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ON SOME FORMULÆ RELATING TO THE DISTANCES OF A POINT FROM THE VERTICES OF A TRIANGLE, AND TO THE PROBLEM OF TACTIONS.

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THE relation between the distances of four points 1, 2, 3, 4 in a plane is

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1 \\ \overline{42}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

where, see my paper "Note on the value of certain Determinants the terms of which are the squared distances of Points in a plane or in space," *Quarterly Journal of Mathematics*, t. III., p. 275 (1859), [286], the determinant is

$$= \Sigma\overline{12}^2 \cdot \overline{23}^2 \cdot \overline{34}^2 - \Sigma\overline{12}^2 \cdot \overline{34}^2 \cdot \overline{43}^2 - \Sigma\overline{12}^2 \cdot \overline{23}^2 \cdot \overline{31}^2,$$

an identity which subsists without the aid of the relations $12 = 21$, &c., and in which the Σ, Σ, Σ contain 24, 12, and 8 terms respectively.

Writing $23 = f$, $31 = g$, $12 = h$, $14 = a$, $24 = b$, $34 = c$, the determinant is

$$\begin{aligned} = 2 \{ \;& g^2h^2(b^2+c^2) + h^2f^2(c^2+a^2) + f^2g^2(a^2+b^2) \\ & + b^2c^2(g^2+h^2) + c^2a^2\ (a^2+f^2) + a^2b^2(f^2+g^2) \\ & - a^2f^2(a^2+f^2) - b^2g^2\ (b^2+g^2) - c^2h^2(c^2\ +h^2) \\ & - b^2c^2f^2 - c^2a^2g^2 - a^2b^2h^2 - f^2g^2h^2 \qquad \} \\ = \;& -2\square, \end{aligned}$$

if $\square$ denote the function in $\{\,\}$ with the signs reversed. The function $\square$ may be expressed in the form

$$\begin{aligned}\square = \quad & a^4f^2 + b^4g^2 + c^4h^2 + f^2h^2g^2 \\ & + (a^2f^2 + b^2c^2)(f^2 - g^2 - h^2) \\ & + (b^2g^2 + c^2a^2)(g^2 - h^2 - f^2) \\ & + (c^2h^2 + a^2b^2)(h^2 - f^2 - g^2),\end{aligned}$$

and also in the form

$$\square = \quad U^2 + (f + g + h)\,V,$$

if for shortness

$$\begin{aligned}U = \quad & a^2f + b^2g + c^2h + fgh, \\ V = \quad & (a^2f^2 + b^2c^2)(f - g - h) \\ & + (b^2g^2 + c^2a^2)(g - h - f) \\ & + (c^2h^2 + a^2b^2)(h - f - g);\end{aligned}$$

and it may be remarked that since $\square$ is an even function of f, g, h, we may in this last formula change at pleasure the signs of these quantities; we thus obtain in all four similar forms of the function $\square$.

It is clear that considering a triangle, and any point in the plane of the triangle, f, g, h may be taken to denote the sides of the triangle, and a, b, c the distances of the point from the vertices: and the equation $\square = 0$ is the relation connecting the sides and distances.

The equation $f + g + h = 0$ denotes that the vertices are *in lineâ*, and when this equation is satisfied we have

$$U = a^2f + b^2g + c^2h + fgh = 0,$$

which is in fact, as it is easy to see, the relation connecting the distances of a point from any three points *in lineâ*.

For a, b, c write $a + x$, $b + x$, $c + x$; x will be the radius of a circle touching the circles, radii a, b, c, described about the vertices as centres. The equation $\square = 0$ becomes after all reductions

$$\begin{aligned}& U^2 - (f + g + h)\,V \\ & + x\,[\quad 4U(af + bg + ch) \\ & \qquad - 2(f + g + h)\{(af^2 + bc(b + c))(f - g - h) \\ & \qquad\qquad + (bg^2 + ca(c + a))(g - h - f) \\ & \qquad\qquad + (ch^2 + ab(a + b))(h - f - g)\}] \\ & + x^2\,[\quad f^2\{-4a^2 + 6a(b + c) - 6bc\} \\ & \qquad + g^2\{-4b^2 + 6b(c + a) - 6ca\} \\ & \qquad + h^2\{-4c^2 + 6c(a + b) - 6ab\}] = 0,\end{aligned}$$

which is a quadratic equation only: the two circles thus obtained are those which touch the given circles all three externally or all three internally. But by changing in every possible manner the signs of a, b, c we obtain in all four equations giving the eight tangent circles. It may be noticed that if as before $f + g + h = 0$, $U = 0$, then not only the constant term vanishes, but the coefficient of x also vanishes or the equation becomes simply $x^2=0$.

In particular, suppose $f=b+c$, $g=c+a$, $h=a+b$; developing this *de novo*, and putting for shortness

$$\begin{aligned} a+b+c &= p, \\ bc+ca+ab &= q, \\ abc &= r, \end{aligned}$$

we find

$$\begin{aligned} U &= 2\{px^2+2qx+pq-2r\}, \\ V &= 2\{px^4+4qx^3+(2pq+12r)x^2+4q^2x+pq^2-4qr\}, \end{aligned}$$

and then the equation $\square = U^2-2pV=0$ gives

$$\begin{aligned} \tfrac{1}{4}\square = (px^2+2qx+pq-2r)^2 - p\{px^4+4qx^3+(2pq+12r)x^2+4q^2x+pq^2-4qr\} \\ = 4\{(q^2-4pr)x^2-2qrx+r^2\}, \end{aligned}$$

so that we have

$$\tfrac{1}{16}\square = (q^2-4pr)x^2-2qrx+r^2 = (qx-r)^2-4prx=0,$$

and thence

$$qx-r=\pm x\sqrt{pr}, \text{ or } x=\frac{r}{q\pm},$$

which gives the radii of the circles inscribed in and circumscribed about the three circles radii a, b, c, whereof each touches the two others: a formula given by Descartes, *Epistolæ* (Ed. 2, Franc. 1792), Pars III., p. 261, in a letter to the Princess Elizabeth, viz. Descartes has

$$(d^2e^2+d^2f^2+e^2f^2-2def^2-2d^2ef-2de^2f)x^2-2(de^2f^2+d^2ef^2+d^2e^2f)x+d^2e^2f^2=0,$$

which putting a, b, c for his d, e, f, becomes *ut suprà*

$$x^2(q^2-4pr)-2qrx+r^2=0.$$

In conclusion I notice the following formula which is obtained without difficulty, viz. if as before we have a triangle the sides whereof are f, g, h, and if a, b, c are the distances of a point from the vertices (so that as before $\square=0$) then the perpendicular distances of the point from the sides, each perpendicular distance divided by the perpendicular distance of the opposite vertex from the same side, are as follows: viz. the quotient for the side f is

$$=\frac{1}{16\Delta^2}[(b^2-c^2)(g^2-h^2)+f^2(b^2+c^2+g^2+h^2-2a^2)-f^4],$$

where Δ is the area of the triangle. It is clear that we ought to have

$$\Sigma\{(b^2-c^2)(g^2-h^2)+f^2(b^2+c^2+g^2+h^2-2a^2)-f^4\}=16\Delta^2,$$

and this equation in fact reduces itself to

$$2g^2h^2+2h^2f^2+2f^2g^2-f^4-g^4-h^4=16\Delta^2,$$

which is right.

2, *Stone Buildings, W.C.*, 17*th September*, 1862.