Arch. Mech., 35, 1, pp. 17–36, Warszawa 1983

Integral representation and uniqueness theorem in generalized thermoelasticity (*)

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THE PAPER deals with the problem of harmonic oscillations according to the generalized linear theory of thermoelasticity, due to LORD and SHULMAN [1], in an exterior domain. Basing on the author's solution given in [2], the integral representation of a pair (ϕ, θ) and suitable radiation conditions are derived. The considerations of this part are expressed as Theorem 1. Next, the uniqueness of the integral representation is proved, Theorem 3. Further, it is shown that the radiation condition in a form coupling the function ϕ and θ at infinity fulfils the sufficient condition. Finally, certain particular exampſes are being worked out and some properties of thermal waves are discussed. It follows that the pure thermal wave is both dissipative and dispersive. In this case some quantitative data are given, (Figs. 3 and 4). At the end a few conclusions of a more general meaning are deduced.

W pracy rozważa się problem drgań harmonicznych w obszarze zewnętrznym wg uogólnionej zlinearyzowanej teorii termosprężystości prezentowanej przez LORDA i SHULMANA w [1]. Na podstawie rozwiązania autora przedstawionego w [2] wyprowadzono reprezentację całkową dla pary (ϕ, θ) wraz z odpowiednimi warunkami wypromieniowania. Tę część rozważań precyzuje Twierdzenie 1. Następnie dowiedziono jednoznaczności otrzymanej reprezentacji całkowej formułując Twierdzenie 3. Dalej wykazano, że warunki wypromieniowania sprzęgające funkcje (ϕ, θ) w nieskończoności spełniają warunek wystarczający. Rozpatrzono również niektóre przypadki szczególne, dyskutując pewne własności płaskich fal cieplnych. Wykazano, że fale czysto cieplne podlegają dysypacji i dyspersji. Dane ilościowe wpływu czasu relaksacji zilustrowano wykresami na rys. 3 i 4. W zakończeniu podano parę wniosków ogólniejszego charakteru.

В работе рассматривается проблема гармонических колебаний во внешней области согласно обобщенной линеаризованной теории термоупругости, представленной Лордом и Шульманом в [1]. На основе решения автора, представленного в [2], выведено интегральное представление для пары (ϕ, θ) совместно с соответствующими условиями излучения. Эту часть рассуждений уточняет Теорема 1. Затем доказывается единственности полученного интегрального представления, формулируя Теорему 3. Дальше показано, что условия излучения, сопрягающие функции (ϕ, θ) в бесконечности, удовлетворяют достаточному условию. Рассмотрены тоже некоторые частные случаи, обсуждая некоторые свойства плоских термических волн. Показано, что чисто термические волны подлежат диссипации и дисперсии. Количественные данные влияния времени релаксации иллюстрированы диаграммами на рис. 3 и 4. В заключении приведено несколько выводов более общего характера.

Introduction

THE PAPER develops the mathematical foundation required for the study of integral representations of thermoelastic harmonic waves in exterior domain with one relaxation time.

(*) The main paragraphs of this work, namely Theorem 1 on representation and Theorem 3 on uniqueness entitlet: "Radiation conditions and a uniqueness theorem in generalized linear thermoelasticity" were submitted to the XVth ICTAM, Toronto 1980, and presented in part at the Polish Solid Mechanics Conference in 1980 and earlier.

The linearized system of partial differential equations according to [1] and the method of analysis as well as the results presented in [2], based on the theory of singular integral equations and Green's identity, are ac o ted.

The text is divided into two chapters. Starting with the governing equations in the first chapter, we establish the integral representation of the functions (ϕ, θ), Theorem 1, where (ϕ, θ) denote the thermoelastic displacement potential and temperature, respectively, Sect. 1.

Next, a uniqueness theorem of the integral representation is formulated and the proof based on the energy concept is carried out, Theorem 3; Sect. 2. The remaining part of this chapter involves a proposal to extend the uniqueness theorem to a certain class of singular surfaces, Sect. 3, what may be especially useful in connection with the crack problems.

Further consideration of the properties of Theorem 1 and 3 is presented in Chapter 2. Thus it is shown that the radiation conditions coupling the functions ϕ, θ fulfil the sufficient condition of Sect. 4. Particular cases of integral representation are derived and discussed in Sect. 5. It is of some interest to note that if the interconvertibility of thermal and mechanical energy is ignored, the thermal wave in the body is found to be both dissipative and dispersive. In this case the relaxation time contribution to the phase velocity and dissipation decreases. The results obtained make it possible to draw conclusions of a more general character as shown in Sect. 6.

The integral representation for generalized thermoelasticity and some aspects of discussion given here bring about a new development. This field theory of integral representations in exterior domain for the Laplace and Helmholtz equations well-known in the potential theory (cf. [3, 6]) and in the theory of elasticity [2, 3, 4, 5, 6, 8, 9], is considerably developed.

Chapter 1

1. Integral representation in generalized thermoelasticity

We assume that an infinite thermoelastic ([1]) body $B_e = E - B_i$ (E is the entire space) has a cavity B_i bounded by a regular surface S with the normal vector from B_i to B_e.

Let $x \in B_e$ and S_r be the spherical surface of a sufficiently large radius r about x containing within it the region B_i. Moreover, we assume that the motion is provoked by a periodic function of time, and once the process is established, the motion of the body is also periodic, with the same frequency. We seek a solution describing harmonic thermoelastic vibration of the body able to propagate the second sound throughout the domain V_r bounded by surface S and S_r. Next, we pass with the radius to infinity and derive the radiation conditions and a regular form of integral representation for our problem.

([1]) A thermoelastic medium is the domain D of the three-dimensional Euclidean space and a set of quantities $\varrho, \lambda, \mu, \gamma, \eta, \varkappa$ satisfying the conditions $\varrho > 0$, $\mu > 0$, $3\lambda + 2\mu > 0$, $\gamma/\eta > 0$, $\varkappa > 0$ (cf. p. 51, [3]).

The linearized system of the generalized theory of thermoelasticity, according to LORD and SHULMAN given by Eqs. (23) and (24) in [1], is employed. The principal model used is seen to be exhibiting a finite heat transport velocity based on a relaxation time τ_0. The energy equation for the body with constant thermal and elastic coefficients may be rewritten as

$$kT_{,ii} = \varrho C_E(\dot{T}+\tau_0\ddot{T})+(3\lambda+2\mu)\alpha T_0(\dot{\epsilon}_{kk}+\tau_0\ddot{\epsilon}_{kk}), \tag{1.1}$$

and the equation of motion of continuous medium without body forces is

$$\varrho\ddot{u}_i = (\lambda+\mu)u_{j,ij}+\mu u_{i,jj}-(3\lambda+2\mu)\alpha T_{,i}. \tag{1.2}$$

Functions u, T, ϵ_{ij} are the displacement vector, absolute temperature and strain tensor, whereas λ and μ, ϱ, α, k, τ_0, T_0 are the Lamés moduli, density, coefficient of thermal expansion, conductivity, the relaxation time and fixed reference temperature, respectively. Finally $C_E = -T\dfrac{\partial^2\Phi}{\partial T^2}$ denotes the specific heat at constant deformation where Φ is Helmholtz free energy defined by the relation

$$\Phi(\epsilon_{ij}, T) = \xi(\epsilon_{ij}, T)-T\zeta(\epsilon_{ij}, T).$$

Here ξ, ζ are known as internal energy density and entropy. The subscripts specify the components of a vector in the directions of the coordinate axes and the summation convention is adopted. We employ a comma to denote partial differentiation with respect to a spatial coordinate while the superposed dot implies partial differentiation with respect to time.

If we look for the solution (ϕ, θ) of Eqs. (1.1) and (1.2) in the form (we are restricted to the coupled dilatation waves since the remaining deformation is only elastic)

$$u = \operatorname{grad}\hat{\phi}, \qquad \hat{\theta} = T-T_0 \text{ (small enough)} \tag{1.3}$$

and

$$\hat{\phi}(x, t) = \phi(x)e^{-i\omega t}, \qquad \hat{\theta}(x, t) = \theta(x)e^{-i\omega t}, \qquad x = x_1, x_2, x_3, \tag{1.4}$$

then Eqs. (1.1) and (1.2) are satisfied provided

$$\Box_1^2\phi-m\theta = 0, \qquad \Box_3^2\theta+\frac{\varepsilon}{m}\hat{h}_3^2\nabla^2\phi = 0, \tag{1.5}$$

where

$$\varrho\frac{C_E}{k} = \frac{1}{\varkappa}, \qquad \frac{\gamma T_0}{k} = \eta, \qquad \gamma = (3\lambda+2\mu)\alpha,$$

$$c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \qquad m = \frac{\gamma}{c_1^2\varrho},$$

$$\Box_1^2 = \nabla^2+h_1^2, \qquad h_1 = \frac{\omega}{c_1} \tag{1.6}$$

$$\Box_3^2 = \nabla^2+\hat{h}_3^2, \qquad \hat{h}_3 = h_3(1-i\omega\tau_0)^{1/2}, \qquad h_3 = \left(\frac{i\omega}{\varkappa}\right)^{1/2},$$

$$\varepsilon = \eta m\varkappa, \qquad \nabla^2\text{—Laplace operator, } \omega\text{—frequency.}$$

Equations (1.5) imply that the amplitudes of oscillations $\phi(x)$ and $\theta(x)$ satisfy two separate equations:

$$\hat{\Box}^2_{\hat{k}_1}\hat{\Box}^2_{\hat{k}_3}\phi = 0, \quad \hat{\Box}^2_{\hat{k}_1}\hat{\Box}^2_{\hat{k}_3}\theta = 0, \quad x \in B_e, \tag{1.7}$$

where

$$\hat{\Box}^2_{\hat{k}_r} = \nabla^2 + \hat{k}_r^2, \quad r = 1, 3. \tag{1.8}$$

The values k_r are selected in a such a way that the following relations are valid:

$$\hat{k}_1^2 + \hat{k}_3^2 = h_1^2 + (1+\varepsilon)\hat{h}_3^2, \quad \hat{k}_1^2\hat{k}_3^2 = h_1^2\hat{h}_3^2. \tag{1.9}$$

Thus they are both the roots of the equation

$$\hat{k}^4 - [h_1^2 + (1+\varepsilon)\hat{h}_3^2]\hat{k}^2 + h_1^2\hat{h}_3^2 = 0. \tag{1.10}$$

Hence, according to the expression (1.4), we have

$$\hat{k}_r = \hat{\alpha}_r + i\hat{\beta}_r, \quad r = 1, 3, \quad \hat{\alpha}_r > 0, \quad \hat{\beta}_r \geqslant 0. \tag{1.11}$$

The explicit formulae for $\hat{\alpha}_r$ and $\hat{\beta}_r$ may be derived but here they are reduced to be the following functions of the material parameters

$$\begin{aligned} \hat{\alpha}_r &= \hat{\alpha}_r(c_1, \varkappa, \varepsilon, \tau_0, \omega), \\ \hat{\beta}_r &= \hat{\beta}_r(c_1, \varkappa, \varepsilon, \tau_0, \omega). \end{aligned} \tag{1.12}$$

Now we wish to find regular solutions of Eq. (1.7) in B_e for the pair (ϕ, θ) by means of surface integrals over S. To do this, we introduce the auxiliary function $\bar{\phi}$ which is established as a fundamental solution of our problem in a whole space E; this means that it is the solution of the equation

$$\hat{\Box}^2_{\hat{k}_1}\hat{\Box}^2_{\hat{k}_3}\bar{\phi}(x|\xi) = -m\delta(x|\xi), \quad x, \xi \in E, \tag{1.13}$$

where $\delta(x)$ is Dirac's function. The solution of Eq. (1.13) is known and may be written in our notations as

$$\bar{\phi}(x|\xi) = -\frac{m}{4\pi(\hat{k}_1^2 - \hat{k}_3^2)}\frac{1}{r}(e^{i\hat{k}_1 r} - e^{i\hat{k}_3 r}), \quad x, \xi \in E, \tag{1.14}$$

where $r^2 = (x_j - \xi_j)(x_j - \xi_j)$, $j = 1, 2, 3$, is the distance between a point x in which the potential is to be determined and a source located at ξ.

A direct calculation shows that $\bar{\phi}$ describes the thermoelastic displacement potential due to a concentrated source of intensity $\varkappa$ acting at ξ and fulfills the radiation condition in exterior domain.

The temperature field coupled with the thermoelastic displacement potential $\bar{\phi}$ denoted as $\bar{\theta} = \bar{\theta}(x|\xi)$ is derived as follows:

$$\bar{\theta}(x|\xi) = \frac{1}{m}\Box_1^2\bar{\phi} = -\frac{1}{4\pi(\hat{k}_1^2 - \hat{k}_3^2)}\frac{1}{r}[(h_1^2 - \hat{k}_1^2)e^{i\hat{k}_1 r} - (h_1^2 - \hat{k}_3^2)e^{i\hat{k}_3 r}]. \tag{1.15}$$

We return to Eq. (1.7) and state that ϕ is a solution of the equation

$$\hat{\Box}^2_{\hat{k}_1}\hat{\Box}^2_{\hat{k}_3}\phi = 0, \quad x \in B_e \tag{1.16}$$

under the assumption that ϕ is not singular in the domain B_e. If we multiply Eq. (1.13) by ϕ and Eq. (1.16) by $\overline{\phi}$, add them up and integrate over a region V_r bounded internally by S and externally by a large sphere S_r of radius r, then we find, that

$$\chi(x)\phi(x) = \frac{1}{m}\int\limits_{V_r} dV(\xi)[\overline{\phi}(x|\xi)\hat{\Box}^2_{\hat{k}_1}\hat{\Box}^2_{\hat{k}_3}\phi(\xi)-\phi(\xi)\hat{\Box}^2_{\hat{k}_1}\hat{\Box}^2_{\hat{k}_3}\overline{\phi}(x|\xi)], \tag{1.17}$$

where

$$\chi(x) = \begin{cases} 1, & x\in V_r, \\ 1/2, & x\in S, \\ 0, & x\notin V_r. \end{cases}$$

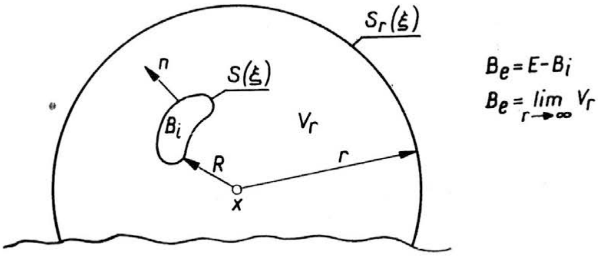

FIG. 1. Symbols commonly used.

Using at first an operator identity for two sufficiently regular functions u and v in the domain of integration

$$v\nabla^4 u - u\nabla^4 v = \operatorname{div} I(u, v),$$

where

$$I(u, v) = v\operatorname{grad}\nabla^2 u - u\operatorname{grad}\nabla^2 v + (\nabla^2 v)\operatorname{grad} u - (\nabla^2 u)\operatorname{grad} v,$$

as well as the divergence theorem, we can find that

$$\chi(x)\phi(x) = \frac{1}{m}\int\limits_{S+S_r} dS(\xi)\left\{\left[\overline{\phi}(x|\xi)\frac{\partial}{\partial n}\hat{\Box}^2\phi(\xi)-\phi(\xi)\frac{\partial}{\partial n}\hat{\Box}^2\overline{\phi}(x|\xi)\right]\right. \\ \left.+\left[(\nabla^2\overline{\phi}(x|\xi))\frac{\partial}{\partial n}\phi(\xi)-(\nabla^2\phi)\frac{\partial}{\partial n}\overline{\phi}(x|\xi)\right]\right\}, \tag{1.18}$$

where

$$\hat{\Box}^2 = \nabla^2 + \hat{k}_1^2 + \hat{k}_3^2.$$

With regard to the relation $(1.5)_1$ between the functions ϕ and θ, Eq. (1.18) is transformed as follows:

$$\chi(x)\phi(x) = \int\limits_{S+S_r} dS\left\{\overline{\phi}\frac{\partial\theta}{\partial n} - \theta\frac{\partial\overline{\phi}}{\partial n} + \frac{1}{m}\left[(\hat{\Box}^2_{\hat{k}}\overline{\phi})\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}(\hat{\Box}^2_{\hat{k}}\overline{\phi})\right]\right\}, \tag{1.19}$$

where

$$'\hat{\Box}^2_{\hat{k}} = \nabla^2 + \hat{k}_1^2 + \hat{k}_3^2 - h_1^2.$$

A more conventional form of Eq. (1.18) is given below:

$$(1.19')\qquad \chi(x)\phi(x) = \int\limits_{S+S_r} dS\Big\{\Big[\bar{\phi}\frac{\partial}{\partial n}\theta - \phi\frac{\partial}{\partial n}\bar{\theta}\Big] + \Big[\bar{\theta}\frac{\partial}{\partial n}\phi - \theta\frac{\partial}{\partial n}\bar{\phi}\Big] + \frac{1}{m}(\hat{k}_1^2+\hat{k}_3^2-2h_1^2)\Big[\bar{\phi}\frac{\partial}{\partial n}\phi - \phi\frac{\partial}{\partial n}\bar{\phi}\Big]\Big\}.$$

We may evaluate the function θ proceeding in a similar way as for ϕ, Eqs. (1.16)–(1.19), but it is simpler to make use of the relation $(1.5)_1$ and apply the operator $\frac{1}{m}\Box_1^2$ to both sides of Eq. (1.19'), to obtain

$$(1.20)\qquad \chi(x)\theta(x) = \int\limits_{S+S_r} dS\Big\{\Big[\bar{\theta}(x|\xi)\frac{\partial}{\partial n}\theta(\xi) - \theta(\xi)\frac{\partial}{\partial n}\bar{\theta}(x|\xi)\Big] + \frac{\varrho\omega^2}{\hat{a}}\Big[\bar{\phi}(x|\xi)\frac{\partial}{\partial n}\phi(\xi) - \phi(\xi)\frac{\partial}{\partial n}\bar{\phi}(x|\xi)\Big]\Big\},$$

where the relation $(1.9)_1$, and the following are applied:

$$\frac{\varrho\omega^2}{\hat{a}} = \frac{\varepsilon}{m^2}\hat{h}_3^2 h_1^2,$$

and

$$\hat{a} = \frac{m\gamma}{\varepsilon\hat{h}_3^2}.$$

Inserting explicit formulae for $\bar{\phi}$, Eq. (1.14), and $\bar{\theta}$, Eq. (1.15), into Eqs. (1.19') and (1.20), one gets for the pair (ϕ, θ)

$$(1.21)\qquad \chi(x)\phi(x) = -\frac{m}{4\pi(\hat{k}_1^2-\hat{k}_3^2)}\Big[\int\limits_{S+S_r} dS\Big\{\frac{r^{i\hat{k}_1 r}}{r}\frac{\partial}{\partial n}\Big[\theta - \frac{1}{m}(h_1^2-\hat{k}_3^2)\phi\Big] - \Big[\theta - \frac{1}{m}(h_1^2-\hat{k}_3^2)\phi\Big]\frac{\partial}{\partial n}\frac{e^{i\hat{k}_1 r}}{r}\Big\} - \int\limits_{S+S_r} dS\Big\{\frac{e^{i\hat{k}_3 r}}{r}\frac{\partial}{\partial n}\Big[\theta - \frac{1}{m}(h_1^2-\hat{k}_1^2)\phi\Big] - \Big[\theta - \frac{1}{m}(h_1^2-\hat{k}_1^2)\phi\Big]\frac{\partial}{\partial n}\frac{e^{i\hat{k}_3 r}}{r}\Big\}\Big],$$

$$(1.22)\qquad \chi(x)\theta(x) = -\frac{m}{4\pi(\hat{k}_1^2-\hat{k}_3^2)}\Big[\int\limits_{S+S_r} dS\Big\{\frac{e^{i\hat{k}_1 r}}{r}\frac{\partial}{\partial n}\Big[\frac{1}{m}(h_1^2-\hat{k}_1^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi\Big] - \Big[\frac{1}{m}(h_1^2-\hat{k}_1^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi\Big]\frac{\partial}{\partial n}\frac{e^{i\hat{k}_1 r}}{r}\Big\} - \int\limits_{S+S_r} dS\Big\{\frac{e^{i\hat{k}_3 r}}{r}\frac{\partial}{\partial n}\Big[\frac{1}{m}(h_1^2-\hat{k}_3^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi\Big] - \Big[\frac{1}{m}(h_1^2-\hat{k}_3^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi\Big]\frac{\partial}{\partial n}\frac{e^{i\hat{k}_3 r}}{r}\Big\}\Big].$$

We recall Eq. (1.11) and analyze the integral over surface S_r occurring in Eq. (1.21). To do this, we write

$$\int_{S_r} (\cdot) = \int_{S_r} \frac{dS}{r} e^{-\hat{\beta}_1 r} e^{i\hat{\alpha}_1 r} \left(\frac{\partial}{\partial r} - i\hat{k}_1\right)\left[\theta - \frac{1}{m}(h_1^2 - \hat{k}_3^2)\phi\right]$$

$$- e^{-\hat{\beta}_3 r} e^{i\hat{\alpha}_3 r}\left(\frac{\partial}{\partial r} - i\hat{k}_3\right)\left[\theta - \frac{1}{m}(h_1^2 - \hat{k}_1^2)\phi\right]\Bigg\}$$

$$+ \int_{S_r} \frac{dS}{r^2}\left\{e^{-\hat{\beta}_1 r} e^{i\hat{\alpha}_1 r}\left[\theta - \frac{1}{m}(h_1^2 - \hat{k}_3^2)\right] - e^{-\hat{\beta}_3 r} e^{i\hat{\alpha}_3 r}\left[\theta - \frac{1}{m}(h_1^2 - \hat{k}_1^2)\phi\right]\right\},$$

(1.23)

$$dS = r^2 \sin\tilde{\theta}\, d\tilde{\theta}\, d\tilde{\varphi}, \quad 0 \leqslant \tilde{\theta} \leqslant \pi, \quad 0 \leqslant \tilde{\varphi} \leqslant 2\pi.$$

In view of the process at infinity, the condition for the integral (1.23) to vanish at $r \to \infty$ can be satisfied if

(1.24)

$$re^{-\hat{\beta}_1 r}\left(\frac{\partial}{\partial r} - i\hat{k}_1\right)\left[\theta - \frac{1}{m}(h_1^2 - \hat{k}_3^2)\phi\right] = o(1), \text{[2]}$$

$$re^{-\hat{\beta}_3 r}\left(\frac{\partial}{\partial r} - i\hat{k}_3\right)\left[\theta - \frac{1}{m}(h_1^2 - \hat{k}_1^2)\phi\right] = o(1),$$

$$e^{-\hat{\beta}_1 r}\left[\theta - \frac{1}{m}(h_1^2 - \hat{k}_3^2)\phi\right] = o(1),$$

$$e^{-\hat{\beta}_3 r}\left[\theta - \frac{1}{m}(h_1^2 - \hat{k}_1^2)\phi\right] = o(1).$$

Similarly, analyzing the integral over the sphere S_r in Eq. (1.22), we find that it disappears at $r \to \infty$ when

(1.25)

$$re^{-\hat{\beta}_1 r}\left(\frac{\partial}{\partial r} - i\hat{k}_1\right)\left[\frac{1}{m}(h_1^2 - \hat{k}_1^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi\right] = o(1),$$

$$re^{-\hat{\beta}_3 r}\left(\frac{\partial}{\partial r} - i\hat{k}_3\right)\left[\frac{1}{m}(h_1^2 - \hat{k}_3^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi\right] = o(1),$$

$$e^{-\hat{\beta}_1 r}\left[\frac{1}{m}(h_1^2 - \hat{k}_1^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi\right] = o(1),$$

$$e^{-\hat{\beta}_3 r}\left[\frac{1}{m}(h_1^2 - \hat{k}_3^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi\right] = o(1).$$

Making use of the fact that

$$-\frac{1}{m^2}(h_1^2 - \hat{k}_1^2)(h_1^2 - \hat{k}_3^2) = \frac{\varrho\omega^2}{\hat{a}} \neq 0,$$

one can show that the asymptotic conditions (1.24) are equivalent to the conditions (1.25).

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(2) $f(r) = o(1)$ at $r \to \infty \Leftrightarrow \lim_{r\to\infty} f(r) = 0,$

$f(r) = 0(1)$ at $r \to \infty \Leftrightarrow \lim_{r\to\infty} f(r) = M < \infty.$

Thus, if both Eqs. (1.24) and (1.25) hold, then Eqs. (1.19) or (1.19′) and (1.20) yield the representations for ϕ and θ in the entire region B_e by means of the surface integrals over the surface S. The above analysis provides us with a proof of the following theorem:

THEOREM 1. *Every solution* (ϕ, θ) *of equations*

$$\hat{\Box}^2_{\hat{k}_1}\hat{\Box}^2_{\hat{k}_3}\phi = 0, \quad \Box^2_1\phi = m\theta \tag{1.26}$$

in the domain B_e, *which satisfies suitable radiation condtions* (1.24) *and* (1.25) *admits the representation of the form*

$$\chi(x)\phi(x) = -\frac{m}{4\pi(\hat{k}_1^2-\hat{k}_3^2)}\Bigg[\int_S dS\left\{\frac{e^{i\hat{k}_1R}}{R}\frac{\partial}{\partial n}\left[\theta-\frac{1}{m}(h_1^2-\hat{k}_3^2)\phi\right] - \left[\theta-\frac{1}{m}(h_1^2-\hat{k}_3^2)\phi\right]\frac{\partial}{\partial n}\frac{e^{i\hat{k}_1R}}{R}\right\} - \int_S dS\left\{\frac{e^{i\hat{k}_3R}}{R}\frac{\partial}{\partial n}\left[\theta-\frac{1}{m}(h_1^2-\hat{k}_1^2)\phi\right]-\left[\theta-\frac{1}{m}(h_1^2-\hat{k}_1^2)\phi\right]\frac{\partial}{\partial n}\frac{e^{i\hat{k}_3R}}{R}\right\}\Bigg], \tag{1.27}$$

$$\chi(x)\theta(x) = -\frac{m}{4\pi(\hat{k}_1^2-\hat{k}_3^2)}\Bigg[\int_S dS\left\{\frac{e^{i\hat{k}_1R}}{R}\frac{\partial}{\partial n}\left[\frac{1}{m}(h_1^2-\hat{k}_1^2)\theta+\frac{\varrho\omega^2}{\hat{a}}\phi\right] - \left[\frac{1}{m}(h_1^2-\hat{k}_1^2)\theta+\frac{\varrho\omega^2}{\hat{a}}\phi\right]\frac{\partial}{\partial n}\frac{e^{i\hat{k}_1R}}{R}\right\} - \int_S dS\left\{\frac{e^{i\hat{k}_3R}}{R}\frac{\partial}{\partial n}\left[\frac{1}{m}(h_1^2-\hat{k}_3^2)\theta+\frac{\varrho\omega^2}{\hat{a}}\phi\right]-\left[\frac{1}{m}(h_1^2-\hat{k}_3^2)\theta+\frac{\varrho\omega^2}{\hat{a}}\phi\right]\frac{\partial}{\partial n}\frac{e^{i\hat{k}_3R}}{R}\right\}\Bigg], \tag{1.28}$$

where $R = |x-\xi|$ *is the distance between points* $x \in B_e$, $\xi \in S$ *and* S *is the surface of cavity.*

2. The uniqueness theorem

We now wish to show that the solution in the form of integral representation (1.27) and (1.28) given in Theorem 1 is unique. The classical method will be used to prove the auxiliary Theorem 2 and then, on the basis of a corollary and Notes 1 and 2, the theorem of uniqueness, Theorem 3, is formulated.

THEOREM 2. *The solution* (ϕ, θ), $(\phi = \phi(x, \omega), \theta = \theta(x, \omega)) \in C^2$ *in* B_e *and* $(\phi(x, \omega), \theta(x, \omega)) \in C'$ *in* $\overline{B}_e = S \cup B_e$, *satisfying the equation system*

$$\Box^2_1\phi - m\theta = 0, \quad \Box^2_3\theta + \frac{\varepsilon}{m}\hat{h}_3^2\nabla^2\phi = 0 \quad \text{in} \quad B_e \tag{2.1}$$

and the homogeneous boundary conditions

$$\left.\frac{\partial\phi}{\partial n}\right|_S = 0, \quad \theta|_S = 0 \tag{2.2}$$

as well as asymptotic relations (radiation conditions) at $r \to \infty$

$$\begin{aligned} \phi &\sim f_1(\tilde{\theta}, \tilde{\varphi}) r^m e^{-(\hat{\beta}_1-\beta_0)r} + f_2(\tilde{\theta}, \tilde{\varphi}) r^m e^{-(\hat{\beta}_3-\beta_0)r}, \\ \theta &\sim g_1(\tilde{\theta}, \tilde{\varphi}) r^m e^{-(\hat{\beta}_1-\beta_0)r} + g_2(\tilde{\theta}, \tilde{\varphi}) r^m e^{-(\hat{\beta}_3-\beta_0)r}, \end{aligned} \tag{2.3}$$

where

$$f_1 = O(r^{-m}), \quad f_2 = O(r^{-m}), \quad g_1 = O(r^{-m}), \quad g_2 = O(r^{-m}) \tag{2.4}$$

and $\beta_0 = \min\ (\hat{\beta}_1, \beta_3)$, m *— finite integer number* ($f \cdot e$ $m = -1$ *for spherical cavity*) *is identically equal to zero in* $\bar{B}_e$,

$$\phi \equiv \theta \equiv 0 \quad in\ \bar{B}_e$$

(cf. [3], p. 156, Theorem 5.6).

Proof. Let us apply the gradient operation ∇ to Eq. $(2.1)_1$. Then we can obtain

$$\begin{aligned} (\nabla^2 + h_1^2)\nabla\phi - m\nabla\theta &= 0, \\ (\nabla^2 + \hat{h}_3)\theta + \frac{\varepsilon}{m}\hat{h}_3^2\nabla^2\phi &= 0. \end{aligned} \tag{2.5}$$

The inner product of Eq. $(2.5)_1$ by $\nabla\bar{\phi}$ where the bar over the symbol denotes its complex conjugate, yields the equation

$$\nabla \cdot (\nabla^2\phi\nabla\bar{\phi}) - \nabla^2\bar{\phi}\nabla\phi + h_1^2\nabla\bar{\phi}\nabla\phi - m\nabla\bar{\phi}\nabla\theta = 0. \tag{2.6}$$

Integration of both sides of Eq. (2.6) over the space V_r and application of the divergence theorem leads to

$$\int_{S+S_r} dS(\nabla^2\phi)\nabla\bar{\phi} \cdot n - \int_{V_r} dV|\nabla^2\phi|^2 + h_1^2 \int_{V_r} dV|\nabla\phi|^2 - m \int_{V_r} dV(\nabla\bar{\phi} \cdot \nabla\theta) = 0. \tag{2.7}$$

Taking the conjugate of the equation, we have

$$\int_{S+S_r} dS(\nabla^2\bar{\phi})\nabla\phi \cdot n - \int_{V_r} dV|\nabla^2\phi| + h_1^2 \int_{V_r} dV|\nabla\phi|^2 - m \int_{V_r} dV(\nabla\phi \cdot \nabla\bar{\theta}) = 0. \tag{2.8}$$

The next step is to multiply Eq. $(2.5)_2$ by $\bar{\theta}$ and to use the identities

$$\begin{aligned} \bar{\theta}\nabla^2\theta &= \nabla \cdot (\bar{\theta}\nabla\theta) - \nabla\bar{\theta} \cdot \nabla\theta, \\ \bar{\theta}\nabla^2\phi &= \nabla(\bar{\theta}\nabla\phi) - \nabla\bar{\theta} \cdot \nabla\phi. \end{aligned}$$

Hence we obtain

$$\nabla \cdot (\bar{\theta}\nabla\theta) - \nabla\bar{\theta} \cdot \nabla\theta + \hat{h}_3^2\bar{\theta}\theta + \frac{\varepsilon}{m}\hat{h}_3^2[\nabla \cdot (\bar{\theta}\nabla\phi) - \nabla\bar{\theta} \cdot \nabla\phi] = 0. \tag{2.9}$$

The equation is integrated over a space V_r and the divergence theorem is used to derive

$$\int_{S+S_r} dS\bar{\theta}\nabla\theta \cdot n - \int_{V_r} dV[|\nabla\theta|^2 - \hat{h}_3^2|\theta|^2] + \frac{\varepsilon}{m}\hat{h}_3^2\left[\int_{S+S_r} dS\bar{\theta}\nabla\phi \cdot n - \int_{V_r} dV\nabla\bar{\theta} \cdot \nabla\phi\right] = 0. \tag{2.10}$$

Corresponding substitution of Eq. (2.8) into Eq. (2.10) and then multiplication by $\frac{m^2}{\varepsilon\hat{h}_3^2}$ leads to the expression

$$(2.11)\qquad \int_{V_r} dV\left\{-h_1^2|\nabla\phi|^2+|\nabla^2\phi|^2+\frac{m^2}{\varepsilon}|\theta|^2-\frac{m^2}{\varepsilon\hat{h}_3^2}|\nabla\theta|^2\right\}$$
$$=\int_{S+S_r} dS\left\{(\nabla^2\bar{\phi})\frac{\partial\phi}{\partial n}-m\bar{\theta}\frac{\partial\phi}{\partial n}-\frac{m^2}{\varepsilon\hat{h}_3^2}\bar{\theta}\frac{\partial\theta}{\partial n}\right\}.$$

If we write the conjugate to Eq. (2.11) and substract the latter from Eq. (2.11), and next use the relation $(2.1)_1$ and the conjugate to it, then we may derive

$$(2.12)\qquad \frac{m^2}{\varepsilon}\left(\frac{1}{\hat{h}_3^2}-\frac{1}{\bar{\hat{h}}_3^2}\right)\int_{V_r} dV|\nabla\theta|^2=\int_{S+S_r} dS\left[h_1^2\left(\bar{\phi}\frac{\partial\phi}{\partial n}-\phi\frac{\partial\bar{\phi}}{\partial n}\right)\right.$$
$$\left.+\frac{m^2}{\varepsilon}\left(\frac{1}{\hat{h}_3^2}\bar{\theta}\frac{\partial\theta}{\partial n}-\frac{1}{\bar{\hat{h}}_3^2}\theta\frac{\partial\bar{\theta}}{\partial n}\right)\right].$$

Let us now discuss the right hand side of Eq. (2.12).

At first we see that the integral over S vanishes because of the homogeneous boundary conditions (2.2).

The remaining part of Eq. (2.12) with r tending to infinity takes the form

$$(2.13)\qquad \lim_{r\to\infty}\int_{S_r} dS(\cdot)=\lim_{r\to\infty}\int_{S_r} dS\left[h_1^2\left(\bar{\phi}\frac{\partial\phi}{\partial n}-\phi\frac{\partial\bar{\phi}}{\partial n}\right)\right.$$
$$\left.+\frac{m^2}{\varepsilon}\left(\frac{1}{\hat{h}_3^2}\bar{\theta}\frac{\partial\theta}{\partial n}-\frac{1}{\bar{\hat{h}}_3^2}\theta\frac{\partial\bar{\theta}}{\partial n}\right)\right].$$

When the asymptotic relations given in Eq. (2.3) are substituted into Eq. (2.13), and some terms are reduced and rearranged, it will be found that

$$(2.14)\qquad \lim_{r\to\infty}\int_{S_r} dS r^{2m}\Big\{\exp[-(\hat{\beta}_1+\hat{\beta}_3-2\beta_0)r]h_1^2(\hat{\beta}_1-\hat{\beta}_3)(\bar{f}_1f_3-f_1\bar{f}_3)$$
$$+\exp[-2(\beta_1-\beta_0)r]\left(\frac{1}{\hat{h}_3^2}-\frac{1}{\bar{\hat{h}}_3^2}\right)\frac{m^2}{\varepsilon}\left(|g_1|^2\left[\frac{1}{r}-(\hat{\beta}_1-\hat{\beta}_0)\right]+|g_3|^2\left[\frac{1}{r}-(\hat{\beta}_3-\beta_0)\right]\right)$$
$$+\exp[-(\hat{\beta}_1-\hat{\beta}_3-2\beta_0)r]\frac{m^2}{\varepsilon}\left[g_1\bar{g}_3\left(\frac{1}{\hat{h}_3^2}\left[\frac{1}{r}-(\hat{\beta}_1-\beta_0)\right]-\frac{1}{\bar{\hat{h}}_3^2}\left[\frac{1}{r}-(\hat{\beta}_3-\beta_0)\right]\right)\right.$$
$$\left.\left.+\bar{g}_1g_3\left(\frac{1}{\hat{h}_3^2}\left[\frac{1}{r}-(\hat{\beta}_3-\beta_0)\right]-\frac{1}{\bar{\hat{h}}_3^2}\left[\frac{1}{r}-(\hat{\beta}_1-\beta_0)\right]\right)\right]\right\}.$$

The integrand in Eq. (2.14) possesses the terms whose components f_i, g_i, $i = 1, 2$ are bounded in the region, whereas the polynomial r^m has a finite power index. They are both multiplied by exponential functions with negative exponents depending on r. Such expressions tend with $r\to\infty$ to zero.

Thus we have

$$\lim_{r\to\infty}\int_{S_r} dS(\cdot) = 0$$

and passing in Eq. (2.12) to the limit with $r \to \infty$ we obtain

$$\frac{m^2}{\varepsilon}\left(\frac{1}{\hat{h}_3^2} - \frac{1}{\bar{\hat{h}}_3^2}\right)\lim_{r\to\infty}\int_{V_r} dV|\nabla\theta|^2 = 0.$$

Since

$$\frac{1}{\hat{h}_3^2} - \frac{1}{\bar{\hat{h}}_3^2} \neq 0,$$

we conclude that

$$\int_{B_e} dV|\nabla\theta|^2 = 0, \quad (\lim_{r\to\infty} V_r = B_e),$$

thus $\theta = \text{const}$ in B_e.

The boundary condition $(2.2)_2$ and the asymptotic condition assumed in infinity imply that

$$\theta \equiv 0 \quad \text{in} \quad \bar{B}_e. \tag{2.15}$$

Substitution of Eq. (2.15) into Eqs. (2.1) leads to

$$\Box_1^2\phi = 0 \quad \text{and} \quad \nabla^2\phi = 0 \quad \text{in } \bar{B}_e.$$

Hence it follows immediately (h_1^2 being real) that

$$\phi \equiv 0 \quad \text{in } \bar{B}_e, \tag{2.16}$$

what completes the proof.

Theorem 2 gives rise to the following

COROLLARY. By virtue of the pair $(\phi, \theta) \in C^1$ in $\bar{B}_e$ and the identities (2.15) and (2.16), it follows that the normal derivatives of the pair (ϕ, θ) tend to zero on S.

Substituting $0 = \phi = \dfrac{\partial\phi}{\partial n} = \theta = \dfrac{\partial\theta}{\partial n}$ on S into the right hand side of the integral representations (1.27) and (1.28) one can thus establish that

$$\chi(x)\phi(x) \equiv 0, \quad \chi(x)\theta(x) \equiv 0.$$

The following remarks should be noted:

NOTE 1. The asymptotic relations (2.3) are constructed, in particular, with the aim to assert the mutual implication of the radiation conditions (1.24) and (1.25). It is obvious that both expressions for ϕ and θ decay exponentially at infinity.

NOTE 2. The general equations $(2.1)_{1,2}$ which are identical with Eqs. $(1.5)_{1,2}$ and $(1.7)_{1,2}$ are equivalent except that in the last equations it must be assumed that the pair (ϕ, θ) is of the class C^4 in B_e.

We are ready to formulate the uniqueness theorem.

THEOREM 3. *If the hypotheses of Theorem* 2 *are fulfilled in view of the Corollary and Notes* 1 *and* 2 *given above, the integral representation* (1.27) *and* (1.28) *is unique.*

3. Uniqueness in the case when the boundary surface is not smooth

Let the surface S bounding region B_e be not smooth but have a vertex (weak singularity !). A typical example of a surface of that kind in two dimensions is given in Fig. 2. In this case, at the vertex A, solutions (ϕ, θ) of equations (or their gradients)

$$\square_1^2\phi - m\theta = 0, \quad \square_3^2\theta + \frac{\varepsilon}{m}\hat{h}_3^2\nabla^2\phi = 0, \quad x \in B_e \tag{3.1}$$

with the boundary conditions

$$\left.\frac{\partial}{\partial n}\right|_S \phi = f(x), \quad \theta|_S = g(x), \tag{3.2}$$

where $f(x) \in C(S)$ and $g(x) \in C^1(S)$ are given functions, may have singularities.

A similar equation has already been considered for a potential theory and extended the Helmholtz equation. Our study of the problem requires further investigation. The singularity in A prevents us from applying Green's identity directly to the domain B_e. Thus we use another region with interior boundary being the surface S_1 (dashed curve in Fig. 2).

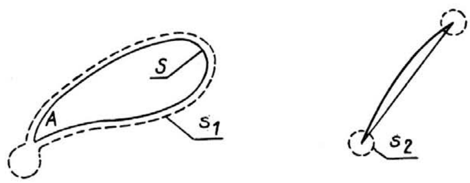

FIG. 2. An example of surface singularity in two dimensions.

The proof that Eqs. (3.1)–(3.2) with $f = g = 0$ have only zero solutions may be carried out along the previous lines subject to the condition that the expression

$$\int_{S_1} dS\left[h_1^2\left(\bar{\phi}\frac{\partial}{\partial n}\phi - \phi\frac{\partial}{\partial n}\bar{\phi}\right) + \frac{m^2}{\varepsilon}\left(\frac{1}{\hat{h}_3^2}\bar{\theta}\frac{\partial}{\partial n}\theta - \frac{1}{\hat{h}_3^2}\theta\frac{\partial}{\partial n}\bar{\theta}\right)\right] = 2i\,\mathrm{Im}\int_{S_1} dS\left[h_1^2\phi\frac{\partial}{\partial n}\bar{\phi} + \frac{m^2}{\varepsilon}\frac{1}{\hat{h}_2^2}\bar{\theta}\frac{\partial}{\partial n}\theta\right] \tag{3.3}$$

(Im — imaginary part) tends to zero, as S_1 shrinks to S. Since ϕ and $\partial\phi/\partial n$ as well as θ and $\partial\theta/\partial n$ are sufficiently regular in a neighbourhood of A, and $\partial\phi/\partial n$ and θ vanishes on S, it suffices to require that

$$\lim_{S_2 \to p.A} \mathrm{Im}\int_{S_1} dS\left[h_1^2\phi\frac{\partial}{\partial n}\bar{\phi} + \frac{m^2}{\varepsilon\hat{h}_3^2}\bar{\theta}\frac{\partial}{\partial n}\theta\right] = 0, \tag{3.4}$$

where S_2 denotes a small sphere with the center at A.

We assume that Eq. (3.4) is satisfied by the difference of two solutions of Eq. (3.1). This can be accomplished by imposing on the solution of Eq. (3.1) the requirement that

$$E = |\nabla\phi|^2 + |\nabla^2\phi|^2 + |\theta|^2 + |\nabla\theta|^2$$

is locally integrable. Physically this means that there is a finite energy in any bounded region of space and that any singularity of the field at the vertex A is sufficiently weak so that no sources are concentrated there.

Hence we see that a singularity produces new effects in the material especially in a neighbourhood of the vertex. This phenomenon is examined in some details in "Mechanics of fracture" devoted to the analysis of three-dimensional cracks in an elastic medium where joints, faults and distribution of stresses near the tip of a crack are dealt with [10].

It appears that various crack propagation models can be described in a more precise way by the thermoelastic theory of harmonic vibrations.

If the geometrical singularity of a tip is sharp, it can be identical with a source of energy depending on the material, stress and crack propagation ability. A moving surface around the singularity is to be introduced to prove the uniqueness theorem in this case. This procedure seems to be useful in the description and interpretation of the crack propagation mechanism.

Chapter 2

4. Discussion of the radiation conditions

The consideration given above provides us with an indication that Note 1 is valid, but not with a proof of its validity. We now wish to do this and to show that other properties of the radiation conditions deserve our attention.

At first we deal with the relation between the asymptotic relation (2.3) and the radiation conditions (1.24) and (1.25). The following symbols are introduced:

$$(4.1)\qquad \begin{aligned} F_1 &\overset{\text{df}}{=} e^{-\hat{\beta}_3 r} H_1, & F_3 &\overset{\text{df}}{=} e^{-\hat{\beta}_1 r} H_3, \\ G_1 &\overset{\text{df}}{=} e^{-\hat{\beta}_1 r} K_1, & G_3 &\overset{\text{df}}{=} e^{-\hat{\beta}_3 r} K_3, \end{aligned}$$

where

$$(4.2)\qquad \begin{aligned} H_1 &= \theta - \frac{1}{m}(h_1^2 - \hat{k}_3^2)\phi, \\ H_3 &= \theta - \frac{1}{m}(h_1^2 - \hat{k}_1^2)\phi, \\ K_1 &= \frac{1}{m}(h_1^2 - \hat{k}_1^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi, \\ K_3 &= \frac{1}{m}(h_1^2 - \hat{k}_3^2)\theta + \frac{\varrho\omega^2}{\hat{a}}\phi. \end{aligned}$$

The radiation conditions (1.24) and (1.25) are transformed to the form

$$
\begin{aligned}
&r\left[\frac{\partial}{\partial r}-(i\hat{k}_1-\hat{\beta}_1)\right]F_3 = o(1),\\
&r\left[\frac{\partial}{\partial r}-(i\hat{k}_3-\hat{\beta}_3)\right]F_1 = o(1),\\
&F_3 = o(1),\\
&F_1 = o(1);
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&r\left[\frac{\partial}{\partial r}-(i\hat{k}_1-\hat{\beta}_1)\right]G_1 = o(1),\\
&r\left[\frac{\partial}{\partial r}-(i\hat{k}_3-\hat{\beta}_3)\right]G_3 = o(1),\\
&G_1 = o(1),\\
&G_3 = o(1).
\end{aligned}
\tag{4.4}
$$

Obviously the asymptotic representation of the functions F_1, F_3 and G_1, G_3 given by Eq. (4.1) is valid if

$$
\begin{aligned}
H_1 &= e^{\beta_0 r}\hat{H}_1, \qquad H_3 = e^{\beta_0 r}\hat{H}_3,\\
K_1 &= e^{\beta_0 r}\hat{K}_1, \qquad K_3 = e^{\beta_0 r}\hat{K}_3
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
&\beta_0 = \min(\hat{\beta}_1, \hat{\beta}_3),\\
&(\hat{H}_1, \hat{H}_3, \hat{K}_1, \hat{K}_3) \in C^1.
\end{aligned}
\tag{4.6}
$$

Hence

$$
\begin{aligned}
\phi &\sim F_3+F_1 = e^{-(\hat{\beta}_1-\beta_0)r}\hat{H}_3+e^{-(\hat{\beta}_3-\beta_0)r}\hat{H}_1,\\
\theta &\sim G_1+G_3 = e^{-(\hat{\beta}_1-\beta_0)r}\hat{K}_1+e^{-(\hat{\beta}_3-\beta_0)r}\hat{K}_3.
\end{aligned}
\tag{4.7}
$$

From the comparison of Eqs. (2.3) and (4.7) we have

$$
\begin{aligned}
\hat{H}_1 &= r^m f_2(\tilde{\theta}, \tilde{\varphi}),\\
\hat{H}_3 &= r^m f_1(\tilde{\theta}, \tilde{\varphi}),\\
\hat{K}_1 &= r^m g_1(\tilde{\theta}, \tilde{\varphi}),\\
\hat{K}_3 &= r^m g_2(\tilde{\theta}, \tilde{\varphi}).
\end{aligned}
$$

Therefore we have shown that the next mutual coupling at $r \to \infty$ exists.

$$
\begin{aligned}
(1.24) &\Leftrightarrow (2.3)_1,\\
(1.25) &\Leftrightarrow (2.3)_2.
\end{aligned}
$$

We proceed to examine the behaviour of energy flux at infinity. To show that Eqs. (1.24) and (1.25) really characterize the radiation conditions, it suffices to remark that the energy flux (outward) through the S_r surface is non-negative.

Using Eq. (4.3) and the notations (4.1) and (4.5) one can calculate the energy radiation flux at infinity.

We write

$$(4.8)\qquad \lim_{r\to\infty} I = \lim_{r\to\infty}\int\limits_0^{\pi}\int\limits_0^{2\pi} r^2\sin\tilde{\theta}d\tilde{\theta}d\tilde{\varphi}\Bigg\{e^{-2(\hat{\beta}_1-\beta_0)r}\Bigg[\left|\frac{\partial}{\partial r}\hat{H}_3\right|^2+(|\hat{k}_1|^2+\hat{\beta}_1^2)|\hat{H}_3|^2 - (i\hat{k}_1-\hat{\beta}_1)\hat{H}_3\frac{\partial}{\partial r}\overline{\hat{H}}_3+(i\overline{\hat{k}}_1+\hat{\beta}_1)\overline{\hat{H}}_3\frac{\partial}{\partial r}\hat{H}_3\Bigg] + e^{-2(\hat{\beta}_3-\beta_0)r}\Bigg[\left|\frac{\partial}{\partial r}\hat{H}_1\right|^2+(|\hat{k}_3|^2+\hat{\beta}_3^2)|\hat{H}_1|^2 - (i\hat{k}_3-\hat{\beta}_3)\hat{H}_1\frac{\partial}{\partial r}\overline{\hat{H}}_1+(i\overline{\hat{k}}_3+\hat{\beta}_3)\overline{\hat{H}}_1\frac{\partial}{\partial r}\hat{H}_1\Bigg]\Bigg\}.$$

Since, according to definition [3], p. 62, $(\phi,\theta)\in C^2(V_r)$ and $(\phi,\theta)\in C^1(\bar{V}_r)$, then, by virtue of Eqs. (4.1), (4.2), (4.5) and (4.7), the functions

$$(4.9)\qquad H_1, H_3, \hat{H}_1, \hat{H}_3, K_1, K_3, \hat{K}_1, \hat{K}_3$$

belong to the class C^2 in the domain V_r and to the class C^2 in $\bar{V}_r$ (cf. Note 2),

$$\bar{V}_r = V_r \cup S \cup S_r.$$

Estimation of all the terms in Eq. (4.8) with Eq. (4.6) being taken into account allow to find in the limiting case as $r\to\infty$ that the integral vanishes, i.e.

$$(4.10)\qquad \lim_{r\to\infty} I = 0.$$

The last expression states that the energy flux vanishes at infinity, and therefore the sufficient condition for the coupled form of the radiation conditions (1.24) and (1.25) connected closely with the integral representation given by Theorem 1 is fulfilled.

We may show that if $\tau_0\to 0$, the radiation conditions (1.24) and (1.25) are equivalent to those of [2], Eqs. (1.8) and (1.9). Moreover, it is possible to form stronger expressions for the asymptotic relations (1.24) and (1.25) analogous to those in Eqs. (1.10)–(1.14) of [2]. Also remarks similar to those in [2] by all adequate accounts are valid here. Further discussion in this matter will be continued in the next section where special cases of generalized thermoelasticity are considered.

5. Particular systems of generalized thermoelastic equations

If the term $\frac{\varepsilon}{m}\hat{h}_3^2\nabla\phi^2$ in Eq. $(1.5)_2$ is small compared to the remaining terms, then this equation is independent of Eq. $(1.5)_1$. Thus, disregarding the coupling in thermoelastic equations, we pass to the generalized thermal stress problem divided into two separate problems to be solved consecutively. Hence we may use known methods for solving Eq. $(1.5)_2$ to obtain temperature distribution and then we should be concerned with the nonhomogeneous Helmholtz equation $(1.5)_1$ with a determined temperature function on the right hand side. Generally, the system of equations for steady harmonic oscillation ω to be solved is then as follows:

$$\square_1^2\phi - m\theta = 0, \quad \square_3^2\theta = 0. \tag{5.1}$$

Eliminating the temperature from Eq. (5.1), one gets the equations we are interested in:

$$\hat{\square}^2_{\hat{k}_1}\hat{\square}^2_{\hat{k}_3}\phi = 0, \quad \hat{\square}^2_{\hat{k}_3}\theta = 0. \tag{5.2}$$

Then the relations (1.9) satisfy the simple equalities

$$\hat{k}_1^2+\hat{k}_3^2 = h_1^2+\hat{h}_3^2, \quad \hat{k}_1^2\hat{k}_3^2 = h_1^2\hat{h}_3^2. \tag{5.3}$$

Hence

$$\hat{k}_1^2 \equiv h_1^2, \quad \hat{k}_3^2 \equiv \hat{h}_3^2. \tag{5.4}$$

By the use of an analogy between Eqs. (5.3) and (1.9), we are able to write the integral representations for the functions ϕ and θ by replacing Eq. (5.4) in the corresponding equations (1.27) and (1.28) only. Thus we have

$$\chi(x)\phi(x) = -\frac{m}{4\pi(h_1^2-\hat{h}_3^2)}\int_S dS\left\{\frac{e^{ih_1R}}{R}\frac{\partial}{\partial n}\left[\theta-\frac{1}{m}(h_1^2-\hat{h}_3^2)\phi\right]\right. \tag{5.5}$$
$$\left.-\left[\theta-\frac{1}{m}(h_1^2-\hat{h}_3^2)\phi\right]\frac{\partial}{\partial n}\left(\frac{e^{ih_1R}}{R}\right)-\left[\frac{e^{i\hat{h}_3R}}{R}\frac{\partial}{\partial n}\theta-\theta\frac{\partial}{\partial n}\left(\frac{e^{i\hat{h}_3R}}{R}\right)\right]\right\},$$

$$\chi(x)\theta(x) = \frac{1}{4\pi}\int dS\left[\frac{e^{i\hat{h}_3R}}{R}\frac{\partial}{\partial n}\theta-\theta\frac{\partial}{\partial n}\left(\frac{e^{i\hat{h}_3R}}{R}\right)\right]. \tag{5.6}$$

The coefficients described by Eq. (5.4) may be expressed explicitly to fulfill the formulae

$$\hat{\alpha}_1 = h_1, \quad \hat{\beta}_1 = 0,$$
$$\hat{\alpha}_3 = \left(\frac{\omega}{2\varkappa}\right)^{1/2}[(\omega^2\tau_0^2+1)^{1/2}+\omega\tau_0], \quad \hat{\beta}_3 = \left(\frac{\omega}{2\varkappa}\right)^{1/2}[(\omega^2\tau_0^2+1)^{1/2}-\omega\tau_0]^{1/2}. \tag{5.7}$$

The relations (5.5) and (5.6) are valid if the radiation conditions satisfy the asymptotics

$$r\left(\frac{\partial}{\partial r}-ih_1\right)\left[\theta-\frac{1}{m}(h_1^2-\hat{h}_3^2)\phi\right] = o(1),$$
$$\theta-\frac{1}{m}(h_1^2-\hat{h}_3^2)\phi = o(1), \tag{5.8}$$

$$re^{-\hat{\beta}_3r}\left(\frac{\partial}{\partial r}-i\hat{h}_3\right)\theta = o(1),$$
$$e^{-\hat{\beta}_3r}\theta = o(1). \tag{5.9}$$

Next, let the relaxation time in Eq. (5.1) be negligible, $(\eta\to 0)$ $\tau_0\to 0$, then Eq. (5.7) implies

$$\begin{matrix}\hat{k}_1^2 = h_1^2 & & \hat{\alpha}_1 = h_1, & \hat{\beta}_1 = 0,\\ & \Rightarrow & & \\ \hat{k}_3^2 = h_3^2 & & \hat{\alpha}_3 = \left(\frac{\omega}{2\varkappa}\right)^{1/2}, & \hat{\beta}_3 = \left(\frac{\omega}{2\varkappa}\right)^{1/2}.\end{matrix} \tag{5.10}$$

The integral representation takes the form

$$(5.11)\qquad \chi(x)\phi(x) = -\frac{m}{4\pi(h_1^2-h_3^2)}\int_S dS\Big\{\frac{e^{ih_1R}}{R}\frac{\partial}{\partial n}\Big[\theta-\frac{1}{m}(h_1^2-h_3^2)\phi\Big] - \Big[\theta-\frac{1}{m}(h_1^2-h_3^2)\phi\Big]\frac{\partial}{\partial n}\Big(\frac{e^{ih_1R}}{R}\Big) - \Big[\frac{e^{ih_3R}}{R}\frac{\partial}{\partial n}\theta-\theta\frac{\partial}{\partial n}\Big(\frac{e^{ih_3R}}{R}\Big)\Big]\Big\},$$

$$(5.12)\qquad \chi(x)\theta(x) = \frac{1}{4\pi}\int dS\Big[\frac{e^{ih_3R}}{R}\frac{\partial}{\partial n}\theta-\theta\frac{\partial}{\partial n}\Big(\frac{e^{ih_3R}}{R}\Big)\Big].$$

Expression (5.12) is in agreement with Eq. (2), p. 569 in [5] providing the outward normal direction like in Fig. 1 is taken into account.

The radiation conditions are now

$$(5.13)\qquad r\Big(\frac{\partial}{\partial r}-ih_1\Big)\Big[\theta-\frac{1}{m}(h_1^2-h_3^2)\phi\Big] = o(1),\qquad \theta-\frac{1}{m}(h_1^2-h_3^2)\phi = o(1),$$

$$(5.14)\qquad re^{-\left(\frac{\omega}{2\varkappa}\right)^{1/2}r}\Big(\frac{\partial}{\partial r}-ih_3\Big)\theta = o(1),\qquad e^{-\left(\frac{\omega}{2\varkappa}\right)^{1/2}r}\theta = o(1).$$

We return again to Eqs. (1.5) and (1.6) and suppose that the relaxation time is to be negligible, $\tau_0 \to 0$.

Obviously the result is closely related to the amplitude equations of classical coupled thermoelasticity, see Eqs. (1.1)–(1.4) in [2]. The outline of the investigations and results of Sects. 1–4 is now applicable, providing $\hat{h}_3$ is replaced by h_3.

Hence

$$\hat{k}_1 = k_1,\qquad \hat{k}_3 = k_3,\qquad \hat{a} = a,$$

where k_1, k_3, a are given by Eq. (1.4) and further in [2].

Introducing the foregoing symbols to Eqs. (1.27), (1.28), (1.24) and (1.25), one gets the integral representations for classical thermoelasticity and the radiation conditions exactly as Eqs. (1.8) and (1.9) in [2].

The proper expressions for the pair (ϕ, θ) may be deduced immediately from Eqs. (1.5) and (1.6) of [2] as well. The appropriate formulae are not specified here since their form is rather obvious.

In order to simplify the equations discussed above, condition $\eta \to 0$ leads to the theory of thermal stresses shown by Eqs. (5.11)–(5.14).

The particular examples allow us to consider some properties of plane waves, moving far away from source of disturbances. Equations (5.6) and (5.12), for instance, describe different but pure thermal waves characterized by the coefficients $\hat{h}_3$ and $h_{\bar{3}}$, respectively. Referring to Eqs. (5.7) and (5.10), it is seen that a damping and phase velocity of plane wave due to the formulae

$$(5.15)\qquad \hat{\vartheta}_r = \operatorname{Im}\hat{k}_r,\qquad \hat{v}_r = \frac{\omega}{\operatorname{Re}\hat{k}_r}$$

may be determined.

Inserting Eq. (5.7) into Eq. (5.15), one can write

$$\text{(5.16)}\qquad \hat{\vartheta}_3 = \hat{\beta}_3 = \left(\frac{\omega}{2\varkappa}\right)^{1/2} [(\omega^2\tau_0^2+1)-\omega\tau_0]^{1/2},$$

$$\hat{v} = \frac{\omega}{\hat{\alpha}_3} = (2\varkappa\omega)^{1/2}[(\omega^2\tau_0^2+1)-\omega\tau_0]^{1/2}.$$

Equation (5.10) yields

$$\text{(5.17)}\qquad \vartheta_3 = \left(\frac{\omega}{2\varkappa}\right)^{1/2},$$

$$\hat{v}_3 = (2\varkappa\omega)^{1/2}.$$

Denoting

$$\text{(5.18)}\qquad z = \omega\tau_0 > 0, \quad 0 < W(z) = [(z^2+1)^{1/2}-z]^{1/2} < 1,$$

the damping and velocity coefficient in Eq. (5.16) can be written as

$$\text{(5.19)}\qquad \hat{\vartheta}_3 = \vartheta_3 W(z),$$

$$\hat{v}_3 = v_3 W(z).$$

By virtue of Eqs. (5.17) and (5.19) we estimate that the thermal wave motion according to Eqs. (5.6) and (5.12) is both dissipative and dispersive. Because of $0 < W(z) < 1$, Eq. (5.18) and the relations (5.19), it is seen that the relaxation properties of a medium reduce the influence on the damping and phase velocity of the wave given by Eq. (5.6) in relation to Eq. (5.12). The comparison of Eqs. (5.17) and (5.19) yields

$$\text{(5.20)}\qquad \vartheta_3 > \hat{\vartheta}_3,$$

$$v_3 > \hat{v}_3.$$

A quantitative contribution of the relaxation time to damping and phase velocity may be given, too. To this end, it is sufficient to take into account the function $W(z)$ in the form of Eq. (5.18) or in the following form written for small and large value of z

$$\text{(5.21)}\qquad W(z) \sim 1-\frac{1}{2}\left(1-\frac{z}{2}\right)z, \quad \text{for } z \ll 1,$$

$$W(z) \sim \left(\frac{1}{2z}\right)^{1/2}, \quad \text{for } z \gg 1.$$

The computed data of W according to the relations (5.18) and (5.21) for a wide range of argument $0 < z = \omega\tau_0 \sim 100$ are plotted in Figs. 3 and 4.

A qualitative explanation of the behaviour of W as a function of τ_0 and ω may be proposed as follows: for fixed $\tau_0 = \tau_0^*$ we calculate $\omega^* = \dfrac{z}{\tau_0^*}$ and thus we obtain $W(z)$. In particular, for $z = 0.1$ and $\tau_0 = \tau_0^* = 9.95\cdot 10^{-12}$ (copper), the value of $W(0.1) = 0.951$ and corresponds to

$$\omega^* = \frac{0.1}{0.95\cdot 10^{-12}} = 1.005\cdot 10^{12}.$$

Assuming the relaxation time in the range $10^{-12}-10^{-11}$ (metals) and the ultrasonic

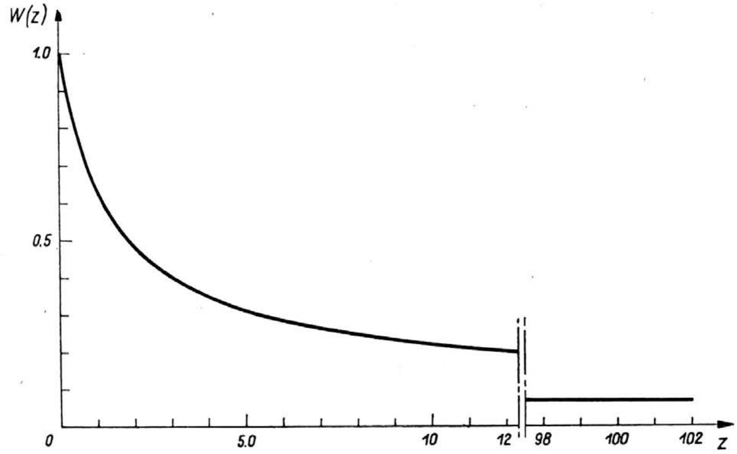

FIG. 3. Dependence of $W(z)$ on $z = \omega\tau_0$, $0 < z < 100$.

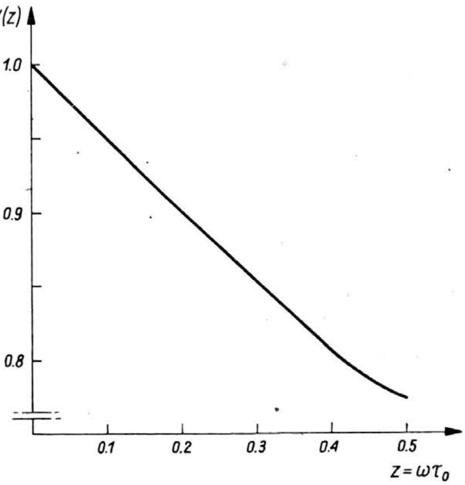

FIG. 4. Plot of function $W(z)$ for $z < 0.5$.

vibration band $0.5 \cdot 10^6 - 5 \cdot 10^{10}$, then the values of z and $W(z)$ are limited as follows (Fig. 4)

$$5 \cdot 10^{-7} < z = \omega\tau_0 < 0.5, \quad 0.786 < W(z) < 1.$$

In this case we see that the dependence W on the oscillation frequency for fixed τ_0 is almost linear.

6. Conclusions

1. The integral representation of the harmonic oscillation in exterior thermoelastic domain with one relaxation time according to the generalized theory of thermoelasticity was established in a three-dimensional coordinate system. Adopting suitable kernels one can specify the proper relations for the two or one-dimensional problem.

2. From the feature of integral representation (1.27) and (1.28) and the radiation conditions (1.24) and (1.25), it follows that the disturbances in a material far enough from the boundary surface of a cavity seem to be vanishing because of the exponentially damping factor in the integrand, exp $[-\hat{\beta}_l r]$, $\hat{\beta}_l > 0, l = 1, 2$.

3. Some modification of the procedure applied here may be useful in the investigation of integral representation for internal domain but it does not refer to the uniqueness theorem.

4. The integral representation facilitates the understanding of fatigue phenomena in the surface layer and in a substractum of a thermoelastic body.

5. If the cavity is assumed to have an extended flat, ellipsoidal form, then the given consideration connected with the vertex, Sect. 3, may be effective in crack problem exploration, especially in the stress analysis and the failure criterion under thermoelastic oscillation of the body.

6. Theorems 1 and 3 hold also if any linear coupled field of a more complicated structure is assumed, for instance, the generalized linear thermoelastic field with two relaxation times.

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Received September 14, 1981.