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Generation of waves on a running stream in electrohydrodynamics

B. K. SHIVAMOGGI (CANBERRA)

THE PRESENT paper is concerned with the electrohydrodynamic effects on the generation of two-dimensional waves by an oscillating pressure acting at the surface of a running stream of a conducting fluid of finite depth. It is found that in the ultimate steady state either two or four waves may exist, depending on the relative values of the speed of the fluid, its depth, the frequency of the applied pressure, and the magnitude of the electrostatic forces at the free surface. The conditions separating these two possible states of the wave system are found to be influenced by the electrostatic forces. Thus the critical stream velocity U_c for the the generation of waves on the downstream side of the source which is a steady pressure acting on the free surface decreases in the presence of electrohydrodynamic effects. Further, the electrohydrodynamic effects slow down some of these waves and speed up some others.

Rozważono wpływy elektrohydrodynamiczne na proces generacji dwuwymiarowych fal w płynącym strumieniu cieczy przewodzącej pod działaniem oscylującego ciśnienia na powierzchni. Stwierdzono, że w stanie ustalonym mogą istnieć dwie lub cztery fale, zależnie od wzajemnych stosunków prędkości cieczy, głębokości strumienia, częstości drgań, ciśnienia i wielkości sił elektrostatycznych. Stwierdzono, że na warunki zapewniające osiągnięcie jednego z tych dwóch stanów mają wpływ siły elektrostatyczne. Tak więc krytyczna prędkość strumienia U_c, generująca fale w dół od źródła stanowiącego ustalone ciśnienie na powierzchni, maleje w obecności efektów elektrohydrodynamicznych. Efekty te spowolniają niektóre z pojawiających się fal, a przyśpieszają inne.

Рассмотрены электрогидродинамические влияния на процесс генерации двухмерных волн в текущем потоке проводящей жидкости под действием осциллирующего давления на поверхности. Констатировано, что в установившимся состоянии могут существовать две или четыре волны в зависимости от взаимных отношений скорости жидкости, глубины потока, частоты колебаний давления и величины электростатических сил. Констатировано, что на условия, обеспечивающие достижение одного из этих двух состояний, имеют влияние электростатические силы. Итак критическая скорость потока U_c генерирующая волны вниз от источника, составляющего установленное давление на поверхности, убывает в присутствии электрогидродинамических эффектов. Эти эффекты замедляют некоторые из появляющихся волн, а ускоряют другие волны.

1. Introduction

BESIDES applications to such areas as electro-fluid dynamics of biological systems, dielectrophoretic orientation and expulsion of liquids in zero-gravity environments, insulation research in liquids and gases, electrohydrodynamics seems to be closely associated with the atmospheric and cloud physics, physicochemical hydrodynamics, bubble and drop dynamics, and the electrostatics of thunderstorms. Early studies of electrostatic effects on the motion of fluids were made by RAYLEIGH [8], who considered the effect of surface charges on the vibration of spherical drops. MICHAEL [5, 6] studied the effects of electrostatic forces on the stability of radial oscillations of a jet of conducting fluid and found that the effect of electrostatic forces on the vibrations of a system of fluid conductors depends only on the geometry of the conducting surfaces and not on the fluid motion produced. MICHAEL [7] and SHIVAMOGGI [9] studied the effects of electrostatic forces on the linear and nonlinear stability of wave-motion at the surface of highly-conducting fluids, and found that the electrostatic forces have destabilising effects on the wave-motion at the surface. The present paper is concerned with the electrohydrodynamic effects on the generation of two-dimensional waves by an oscillatory pressure acting at the surface of a running stream of a conducting fluid of finite depth, (the hydrodynamic counterpart of this problem was studied by DEBNATH and ROSENBLAT [1]). It is found that in the ultimate steady state either two of four waves may exist, depending on the relative values of the speed of the fluid, its depth, the frequency of the applied pressure, and the magnitude of the electrostatic forces at the free surface. The conditions separating these two possible states of the wave system are found to be influenced by the electrostatic forces.

2. The initial-value problem

Let us assume that in its undisturbed state the conducting fluid which is of infinite horizontal extent, has uniform depth h (the conducting fluid being supported on a conducting plate at $z = -h$, see Fig. 1). The wave-generating mechanism is a periodic pressure, of frequency ω, applied at the free surface given by $z = 0$. The applied pressure is two-dimensional so that the resulting wave-motion occurs wholly parallel to the flow. The conducting fluid is taken to be inviscid, incompressible, and the capillary effects (produced by the surface tension) at the free surface are ignored.

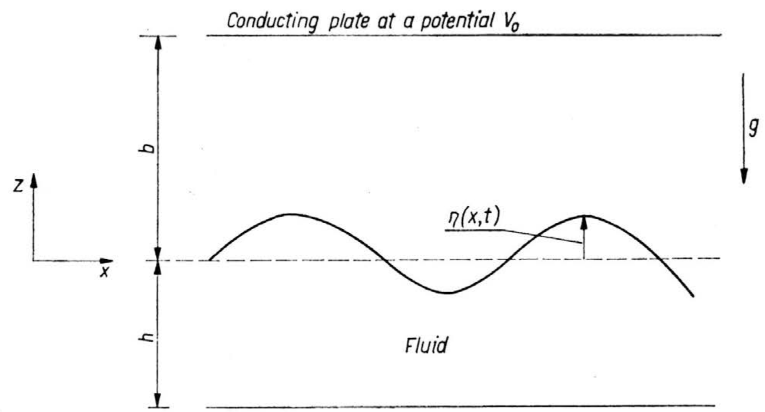

FIG. 1.

In treating steady-wave problems by the method of Fourier-transforms, one has to impose an appropriate radiation condition (LIGHTHILL [3]) to ensure uniqueness. One may avoid this by posing a more realistic initial-value problem, the applied pressure being "switched on" at time $t = 0$. An asymptotic development in time then leads to the ultimate steady-state solution.

It is convenient to pose the problem in a coordinate frame which is at rest with respect to the applied pressure at the surface. In this frame the fluid streams with speed U in the x-direction. The applied pressure is taken to be of the form

$$p(x, t) = p_0 \delta(x) e^{i\omega t}, \tag{2.1}$$

$\delta(x)$ being the Dirac-delta function. A conducting plate maintained at a potential V_0 lies at a distance b above the free surface of the fluid (see Fig. 1) so that in the undisturbed state the electrostatic potential is given by

$$\Psi = V_0 \frac{z}{b}. \tag{2.2}$$

Let $\phi(x, z, t)$, $\psi(x, z, t)$ be perturbations respectively in the velocity potential, electrostatic potential, and $\eta(x, t)$ the elevation of the free surface in the disturbed state. Then one has the following linearised initial-boundary-value problem:

$$z < 0: \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{2.3}$$

$$z > 0: \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{z^2} = 0, \tag{2.4}$$

$$z = -h: \quad \frac{\partial\phi}{\partial z} = 0, \tag{2.5}$$

$$z = b: \quad \psi = 0, \tag{2.6}$$

$$z = 0: \quad \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial z}, \tag{2.7}$$

$$\psi + \eta\Psi_z = 0, \tag{2.8}$$

$$\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x} + g\eta + \frac{1}{4\pi}\Psi_z\psi_z = -\frac{p_0}{\varrho}\delta(x)e^{i\omega t}, \tag{2.9}$$

$$t = 0: \quad \phi = \psi = \eta = 0, \tag{2.10}$$

where g is the acceleration due to gravity. The relation (2.5) describes the condition of impenetrability of the fluid at the boundary $z = -h$. The relation (2.7) describes the kinematic condition on the velocity field at the interface. The relations (2.6) and (2.8) describe the constraint that the electrostatic potentials at the conducting surfaces are kept fixed, (alternately, one may keep the charge fixed at these surfaces, but this situation is not considered in this paper). The relation (2.9) describes the force balance at the interface.

Let us Fourier-transform the various quantities according to

$$\bar{q}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(x)e^{-ikx}dx \tag{2.11}$$

so that one obtains

$$\begin{aligned} \bar{\phi}(k, z, t) &= \bar{A}(k, t)\cdot \cosh k(z+h), \\ \bar{\psi}(k, z, t) &= \bar{B}(k, t)\cdot \frac{V_0}{b}\sinh k(z-b) \end{aligned} \tag{2.12}$$

and using Eqs. (2.12) in the relations (2.7), (2.8) and (2.9), one obtains

$$\frac{d\bar{\eta}}{dt}+ikU\bar{\eta}-k\bar{A}\sinh kh = 0, \tag{2.13}$$

$$\bar{B}\sinh kb = -\bar{\eta}, \tag{2.14}$$

$$\frac{V_0^2}{4\pi b^2}\bar{B}k\cosh kb+\left(\frac{d\bar{A}}{dt}+ikU\bar{A}\right)\cosh kh+g\bar{\eta} = -\frac{p_0}{\varrho\sqrt{2\pi}}e^{i\omega t}, \tag{2.15}$$

$$t=0: \quad \bar{\eta}=\bar{A}=\bar{B}=0 \tag{2.16}$$

from which there follows

$$\bar{\eta}(k,t) = \frac{p_0\sqrt{k\tanh kh}}{2\varrho\sqrt{2\pi\left(g-\frac{V_0^2 k}{4\pi b^2}\coth kb\right)}}\left[\frac{e^{i\omega t}-e^{im_1 t}}{\omega-m_1}-\frac{e^{i\omega t}-e^{im_2 t}}{\omega-m_2}\right], \tag{2.17}$$

where

$$m_1, m_2 = -kU\pm\sqrt{\left(gk-\frac{V_0^2 k}{4\pi b^2}\coth kb\right)\tanh kh}\,. \tag{2.18}$$

Upon inverting the Fourier integral, one obtains

$$\eta(x,t) = \frac{p_0}{4\pi\varrho}\int_{-\infty}^{\infty}\left[\frac{k\tanh kh}{g-\frac{V_0^2 k}{4\pi b^2}\coth kb}\right]^{1/2}\left[\frac{e^{i\omega t}-e^{im_1 t}}{\omega-m_1}-\frac{e^{i\omega t}-e^{im_2 t}}{\omega-m_2}\right]e^{ikx}dk. \tag{2.19}$$

3. Asymptotic development of the solutions

Let us rewrite Eq. (2.19) in the form

$$\eta(x,t) = \frac{p_0}{4\pi\varrho}(Ie^{i\omega t}-J), \tag{3.1}$$

where

$$I = \int_{-\infty}^{\infty}\left[\frac{k\tanh kh}{g-\frac{V_0^2 k}{4\pi b^2}\coth kb}\right]^{1/2}\left[\frac{1}{\omega-m_1}-\frac{1}{\omega-m_2}\right]e^{ikx}dk, \tag{3.2}$$

$$J = \int_{-\infty}^{\infty}\left[\frac{k\tanh kh}{g-\frac{V_0^2 k}{4\pi b^2}\coth kb}\right]^{1/2}\left[\frac{e^{im_1 t}}{\omega-m_1}-\frac{e^{im_2 t}}{\omega-m_2}\right]e^{ikx}dk. \tag{3.3}$$

The dominant contributions to I as $x \Rightarrow \infty$ come from the poles of the integrand, which are given by

$$\begin{aligned} \omega-m_1 &= \omega+kU-\sqrt{\left(gk-\frac{V_0^2 k^2}{4\pi b^2}\coth kb\right)\tanh kh} = 0,\\ \omega-m_2 &= \omega+kU+\sqrt{\left(gk-\frac{V_0^2 k^2}{4\pi b^2}\coth kb\right)\tanh kh} = 0. \end{aligned} \tag{3.4}$$

These points are shown in Figs. 2 and 3 as points of intersection of the curve $\sqrt{\left(gk-\frac{V_0^2k^2}{4\pi b^2}\coth kb\right)\tanh kh}$ and with the lines $(\omega+kU)$ and $-(\omega+kU)$, for different sets of values of U, ω, h, V_0 and b, (for comparison the curves for the hydrodynamic case are also shown in dashed lines).

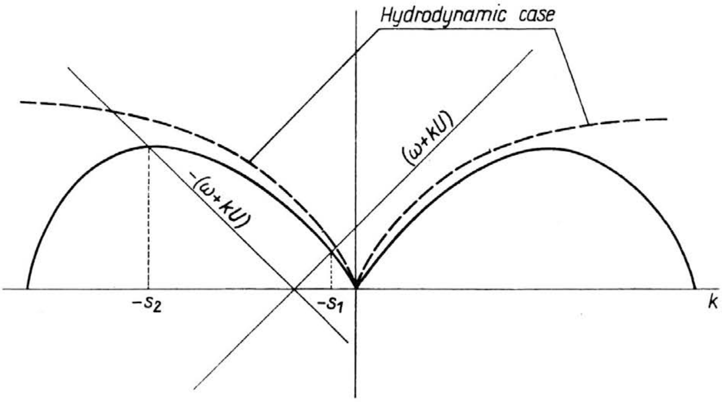

FIG. 2.

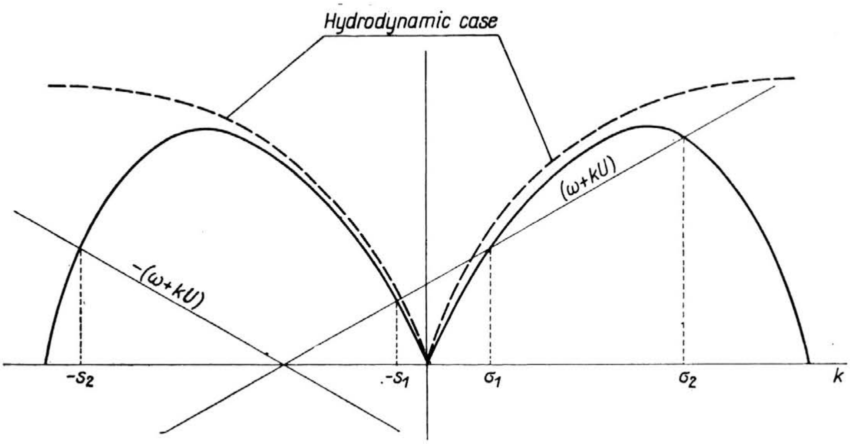

FIG. 3.

Corresponding to a situation represented in Fig. 2, Eqs. (3.4) each produce one pole at locations,

$$k = -s_1, \quad k = -s_2, \tag{3.5}$$

respectively. Corresponding to a situation represented in Fig. 3, in addition to the poles at locations given by Eq. (3.5), Eq. $(3.4)_1$ produces two more poles at locations

$$k = \sigma_1, \quad k = \sigma_2. \tag{3.6}$$

Corresponding to a certain value U_c (which will be calculated in Sect. 4) of U, σ_1 and σ_2 degenerate into a double pole, $\sigma_1 = \sigma_2 = \sigma$. Note then that the poles σ_1, σ_2 exist only if

(3.7) $$U < U_c.$$

It may seem from Eq. (3.2) that there are at least two poles produced by

$$g - \frac{V_0^2 k}{4\pi b^2} \coth kb = 0.$$

However, the contributions from these poles will be zero because inspection of the integrand in Eq. (3.2) shows that the corresponding residues vanish.

Using the theorem of residues (LIGHTHILL [4]), one then obtains

(3.8) $$I \sim \pi i \operatorname{sgn} x[\xi(-s_1)e^{-is_1 x} - \zeta(-s_2)e^{-is_2 x} + H(U_c - U)\{\xi(\sigma_1)e^{i\sigma_1 x} + \zeta(\sigma_2)e^{i\sigma_2 x}] + O\left(\frac{1}{|x|}\right),$$

where $H(x)$ is the Heaviside step function, and

(3.9)
$$\xi(x) \equiv -\left[\frac{x \tanh xh}{g - \frac{V_0^2 x}{4\pi b^2} \coth kb}\right]^{1/2} \Bigg/ \left(\frac{dm_1}{dk}\right)_{k=x},$$
$$\zeta(x) \equiv -\left[\frac{x \tanh xh}{g - \frac{V_0^2 x}{4\pi b^2} \coth kb}\right]^{1/2} \Bigg/ \left(\frac{dm_2}{dk}\right)_{k=x}.$$

The dominant contributions to J as $t \Rightarrow \infty$ come from the poles of the integrand in Eq. (3.3) at locations given by Eqs. (3.4), and the point where the phase of the integrand is stationary, which occurs at

(3.10)
$$\frac{d}{dk}\left(m_1 + \frac{kx}{t}\right) = 0,$$
$$\frac{d}{dk}\left(m_2 + \frac{kx}{t}\right) = 0.$$

Following DEBNATH and ROSENBLAT [1], let us consider the asymptotic limit $|x| \Rightarrow \infty$, $t \Rightarrow \infty$ such that $|x| \ll Ut$. Then Eqs. (3.10) become

(3.11)
$$\frac{d}{dk}\sqrt{\left(gk - \frac{V_0^2 k^2}{4\pi b^2}\coth kb\right)\tanh kh} = U,$$
$$\frac{d}{dk}\sqrt{\left(gk - \frac{V_0^2 k^2}{4\pi b^2}\coth kb\right)\tanh kh} = -U.$$

It is clear from Figs. 2 and 3 that Eqs. (3.11) each have at most one real root given by

(3.12) $$k = \sigma \quad \text{and} \quad k = -\varrho,$$

respectively. A necessary and sufficient condition for these roots to exist is seen from Figs. 2 and 3 to be

(3.13) $$U \leqslant \sqrt{gh}$$

(which is the same as the one for the hydrodynamic case found by DEBNATH and ROSENBLAT [1]), $\sqrt{gh}$ being the slope at the curve

$$\sqrt{\left(gk-\frac{V_0^2k^2}{4\pi b^2}\coth kb\right)\tanh kh} \quad \text{at} \quad k=0.$$

The contributions to J from the points of stationary phase at locations given by Eq. (3.12) are then

$$(3.14)\quad J_1 \sim H(\sqrt{gh}-U)\left[\left(-\frac{2\pi}{tm_1''(\sigma)}\right)^{1/2}\left\{\frac{\sigma\tanh\sigma h}{g-\frac{V_0^2\sigma}{4\pi b^2}\coth kb}\right\}^{1/2} \times\frac{e^{im_1(\sigma)t+i\sigma x-\frac{\pi i}{4}}}{\omega-m_1(\sigma)}-\left(-\frac{2\pi}{tm_2''(-\varrho)}\right)^{1/2} \times\left\{\frac{\varrho\tanh\varrho h}{g-\frac{V_0^2\varrho}{4\pi b^2}\coth\varrho b}\right\}^{1/2}\frac{e^{im_2(-\varrho)t-i\varrho x-\frac{\pi i}{4}}}{\omega-m_2(-\varrho)}\right]+O\left(\frac{1}{t}\right).$$

Note that the point of stationary phase at $k=\sigma$ necessarily exists if the poles σ_1, σ_2 given by Eq. (3.6) exist because Eq. (3.13) is automatically satisfied if Eq. (3.7) is true (U_c being less than $\sqrt{gh}$, see Sect. 4).

Next, in order to evaluate the contributions to J from the poles of the integrand in Eq. (3.3), one changes the variable of integration from k to $m_{1,2}=m_{1,2}(k)$, as in DEBNATH and ROSENBLAT [1], and uses the theorem of residues (LIGHTHILL [4]) as before to obtain

$$(3.15)\quad J_2 \sim \pi ie^{i\omega t}[-\xi(-s_1)e^{-is_1x}+\zeta(s_2)e^{-is_2x} +H(U_c-U)\{\xi(\sigma_1)e^{i\sigma_1x}-\xi(\sigma_2)e^{i\sigma_2x}\}]+O\left(\frac{1}{|x|}\right).$$

Note that Eq. (3.14) is a transient contribution while Eq. (3.15) is a steady-state contribution. Using Eqs. (3.8) and (3.15), one obtains from Eq. (3.1) for the ultimate steady state

(3.16)

$$\eta_s(x,t)\sim\begin{cases}-\dfrac{ip_0}{2\varrho}e^{i\omega t}[\xi(-s_1)e^{-is_1x}-\zeta(-s_2)e^{-is_2x}+H(U_c-U)\cdot\xi(\sigma_2)e^{i\sigma_2x}], & x>0,\\ -\dfrac{ip_0}{2\varrho}H(U_c-U)\cdot\xi(\sigma_1)e^{i\omega t+i\sigma_1x}, & x<0.\end{cases}$$

Equation (3.16) shows that

(i) if $U>U_c$, there are two waves propagating downstream of the origin with speeds ω/s_1 and ω/s_2 in the positive x-direction; whereas the former wave is slowed down, the latter wave moves faster in the presence of electrohydrodynamic effects;

(ii) if $U<U_c$, in addition to the above two waves, there are two waves — one moving with speed ω/σ_1 on the upstream side of the origin, and the other moving with speed

ω/σ_2 on the downstream side of the origin; whereas the former wave is slowed down, the latter wave moves faster in the presence of electrohydrodynamic effects;

(iii) for the case $\omega = 0$, one has

$$s_1 = s_2 = \sigma_1 = 0$$

and the wave system degenerates to only one wave moving on the downstream side with speed ω/σ_2 if $U < U_c$; the hydrodynamic counterpart of this wave is the one found by STOKER ([4] Ch. 7).

4. Determination of U_c

Corresponding to $U = U_c$, the poles σ_1, σ_2 degenerate into $k = \sigma_1 = \sigma_2 = \sigma$ so that $k = \sigma$ simultaneously satisfies Eqs. $(3.4)_1$ and $(3.11)_1$. Putting

$$\lambda = \sigma h, \tag{4.1}$$

equations $(3.4)_1$ and $(3.11)_1$ give

$$(gh)\left(\frac{\omega}{g}\right) + \lambda U_c - \sqrt{\left(gh - \frac{V_0^2 \lambda}{4\pi b^2} \coth \sigma b\right)} \sqrt{\lambda \tanh \lambda h} = 0, \tag{4.2}$$

$$\sqrt{gh} \left[\frac{1 - \frac{V_0^2 \sigma}{4\pi b^2 g} \coth \sigma b + \frac{V_0^2 \sigma^2}{4\pi b g} \operatorname{cosech}^2 \sigma b}{2\sqrt{\lambda \tanh \lambda h} \sqrt{1 - \frac{V_0^2 \sigma}{4\pi b^2 g} \coth \sigma b}} \tanh \lambda + \sqrt{\left(1 - \frac{V_0^2 \sigma}{4\pi b^2 g} \coth \sigma b\right)} \left(\frac{\lambda \operatorname{sech}^2 \lambda}{2\sqrt{\lambda \tanh \lambda}}\right) \right] = U_c. \tag{4.3}$$

Corresponding to the case with $h \Rightarrow \infty$, Eqs. (4.2) and (4.3) give

$$U_c = \frac{g}{4\omega} \left[\frac{\left(1 - \frac{V_0^2 \sigma^2}{4\pi b g} \operatorname{cosech}^2 \sigma b\right)}{\left(1 - \frac{V_0^2 \sigma}{4\pi b^2 g} \coth \sigma b\right)} \left(1 - \frac{V_0^2 \sigma}{2\pi b^2 g} \coth \sigma b + \frac{V_0^2 \sigma^2}{4\pi b} \operatorname{cosech}^2 \sigma b\right) \right]. \tag{4.4}$$

In the hydrodynamic limit ($V_0 \Rightarrow 0$), Eq. (4.4) reduces to the result derived by KAPLAN [2].

Corresponding to the case with $\omega = 0$, Eqs. (4.2) and (4.3) give

$$U_c = \sqrt{gh} \left[1 - \frac{V_0^2 \sigma}{4\pi b^2 gh} \coth \sigma b\right]^{1/2}. \tag{4.5}$$

In the hydrodynamic limit ($V_0 \Rightarrow 0$), Eq. (4.5) reduces to the result derived by STOKER ([4], Ch. 7). Equation (4.5) shows that for the critical stream velocity U_c for the generation of waves on the downstream side of the source which is a steady pressure acting on the free surface decreases in the presence of electrohydrodynamic effects.

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RESEARCH SCHOOL OF PHYSICAL SCIENCES
THE AUSTRALIAN NATIONAL UNIVERS TY, CANBERRA, AUSTRALIA.

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