurkacji Hopfa. Istnienie wielu różnych stabilnych form konwekcji może wyjaśniać trudności w jednoznacznym określeniu drugiej krytycznej liczby Rayleigha przejścia od laminarnego do fluktuacyjnego przepływu.

Исследованы свойства приближенных решений уравнений Дарси-Буссинеска, полученных для малых амплитуд при помощи пертурбационного метода, а в случае конечных амплитуд при использовании метода Галеркина. Доказано, что две ветви, из трех, которые разветвляются для $Ra = 4\pi^2$, представляющие двухмерные установившиеся течения, стабильны. Третья ветвь трехмерной конвекции нестабильна. Однако ветвь трехмерного течения, исходящая из $Ra = 4,5\pi^2$, стабильна. Анализ Галеркина показал, что три упомянутые стабильные ветви теряют стабильность в точках бифуркации Хопфа. Существование многих разных стабильных форм конвекции может выяснять трудности в однозначном определении второго критического числа Релея перехода от ламинарного к турбулентному течениям.

1. Introduction

THE NATURAL convection in a saturated porous layer of infinite horizontal extent has received considerable attention in recent years mainly because of its geophysical interest. In terms of mathematics convection in a porous medium is simpler than the ordinary Bénard problem, which results from the replacement of the viscosity term in the Navier–Stokes equation by Darcy's law.

Similarly to the classical Bénard problem, the Rayleigh number Ra assumes in this case two critical values. The first of them $Ra_1 = 4\pi^2$, given by LAPWOOD [1], determines transition from conductive to convective heat transfer in a porous layer and has been well established by many laboratory experiments. The second critical Rayleigh number

Ra_2 determines transition from laminar to turbulent flow, but its exact value is not known at present.

Experiments carried out by COMBARNOUS and LE FUR [2] suggest that the second critical Rayleigh number lies between 240 and 280. Numerical calculations of Ra_2 [3, 4, 5] and [6] were also carried out, but the results are not as yet satisfactory because of their non-uniqueness. The recent SCHUBERT and STRAUS calculations [6] suggest that the transition to fluctuating convection occurs at a value of Ra between 300 and 320. The authors [7] have shown that the transition of two-dimensional convection in a square cell to fluctuations takes place at $Ra_2 = 30\pi^2$.

The nonuniqueness of the second critical Rayleigh number results probably from the multiplicity of stationary states of convection for large Ra numbers. If the Rayleigh number increases, the branches of steady-state solutions lose stability for different values of Ra_2. Hence we observe experimentally and numerically different values of the second critical Rayleigh number, depending on the particular realization of the pattern flow. Moreover, the loss of stability of some branch of steady-state solutions need not lead to turbulence, the transition to other steady-states being possible. At present the behaviour of thermal convection in a porous layer for large Ra is considered to be extremely complex, and it is still impossible to completely analyse the stability of flow.

In this paper we give a perturbation analysis of small solutions which emanate from the two first points of bifurcation $Ra = 4\pi^2$ and $4.5\pi^5$. In the sequel we use the Galerkin method for obtaining the finite amplitude solutions and analyse their stability. The obtained results show the possibility of the existence of two- and three-dimensional stable flows which lose stability at Ra close to the experimental data.

2. Formulation of the problem

Consider a saturated porous layer of infinite extent heated from below. The layer has a thickness equal to unity and is bounded by two nonpermeable, perfectly conducting, horizontal plates. We assume that fluid motion and heat transfer including convection in a porous medium are described by the dimensionless Darcy–Boussinesq equations

$$\frac{\partial\theta}{\partial t} = \nabla^2\theta + u_z - \bar{u}\nabla\theta, \tag{2.1}$$

$$-\bar{u} - \nabla p + \mathrm{Ra}\,\theta\bar{z} = 0, \qquad \nabla\bar{u} = 0,$$

with the boundary conditions on the lower and upper plates

$$z = 0,\ 1:\ \theta = u_z = 0. \tag{2.2}$$

Here θ is the temperature, $\bar{u} = (u_z, u_x, u_y)$ is the velocity vector, p is the pressure, $\bar{z} = (0, 0, 1)$ is the unit vector directed upwards and ∇^2 is the Laplace operator.

The Rayleigh number is defined as follows:

$$\mathrm{Ra} = kg\alpha h\,\Delta T/k_m\nu,$$

where k denotes the coefficient of permeability, g — acceleration due to gravity, α — the coefficient of thermal expansion, h — wave number, ΔT — difference of temperature,

ν — viscosity, k_m — coefficient of thermal diffusion. The physical parameters of the porous medium as well as the fluid (viscosity, permeability, thermal expansion etc.) are constant and do not depend on temperature and pressure.

When the Rayleigh number is sufficiently small, there is only a conductive, steady state solution of Eqs. (2.1) and (2.2) $\theta = \bar{u} = p = 0$. This solution loses stability at the bifurcation point when convection appears. The necessary condition of bifurcation occurrence is that the linearized form of the steady-state problem (2.1) and (2.2)

$$\begin{gathered} \nabla^2\varphi+\psi_z = 0, \\ -\bar{\psi}-\nabla q+\mathrm{Ra}\,\varphi\bar{z} = 0, \\ \nabla\bar{\psi} = 0, \end{gathered} \tag{2.3}$$

$$z = 0, \quad 1: \varphi = \psi_z = 0 \tag{2.4}$$

has a nontrivial eigenvector $[\varphi, \bar{\psi}, q]^T$, where $\varphi, \bar{\psi}, q$ denote the temperature, the velocity and the pressure, respectively. Combining the particular equations of the set (2.3), it is easy to obtain the linear eigenvalue problem in the form of a single equation:

$$\nabla^4 q+\mathrm{Ra}\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}\right) = 0. \tag{2.5}$$

From the relations (2.4) and the momentum balance equation in the z direction (2.3) there result the boundary conditions

$$z = 0, 1: \frac{\partial q}{\partial z} = 0. \tag{2.6}$$

The eigenvectors of the linear eigenvalue problem (2.5) and the relations (2.6) have the form

$$q = 2\pi\cos(i\pi z)\cdot\cos(kh_x\pi x)\cdot\cos(lh_y\pi y).$$

Eigenvalues, corresponding to these eigenvectors, are given by

$$\mathrm{Ra}_{ikl} = \pi^2\frac{(i^2+k^2h_x^2+l^2h_y^2)^2}{k^2h_x^2+l^2h_y^2},$$

where h_x^{-1} and h_y^{-1} are the horizontal wave numbers in the x and y directions, respectively. The smallest eigenvalue $\mathrm{Ra}_1 = 4\pi^2$ follows for $h_x = h_y = 1$ and two sets of numbers (i, k, l)

$$(1, 1, 0) \quad \text{and} \quad (1, 0, 1).$$

The eigenvalue $\mathrm{Ra} = 4\pi^2$ is double because there are two corresponding eigenvalues which can be easily calculated from Eqs. (2.5) and (2.6)

$$\begin{gathered} \varphi_1 = \sin\pi z\cos\pi x, \quad q_1 = -2\pi\cos\pi z\cos\pi x, \\ \psi_{1z} = 2\pi^2\sin\pi z\cos\pi x, \quad \psi_{1x} = -2\pi^2\cos\pi z\sin\pi x, \end{gathered} \tag{2.7}$$

$$\begin{gathered} \varphi_2 = \sin\pi z\cos\pi y, \quad q_2 = -2\pi\cos\pi z\cos\pi y, \\ \psi_{2z} = 2\pi^2\sin\pi z\cos\pi y, \quad \psi_{2y} = -2\pi^2\cos\pi z\sin\pi y. \end{gathered} \tag{2.8}$$

These eigenfunctions are two-dimensional, hence we conclude that the small nonlinear solutions of Eqs. (2.1) and (2.2) corresponding to them also are two-dimensional. However, every linear combination of these gives a three-dimensional eigenvector and can lead to a three-dimensional nonlinear solution.

It is well known [9] that the odd multiplicity of the eigenvalue is a sufficient condition for the existence of the bifurcation point. When the multiplicity is even, then every case must be analysed precisely because we do not know the general rules for the determination of the emanating branches. The next section is an illustration of this problem.

We should also notice that close to the first bifurcation point $\mathrm{Ra}_1 = 4\pi^2$ there is the next point of bifurcation $\mathrm{Ra} = 4.5\pi^2$. Because the eigenvalue $\mathrm{Ra} = 4.5\pi^2$ is simple at this point, only one branch of the nonlinear solution bifurcates. However, it is interesting from the physical point of view that the corresponding eigenvector is three-dimensional, which suggests that it is possible for stable three-dimensional flow to occur very close to the first critical Rayleigh number.

In the sequel we assume the periodicity of the solutions of Eqs. (2.1) and (2.2) in the x and y directions, and restrict our considerations to the box with perfectly insulated walls. Hence the boundary conditions on the sidewalls become

$$\frac{\partial\theta}{\partial\bar{n}} = u_n = 0, \tag{2.9}$$

where $\bar{n}$ denotes the normal direction to the wall. The horizontal dimensions h_x and h_y are assumed to be equal to unity.

3. Bifurcation of steady-state solutions

For the evaluation of the branches of steady-state solutions, which emanate at $\mathrm{Ra}_1 = 4\pi^2$, we will use the perturbation analysis [10]. Hence we assume steady-state solutions of Eqs. (2.1) and (2.2) in the following power series:

$$\begin{bmatrix}\theta\\ \bar{u}\\ p\end{bmatrix} = \varepsilon\left\{\alpha_1\begin{bmatrix}\varphi_1\\ \bar{\psi}_1\\ q_1\end{bmatrix}+\alpha_2\begin{bmatrix}\varphi_2\\ \bar{\psi}_2\\ q_2\end{bmatrix}+\varepsilon\sum_{i=2}^{\infty}\varepsilon^{i-2}\begin{bmatrix}\theta_i\\ \bar{u}_i\\ p_i\end{bmatrix}\right\}, \tag{3.1}$$

$$\nabla\cdot\bar{u}_i = 0,$$

$$\mathrm{Ra} = \mathrm{Ra}_1+\sum_{i=1}^{\infty}\varepsilon^i r_i, \tag{3.2}$$

where ε is a small parameter.

Now our main effort is directed to evaluating the coefficients α_1 and α_2, which determine the multiplicity of the solutions. It is convenient to normalize these coefficients

$$\alpha_1^2+\alpha_2^2 = 1.$$

Putting Eqs. (3.1) and (3.2) into Eqs. (2.1) and (2.2) and expanding the nonlinear term $u\cdot\nabla\theta$ into the Taylor series, we obtain the equations of perturbation, after comparing the terms to zero powers of ε. The equation of the second perturbation has the form

$$\begin{aligned}\nabla^2\theta_2+u_{2z} &= (\alpha_1\bar{\psi}_1+\alpha_2\bar{\psi}_2)\nabla(\alpha_1\varphi_1+\alpha_2\varphi_2),\\ -\bar{u}_2-\nabla p_2+\mathrm{Ra}_1\theta_2\bar{z} &= -r_1(\alpha_1\varphi_1+\alpha_2\varphi_2)\bar{z},\\ \nabla\cdot\bar{u}_2 &= 0\end{aligned} \tag{3.3}$$

with the boundary conditions (2.2) and (2.9). The necessary condition for the existence of the solution $[\theta_2, \bar{u}_2, p_2]^T$ is that the right-hand side of Eq. (3.3) be orthogonal to the eigenvector of the adjoint to Eqs. (3.5) and (3.6) linear eigenvalue problem [10]. The adjoint to Eqs. (3.5) and (3.6) linear eigenvalue problem has the form

$$(3.4)\qquad \begin{aligned} \nabla^2\varphi^* + \mathrm{Ra}_1\psi_z^* &= 0,\\ -\bar{\psi}^* - \nabla p^* + \varphi_z^* &= 0,\\ \nabla\cdot\bar{\psi}^* &= 0 \end{aligned}$$

with the eigenvector

$$[\varphi^*, \bar{\psi}^*, p^*]^T = [\varphi, \psi/\mathrm{Ra}_1, p/\mathrm{Ra}_1]^T.$$

Multiplying the right-hand side of Eqs. (3.3) by the eigenvectors $[\varphi_i^*, \psi_i^*, p^*]^T$ $(i = 1, 2)$, we obtain two equations of bifurcation:

$$\langle(\alpha_1\psi_1 + \alpha_2\psi_2), \varphi_i^*\rangle - r_1\langle(\alpha_1\varphi_1 + \alpha_2\varphi_2), \psi_{iz}^*\rangle = 0,$$

where $\langle .,.\rangle$ denotes the scalar product in L_2 [(0,1) (0,1) (0,1)]. Since $\varphi_i = \varphi_i^*$ $(i = 1, 2)$, multiplying the above equations by α_1 and α_2, respectively, and adding, we obtain

$$r_1 \frac{\pi^2}{\mathrm{Ra}_1} = 0$$

which implies $r_1 = 0$.

Further we consider the equations of the third perturbation

$$(3.5)\qquad \begin{aligned} \nabla^2\theta_3 + u_{z3} &= (\alpha_1\bar{\psi}_1 + \alpha_2\bar{\psi}_2)\theta_2 + \bar{u}_2\nabla(\alpha_1\varphi_1 + \alpha_2\varphi_2),\\ -\bar{u}_3 - \nabla p_3 + \mathrm{Ra}_1\theta_3\bar{z} &= -r_2(\alpha_1\varphi_1 + \alpha_2\varphi_2)\bar{z},\\ \nabla\cdot\bar{u}_3 &= 0 \end{aligned}$$

which lead to the bifurcation equations in the following form:

$$(3.6)\qquad \langle(\alpha_1\bar{\psi}_1 + \alpha_2\bar{\psi}_2)\nabla\theta_2 + \bar{u}_2\nabla(\alpha_1\varphi_1 + \alpha_2\varphi_2), \varphi_i^*\rangle - r_2\langle\alpha_1\varphi_1 + \alpha_2\varphi_2, \psi_{iz}^*\rangle = 0, \quad i = 1, 2.$$

These equations allow the evaluation of α_1 and α_2, but before we must calculate θ_2 and $\bar{\psi}_2$ from Eq. (3.3). We neglect longish calculations and present only the results:

$$\theta_2 = -\frac{\pi}{4}\sin(2\pi z) - \frac{3\pi}{7}\alpha_1\alpha_2\sin(2\pi z)\cos(\pi x)\cos(\pi y),$$

$$u_{2z} = \mathrm{Ra}_1\frac{\pi}{7}\alpha_1\alpha_2\cos(2\pi z)\cos(\pi x)\sin(\pi y),$$

$$u_{2x} = \mathrm{Ra}_1\frac{\pi}{7}\alpha_1\alpha_2\cos(2\pi z)\sin(\pi x)\cos(\pi y),$$

$$u_{2y} = \mathrm{Ra}_1\frac{\pi}{7}\alpha_1\alpha_2\sin(2\pi z)\cos(\pi x)\cos(\pi y),$$

$$p_2 = \mathrm{Ra}_1\frac{1}{8}\cos(2\pi z) + \frac{1}{7}\mathrm{Ra}_1\alpha_1\alpha_2\cos(2\pi z)\cos(\pi x)\cos(\pi y).$$

Consequently, the functionals existing in Eqs. (3.6) are given by

$$\langle \bar{\psi}_1 \nabla \theta_2, \varphi_i^* \rangle = \pi^4/8, \quad i = 1, 2,$$
$$\langle \bar{\psi}_2 \nabla \theta_2, \varphi_1^* \rangle = \langle \bar{\psi}_1 \nabla \theta_2, \varphi_2^* \rangle = 3\alpha_1 \alpha_2 \pi^4/56,$$
(3.7)
$$\langle \bar{u}_2 \nabla \varphi_1, \varphi_2^* \rangle = \langle \bar{u}_2 \nabla \varphi_2, \varphi_1^* \rangle = 0,$$
$$\langle \varphi_i, \psi_{iz}^* \rangle = \frac{\pi^2}{2\mathrm{Ra}_1},$$
$$\langle \bar{u}_2 \nabla \varphi_i, \varphi_i^* \rangle = 0, \quad i = 1, 2.$$

Introducing Eq. (3.7) into Eq. (3.6) and adjoining the condition of normalization we obtain equations for three variables α_1, α_2 and r_2:

$$\alpha_1\left(\frac{\pi^2}{4} + \alpha_2^2 \frac{3\pi^2}{28} - r_2/\mathrm{Ra}_1\right) = 0,$$
$$\alpha_2\left(\alpha_1^2 \frac{3\pi^2}{28} + \frac{\pi^2}{4} - r_2/\mathrm{Ra}_1\right) = 0,$$
$$\alpha_1^2 + \alpha_2^2 = 1.$$

There are three sets of nontrivial solutions of the above equations:

1) $\alpha_1 = \pm 1, \quad \alpha_2 = 0, \quad r_2 = \dfrac{\pi^2}{4}\mathrm{Ra}_1,$

2) $\alpha_1 = 0, \quad \alpha_2 = \pm 1, \quad r_2 = \dfrac{\pi^2}{4}\mathrm{Ra}_1,$

3) $\alpha_1 = \alpha_2 = \pm 1/\sqrt{2},$

$\alpha_1 = -\alpha_2 = \pm 1/\sqrt{2},$

$r_2 = \dfrac{17}{56}\pi^2 \mathrm{Ra}_1.$

Each of these solutions corresponds to the branch of nonlinear solutions of Eqs. (2.1) and (2.2) which emanate at the first point of bifurcation $\mathrm{Ra}_1 = 4\pi^2$.

From Eqs. (3.1) and (3.2) it follows that

$$\varepsilon = \left(\frac{\mathrm{Ra} - \mathrm{Ra}_1}{\mathrm{Ra}_1}\right)^{1/2} + \text{terms of higher order}$$

and three distinct sets of solutions of Eqs. (2.1) and (2.2) assume the form

1), 2) $$\begin{bmatrix} \theta \\ \bar{u} \\ p \end{bmatrix} = \pm \frac{2}{\pi}\left(\frac{\mathrm{Ra} - \mathrm{Ra}_1}{\mathrm{Ra}_1}\right)^{1/2} \begin{bmatrix} \varphi_i \\ \bar{\psi}_i \\ q_i \end{bmatrix} + \frac{4}{\pi}\,\frac{\mathrm{Ra} - \mathrm{Ra}_1}{\mathrm{Ra}_1} \begin{bmatrix} \theta_2 \\ \bar{u}_2 \\ p_2 \end{bmatrix} + \ldots, \quad i = 1, 2.$$

3) $$\begin{bmatrix} \theta \\ \bar{u} \\ p \end{bmatrix} = \pm \frac{1}{\pi}\sqrt{\frac{28}{17}}\left(\frac{\mathrm{Ra} - \mathrm{Ra}_1}{\mathrm{Ra}_1}\right)^{1/2} \begin{bmatrix} \varphi_1 \\ \bar{\psi}_1 \\ q_1 \end{bmatrix} + \begin{bmatrix} \psi_2 \\ \bar{\psi}_2 \\ q_2 \end{bmatrix} + \frac{56}{17\pi^2}\,\frac{\mathrm{Ra} - \mathrm{Ra}_1}{\mathrm{Ra}_1} \begin{bmatrix} \theta_2 \\ \bar{u}_2 \\ p_2 \end{bmatrix} + \ldots.$$

The solutions of the next perturbation equations depend on α_1 and α_2. For example, when $\alpha_1 = \pm 1$, $\alpha_2 = 0$ $[\theta_3, \bar{u}_3, p_3]^T$ does not depend on the y coordinate, when $\alpha_1 = 0$, $\alpha_2 = \pm 1$ $[\theta_3, \bar{u}_3, p_3]^T$ does not depend on x. This is true for the next perturbation solutions $[\theta_4, \bar{u}_4, p_4]^T, \ldots$. Hence the first two solutions: 1) and 2) determine exactly the two-dimensional rolls, the axes of which are parallel to the x or y coordinate, respectively. The third solution represents three-dimensional convection. The stability of this solution will be analysed in the next section.

The case of bifurcation at the point $\mathrm{Ra} = 4.5\pi^2$ is the classical one because the eigenvalue Ra is simple. Hence only one branch emanates from this point and its amplitude is proportional to $(\mathrm{Ra}/4.5\pi^2-1)^{1/2}$. The stability analysis of this branch is given in the fifth section.

4. Stability of small amplitude solutions

The significance of the stability analysis is due to the fact that only stable physical states can be observed experimentally. For the thermal convection phenomena it is interesting which stable pattern flow, two- or three-dimensional, leads to turbulence. In this section we analyse the stability of branches emanated at $\mathrm{Ra} = 4\pi^2$.

We assume the solution of the nonlinear problem (2.1) and (2.2) in the form of the following power series:

$$\begin{bmatrix} \theta \\ \bar{u} \\ p \end{bmatrix} = \gamma \left\{ \alpha_1(\tau) \begin{bmatrix} \varphi_1 \\ \bar{\psi}_1 \\ q_1 \end{bmatrix} + \alpha_2(\tau) \begin{bmatrix} \varphi_2 \\ \bar{\psi}_2 \\ q_2 \end{bmatrix} + \gamma \sum_{i=2}^{\infty} \gamma^{i-2} \begin{bmatrix} \theta_i(t, x, y, z) \\ \bar{u}_i(t, x, y, z) \\ p(t, x, y, z) \end{bmatrix} \right\},$$

$$\nabla u_i = 0, \qquad \mathrm{Ra} = \mathrm{Ra}_1 + \gamma^2, \qquad \tau = \gamma^2 t.$$

The equations of the second perturbation assume exactly the same form as in the previous section, but θ_2 is changed slightly because α_1 and α_2 are not normalized:

$$\theta_2 = -\frac{\pi}{4}(\alpha_1^2+\alpha_2^2)\sin(2\pi z) - \frac{3\pi}{7}\alpha_1\alpha_2 \sin(2\pi z)\cos(\pi x)\cos(\pi y).$$

The equations of the third perturbation yield the following form:

$$\frac{\partial(\alpha_1\varphi_1+\alpha_2\varphi_2)}{\partial t} = \nabla^2\theta_3 + u_{3z} - [(\alpha_1\psi_1+\alpha_2\psi_2)\nabla\theta_2 + \bar{u}_2\nabla(\alpha_1\varphi_1+\alpha_2\varphi_2)],$$

$$-\bar{u}_3 - \nabla p_3 + \mathrm{Ra}_1\theta_3\bar{z} = -(\alpha_1\varphi_1+\alpha_2\varphi_2)\bar{z}.$$

Consequently, the equations of bifurcation may be written as

$$\frac{1}{4}\frac{d\alpha_i}{dt} = -\langle(\alpha_1\bar{\psi}_1+\alpha_2\bar{\psi}_2)\nabla\theta_2 + \bar{u}_2\nabla(\alpha_1\varphi_1+\alpha_2\varphi_2), \varphi_i^*\rangle$$
$$+\langle(\alpha_1\varphi_1+\alpha_2\varphi_2), \psi_{iz}\rangle, \qquad i = 1, 2.$$

The functionals which occur in the above equations are also given by Eqs. (3.7) except two of them:

$$\langle\bar{\psi}_1\nabla\theta_2, \varphi_1^*\rangle = \langle\bar{\psi}_2\nabla\theta_2, \varphi_2^*\rangle = \frac{\pi^4}{8}(\alpha_1^2+\alpha_2^2).$$

Therefore we have

$$(4.1)\quad \begin{aligned} \frac{1}{4}\frac{d\alpha_1}{dt} &= -\frac{\pi^4}{8}(\alpha_1^2+\alpha_2^2)\alpha_1-\alpha_1\alpha_2^2\frac{3\pi^4}{56}+\alpha_1\frac{\pi^2}{2\mathrm{Ra}_1},\\ \frac{1}{4}\frac{d\alpha_2}{dt} &= -\frac{\pi^4}{8}(\alpha_1^2+\alpha_2^2)\alpha_2-\alpha_1^2\alpha_2\frac{3\pi^4}{56}+\alpha_2\frac{\pi^2}{2\mathrm{Ra}_1}.\end{aligned}$$

According to the previous analysis there are three sets of nontrivial fixed points:

1), 2)
$$\alpha_1 = \pm\frac{2}{\pi\sqrt{\mathrm{Ra}_1}},\qquad \alpha_2 = 0,$$
$$\alpha_1 = 0,\qquad \alpha_2 = \pm\frac{2}{\pi\sqrt{\mathrm{Ra}_1}},$$

3)
$$\alpha_1 = \alpha_2 = \pm\frac{1}{\pi\sqrt{\mathrm{Ra}_1}}\sqrt{\frac{28}{17}},$$
$$\alpha_1 = -\alpha_2 = \pm\frac{1}{\pi\sqrt{\mathrm{Ra}_1}}\sqrt{\frac{28}{17}}.$$

The standard stability analysis of fixed points shows that the solutions belonging to sets 1) and 2) are stable, whereas the trivial solutions and solutions belonging to set 3) are unstable. Figure 1 presents the geometrical properties of the particular solutions. It is an obvious conclusion that the steady-state form of flow depends only on a choice of the initial conditions for $\alpha_1(\tau)$ and $\alpha_2(\tau)$.

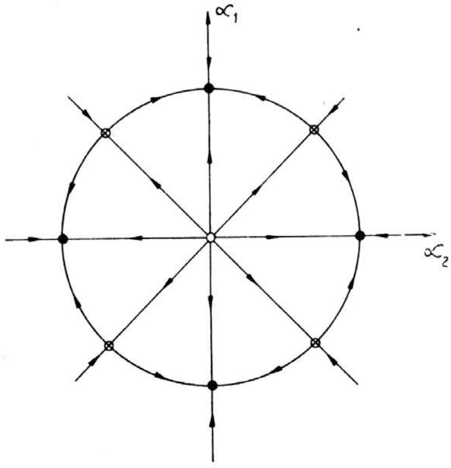

FIG. 1. Stability of the fixed points of Eqs. (4.1) (● — stable, ○ — unstable).

5. Galerkin's analysis of finite amplitude solutions

By taking the curl of Darcy's law in the relations (2.1), it can be seen that the vertical component of verticity is zero:

$$(5.1)\quad \frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}=0.$$

This is identically satisfied if

$$(5.2) \qquad u_x = \frac{\partial^2\phi}{\partial x^2}, \qquad u_y = \frac{\partial^2\phi}{\partial x\partial y}.$$

Furthermore, with $u_z = -\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}$ the continuity equation will be satisfied.

With the help of Eqs. (5.1) and (5.2), we can obtain two equations for unknown θ and ϕ from the relations (2.1)

$$(5.3) \qquad \nabla^2\phi = -\mathrm{Ra}\,\theta,$$

$$\frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial x}\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial\theta}{\partial y}\frac{\partial^2\phi}{\partial y\partial z} - \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\frac{\partial\theta}{\partial z} = \nabla^2\theta - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}.$$

The boundary conditions on θ and ϕ are

$$\theta = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{on} \quad z = 0, 1.$$

We use the Galerkin technique to determine ϕ and θ in this three-dimensional situation.

The solutions of the problem (5.3) will be sought in the form of the Fourier series:

$$(5.4) \qquad \phi = \sum_{n,j,m}^{\infty} \beta_{njm} F_{njm}, \qquad \theta = \sum_{n,j,m}^{\infty} \alpha_{njm} F_{njm},$$

where F_{njm} is a complete orthonormal sequence

$$F_{njm} = \begin{cases} \sqrt{2}\sin(n\pi z), & m = 0, \quad j = 0, \\ 2\sin(n\pi z)\cos(j\pi x), & m = 0, \quad j \neq 0, \\ 2\sin(n\pi z)\cos(m\pi y), & m \neq 0, \quad j = 0, \\ 2\sqrt{2}\sin(n\pi z)\cos(j\pi x)\cos(m\pi y), & m \neq 0, \quad j \neq 0. \end{cases}$$

Truncating the sequences (5.4) to functions

$$(5.5) \qquad F_{100}, F_{200}, F_{110}, F_{101}, F_{210}, F_{201}, F_{111}, F_{211},$$

we obtain a finite system of ordinary differential equations:

$$(5.6) \qquad \dot{\alpha}_{100} = -\pi^2\alpha_{100} + \frac{7\mathrm{Ra}\pi}{10\sqrt{2}}\alpha_{110}\alpha_{210} + \frac{7\mathrm{Ra}\pi}{10\sqrt{2}}\alpha_{101}\alpha_{210} + \frac{\mathrm{Ra}\pi}{\sqrt{2}}\alpha_{111}\alpha_{211},$$

$$\dot{\alpha}_{200} = -4\pi^2\alpha_{200} - \frac{\mathrm{Ra}\pi}{\sqrt{2}}\alpha_{110}^2 - \frac{\mathrm{Ra}\pi}{\sqrt{2}}\alpha_{101}^2 - \frac{2\sqrt{2}\pi}{3}\mathrm{Ra}\,\alpha_{111}^2,$$

$$\dot{\alpha}_{110} = \frac{\mathrm{Ra} - 4\pi^2}{2}\alpha_{110} + \frac{\mathrm{Ra}\pi}{\sqrt{2}}\alpha_{110}\alpha_{200} - \frac{\mathrm{Ra}\pi}{5\sqrt{2}}\alpha_{210}\alpha_{100} + \frac{6\mathrm{Ra}\pi}{5\sqrt{2}}\alpha_{201}\alpha_{111} + \frac{\mathrm{Ra}\pi}{2\sqrt{2}}\alpha_{101}\alpha_{211},$$

(5.6) [cont.]

$$\dot{\alpha}_{101} = \frac{\mathrm{Ra}-4\pi^2}{2}\alpha_{101} + \frac{\mathrm{Ra}\,\pi}{\sqrt{2}}\alpha_{101}\alpha_{100} - \frac{\mathrm{Ra}\,\pi}{5\sqrt{2}}\alpha_{201}\alpha_{100} + \frac{6\,\mathrm{Ra}\,\pi}{5\sqrt{2}}\alpha_{210}\alpha_{111} + \frac{\mathrm{Ra}\,\pi}{2\sqrt{2}}\alpha_{110}\alpha_{211},$$

$$\dot{\alpha}_{210} = \frac{\mathrm{Ra}-25\pi^2}{5}\alpha_{210} - \frac{\mathrm{Ra}\,\pi}{2\sqrt{2}}\alpha_{110}\alpha_{100} - \sqrt{2}\,\mathrm{Ra}\,\pi\alpha_{101}\alpha_{111},$$

$$\dot{\alpha}_{201} = \frac{\mathrm{Ra}-25\pi^2}{5}\alpha_{201} - \frac{\mathrm{Ra}\,\pi}{2\sqrt{2}}\alpha_{101}\alpha_{100} - \sqrt{2}\,\mathrm{Ra}\,\pi\alpha_{110}\alpha_{111},$$

$$\dot{\alpha}_{111} = \frac{2\mathrm{Ra}-9\pi^2}{3}\alpha_{111} + \frac{2\sqrt{2}}{3}\mathrm{Ra}\,\pi\alpha_{101}\alpha_{110} - \frac{\mathrm{Ra}\,\pi}{3\sqrt{2}}\alpha_{211}\alpha_{101} + \frac{\pi 4\,\mathrm{Ra}}{5\sqrt{2}}\alpha_{110}\alpha_{201} + \frac{\pi 4\,\mathrm{Ra}}{5\sqrt{2}}\alpha_{101}\alpha_{210},$$

$$\dot{\alpha}_{211} = \frac{\mathrm{Ra}-18\pi^2}{3}\alpha_{211} - \frac{\mathrm{Ra}\,\pi}{\sqrt{2}}\alpha_{101}\alpha_{110} - \frac{\sqrt{2}}{3}\mathrm{Ra}\,\pi\alpha_{111}\alpha_{100}.$$

The eight distinct steady-states of the convection were designated (eight sets of stationary solutions of the system (5.6)). Figure 2 shows the norms of the branches of these stationary solutions. The branches A, B and E starting from point $\mathrm{Ra} = 4\pi^2$ are the same as were determined in the perturbation analysis in Sect. 3. It can be seen that branch E consists of two separate pieces for $4\pi^2 < \mathrm{Ra} < 18\pi^2$ and for $\mathrm{Ra} > 21\pi^2$, respectively. $\mathrm{Ra} = 21\pi^2$ is a singularity point of this branch. The branches D, F and G, which do not bifurcate from the trivial solution, emanate from the bifurcation point of limit-point type.

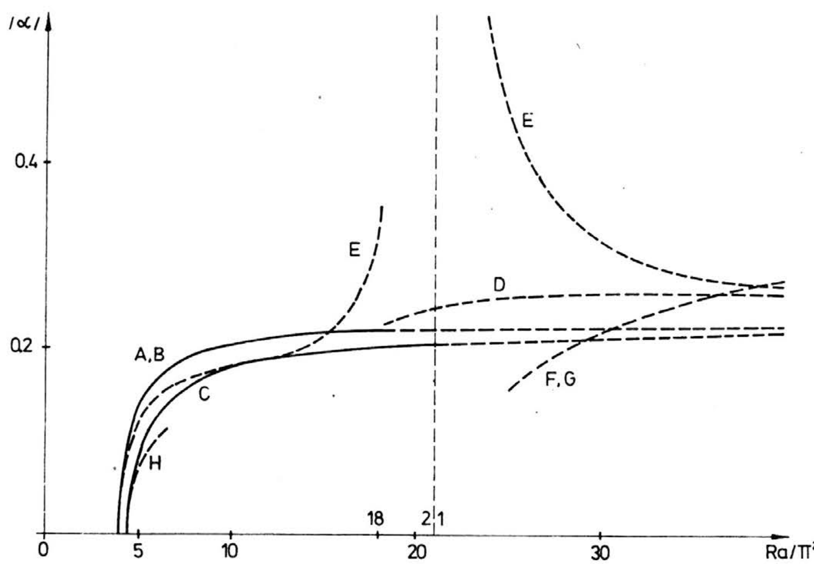

FIG. 2. Norms of stationary solutions of the system (5.6).

The steady-states A, B, F, G represent two-dimensional convective rolls. The other branches: C, D, E, H correspond to three-dimensional convection. Figure 3a shows the isotherm pattern for three-dimensional flow represented by branch E while Fig. 3b — the isotherm pattern for three-dimensional flow represented by branch C. The linearized stability analysis of steady-states of convection was examined.

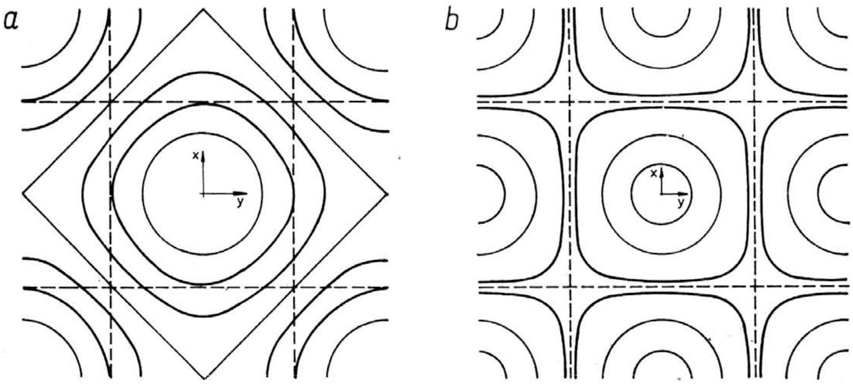

FIG. 3. The isotherm pattern for three-dimensional flows represented by a) E and b) C branches, respectively.

For the purpose of the stability analysis the real part of the eigenvalues of the Jacobi matrix of the system (5.6) for fixed points were designated.

The stability of branches A, B, C could be directly determined. The characteristic equations of other branches were solved in the numerical way. The stability analysis shows that the branches of three-dimensional convection, D, E and two-dimensional convection F, G are entirely unstable. The steady-states A and B are at first stable, and lose stability when Ra crosses the Hopf bifurcation point $\mathrm{Ra} = 18\pi^2$.

The stationary solution C, starting from $\mathrm{Ra} = 4.5\pi^2$, is unstable in the range of $4.5\pi^2 < \mathrm{Ra} < 4.78\pi^2$ and stable in the range of $4.78\pi^2 < \mathrm{Ra} < 21\pi^2$.

According to the basis (5.5), it has the form

$$C = \left(0, -\frac{\mathrm{Ra}-4.5\pi^2}{\sqrt{2}\,\mathrm{Ra}\,\pi}, 0, 0, 0, 0, \pm\frac{\sqrt{3}}{\mathrm{Ra}}\sqrt{\mathrm{Ra}-4.5\pi^2}, 0\right).$$

$\mathrm{Ra} = 21\pi^2$ is also the Hopf bifurcation point of this branch. When the Rayleigh number exceeds $21\pi^2$, the solution C becomes unstable.

6. Conclusions

At $\mathrm{Ra} = 4\pi^2$ three branches of steady-state solutions of the Darcy–Boussinesq equations emanate. Two of them representing two-dimensional flows are stable. The third which describes three-dimensional flow is unstable. However, very close to the first critical Rayleigh number, at $\mathrm{Ra} = 4.5\pi^2$, one branch of stable three-dimensional flow emanates.

The Galerkin analysis, made for a finite basis truncated to eight elements, increased the number of branches of steady-state solutions to seven, but only three of them having been discovered earlier by the perturbed method are stable. The stable branches starting at $Ra = 4\pi^2$ and $Ra = 4.5\pi^2$ lose stability at the Hopf bifurcation points $Ra = 18\pi^2$ and $21\pi^2$, respectively.

At present the stability of bifurcating periodic solutions from these points is not known. It is an interesting problem because the transition from laminar to turbulent flow is abrupt when subcritical bifurcation takes place [11] (unstable orbits) and is preceded by complicated periodic and quasi-periodic states [12] for the supercritical point of bifurcation (stable orbits).

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