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ON THE CONICS WHICH PASS THROUGH THE FOUR FOCI OF A GIVEN CONIC.

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THE foci of a conic are the points of intersection of the tangents through the circular points at infinity; the pair of tangents through each of the circular points at infinity is a conic through the four foci; and we have thus two conics $P=0$, $Q=0$ passing through the four foci; the equation of any other conic through the four foci is of course $P+\lambda Q=0$; and in particular if λ be suitably determined this equation gives the axes of the conic.

I was led to develope the solution, in seeking to obtain the elegant formulæ given in Mr P. J. Hensley's paper "Determination of the foci of the conic section expressed by trilinear coordinates," *Journal*, t. v., pp. 177—183, (March, 1862).

I take the coordinates to be proportionate to the perpendicular distances of the point from the sides of the fundamental triangle, each distance divided by the perpendicular distance of the side from the opposite angle. This being so, the equation of the line infinity is

$$x+y+z=0,$$

and, α, β, γ denoting the sides of the fundamental triangle, the equation of the circle circumscribed about the triangle is

$$\frac{\alpha^2}{x}+\frac{\beta^2}{y}+\frac{\gamma^2}{z}=0.$$

The foregoing two equations determine the circular points at infinity; and if (x_1, y_1, z_1) are the coordinates, there is no difficulty in obtaining the system of values

$$\begin{aligned} x_1 : y_1 : z_1 &= -\alpha \quad : \beta(\cos C+i\sin C) : \gamma(\cos B+i\sin B) \\ &= \alpha(\cos C-i\sin C) : -\beta \quad : \gamma(\cos A+i\sin A) \\ &= \alpha(\cos B+i\sin B) : \beta(\cos A-i\sin A) : -\gamma \quad , \end{aligned}$$

where as usual $i=\surd(-1)$, and where A, B, C denote the angles of the triangle, so that the cosines and sines of these angles denote given functions of α, β, γ. The coordinates (x_2, y_2, z_2) of the other circular point at infinity are of course obtained by merely writing $-i$ for i. We find also

$$x_1x_2 : y_1y_2 : z_1z_2 : y_1z_2+y_2z_1 : z_1x_2+z_2x_1 : x_1y_2+x_2y_1$$
$$=-\alpha^2 : -\beta^2 : -\gamma^2 : \beta^2+\gamma^2-\alpha^2 : \gamma^2+\alpha^2-\beta^2 : \alpha^2+\beta^2-\gamma^2,$$

which are the formula chiefly made use of in the sequel.

Suppose now that the equation of the conic is

$$U=(a, b, c, f, g, h)(x, y, z)^2=0;$$

then putting for a moment

$$U_1=(a, b, c, f, g, h)(x_1, y_1, z_1)^2,$$
$$W_1=(a, b, c, f, g, h)(x, y, z)(x_1, y_1, z_1),$$

and the like as regards U_2 and W_2; the tangents from the points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively are

$$UU_1-W_1^2=0,$$
$$UU_2-W_2^2=0,$$

which are respectively pairs of lines intersecting in the four foci. And it is moreover clear that the equation of the axes is

$$U_2W_1^2-U_1W_2^2=0.$$

The foregoing equations may be written

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(yz_1-y_1z, zx_1-z_1x, xy_1-x_1y)^2=0,$$
$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(yz_2-y_2z, zx_2-z_2x, xy_2-x_2y)^2=0,$$

where $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ are the inverse system of coefficients.

These may be written

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(x_1, y_1, z_1)^2=0,$$
$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(x_2, y_2, z_2)^2=0,$$

where a, b, c, f, g, h are quadric functions of x, y, z, viz.

$$\begin{aligned}
\mathrm{a} &= \mathfrak{B}z^2+\mathfrak{C}y^2-2\mathfrak{F}yz,\\
\mathrm{b} &= \mathfrak{C}x^2+\mathfrak{A}z^2-2\mathfrak{G}zx,\\
\mathrm{c} &= \mathfrak{A}y^2+\mathfrak{B}x^2-2\mathfrak{H}xy,\\
\mathrm{f} &= -\mathfrak{A}yz-\mathfrak{F}x^2+\mathfrak{G}xy+\mathfrak{H}xz,\\
\mathrm{g} &= -\mathfrak{B}zx-\mathfrak{G}y^2+\mathfrak{H}yz+\mathfrak{F}yx,\\
\mathrm{h} &= -\mathfrak{C}xy-\mathfrak{H}z^2+\mathfrak{F}zx+\mathfrak{G}xy,
\end{aligned}$$

and this being so, I combine the two equations as follows:

$$\begin{aligned}
&x_2^2\,(\mathrm{a}, \ldots)(x_1, y_1, z_1)^2 + x_1^2\,(\mathrm{a}, \ldots)(x_2, y_2, z_2)^2 = 0,\\
&y_2^2 \quad ,, \qquad + y_1^2 \quad ,, \qquad = 0,\\
&z_2^2 \quad ,, \qquad + z_1^2 \quad ,, \qquad = 0,\\
&y_2 z_2\,(\mathrm{a}, \ldots)(x_1, y_1, z_1)^2 + y_1 z_1\,(\mathrm{a}, \ldots)(x_2, y_2, z_2)^2 = 0,\\
&z_2 x_2 \quad ,, \qquad + z_1 x_1 \quad ,, \qquad = 0,\\
&x_2 y_2 \quad ,, \qquad + x_1 y_1 \quad ,, \qquad = 0,
\end{aligned}$$

any one of which is the equation of a conic passing through the four foci; the current coordinates being always (x, y, z).

The first of these equations is

$$\begin{aligned}
&\mathrm{a}\,(-2x_1^2 x_2^2) + \mathrm{b}\,(x_2^2 y_1^2 + x_1^2 y_2^2) + \mathrm{c}\,(x_2^2 z_1^2 + x_1^2 z_2^2)\\
&\qquad + 2\mathrm{f}\,(x_2^2 y_1 z_1 + x_1^2 y_2 z_2) + 2\mathrm{g}\,(x_2^2 z_1 x_1 + x_1^2 z_2 x_2) + 2\mathrm{h}\,(x_2^2 x_1 y_1 + x_1^2 x_2 y_2) = 0,
\end{aligned}$$

where the quantities multiplied by a, b, &c. are all of them easily expressible in terms of x_1x_2, y_1y_2, z_1z_2, $y_1z_2 + y_2z_1$, $z_1x_2 + z_2x_1$, $x_1y_2 + x_2y_1$, which are respectively proportional to given functions of (α, β, γ); and replacing for a, b, &c. their values, the equation is

$$\begin{aligned}
&\;\;(\mathfrak{B}z^2 + \mathfrak{C}x^2 - 2\mathfrak{F}yz) \;.\; 2\alpha^4\\
&+\;\;(\mathfrak{C}x^2 + \mathfrak{A}y^2 - 2\mathfrak{G}zx) \;.\; (\alpha^2 + \beta^2 - \gamma^2)^2 - 2\alpha^2\beta^2\\
&+\;\;(\mathfrak{A}y^2 + \mathfrak{B}x^2 - 2\mathfrak{H}xy) \;.\; (\gamma^2 + \alpha^2 - \beta^2)^2 - 2\gamma^2\alpha^2\\
&+ 2\,(-\mathfrak{A}yz - \mathfrak{F}x^2 + \mathfrak{G}xy + \mathfrak{H}xz) \;.\; \alpha^2(\beta^2 + \gamma^2) - (\beta^2 - \gamma^2)^2\\
&+ 2\,(-\mathfrak{B}zx - \mathfrak{G}y^2 + \mathfrak{H}yz + \mathfrak{F}yz) \;.\; -\alpha^2(\gamma^2 + \alpha^2 - \beta^2)\\
&+ 2\,(-\mathfrak{C}xy - \mathfrak{H}z^2 + \mathfrak{F}zx + \mathfrak{G}zy) \;.\; -\alpha^2(\alpha^2 + \beta^2 - \gamma^2) = 0.
\end{aligned}$$

Now putting for shortness

$$\square = \alpha^4 + \beta^4 + \gamma^4 - 2\beta^2\gamma^2 - 2\gamma^2\alpha^2 - 2\alpha^2\beta^2,$$

so that $-\square$ is equal to sixteen times the square of the area of the fundamental triangle, the coefficient of x^2 is

$$= \mathfrak{B}(\square + 2\alpha^2\gamma^2) + \mathfrak{C}(\square + 2\alpha^2\beta^2) - 2\mathfrak{F}\{-\square - \alpha^2(\beta^2 + \gamma^2 - \alpha^2)\},$$

which is

$$= (\mathfrak{B} + \mathfrak{C} + 2\mathfrak{F})\,\square + 2\alpha^2\{\mathfrak{B}\gamma^2 + \mathfrak{C}\beta^2 + \mathfrak{F}(\beta^2 + \gamma^2 - \alpha^2)\}.$$

And reducing in a similar manner the other coefficients, the equation is

$$\square\,\{(\mathfrak{B} + \mathfrak{C} + 2\mathfrak{F})\,x^2 + \mathfrak{A}\,(y + z)^2 - 2\,(\mathfrak{H} + \mathfrak{G})\,x\,(y + z)\} + 2\alpha^2\Theta = 0,$$

where for shortness

$$\begin{aligned}
\Theta = \;& x^2 \;.\; \mathfrak{B}\gamma^3 + \mathfrak{C}\,\beta^2 + \mathfrak{F}\,(\beta^2 + \gamma^2 - \alpha^2)\\
&+ y^2 \;.\; \mathfrak{C}\alpha^2 + \mathfrak{A}\,\gamma^2 + \mathfrak{G}\,(\gamma^2 + \alpha^2 - \beta^2)\\
&+ z^2 \;.\; \mathfrak{A}\beta^2 + \mathfrak{B}\,\alpha^2 + \mathfrak{H}\,(\alpha^2 + \beta^2 - \gamma^2)\\
&+ yz \;.\; -2\mathfrak{F}\alpha^2 + \mathfrak{A}\,(\beta^2 + \gamma^2 - \alpha^2) - \mathfrak{H}\,(\gamma^2 + \alpha^2 - \beta^2) - \mathfrak{G}\,(\alpha^2 + \beta^2 - \gamma^2)\\
&+ zx \;.\; -2\mathfrak{G}\beta^2 - \mathfrak{H}\,(\beta^2 + \gamma^2 - \alpha^2) + \mathfrak{B}\,(\gamma^2 + \alpha^2 - \beta^2) - \mathfrak{F}\,(\alpha^2 + \beta^2 - \gamma^2)\\
&+ xy \;.\; -2\mathfrak{H}\gamma^2 - \mathfrak{G}\,(\beta^2 + \gamma^2 - \alpha^2) - \mathfrak{F}\,(\gamma^2 + \alpha^2 - \beta^2) + \mathfrak{C}\,(\alpha^2 + \beta^2 - \gamma^2),
\end{aligned}$$

or, what is the same thing, the equation is

$$\square\,\{(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+2\mathfrak{F}+2\mathfrak{G}+2\mathfrak{H})\,x^2-2\,(\mathfrak{A}+\mathfrak{H}+\mathfrak{G})\,x\,(x+y+z)+\mathfrak{A}\,(x+y+z)^2\}+2\alpha^2\Theta=0,$$

or putting for shortness

$$\begin{aligned} \mathfrak{A}+\mathfrak{B}+\mathfrak{C}+2\mathfrak{F}+2\mathfrak{G}+2\mathfrak{H} &= \mathfrak{P}, \\ \mathfrak{A}+\mathfrak{H}+\mathfrak{G} &= \mathfrak{L}, \\ \mathfrak{H}+\mathfrak{B}+\mathfrak{F} &= \mathfrak{M}, \\ \mathfrak{G}+\mathfrak{F}+\mathfrak{C} &= \mathfrak{N}, \end{aligned}$$

so that in fact

$$\mathfrak{P}=\mathfrak{L}+\mathfrak{M}+\mathfrak{N},$$

the equation is

$$\square\,\{\mathfrak{P}x^2-2\mathfrak{L}x\,(x+y+z)+\mathfrak{A}\,(x+y+z)^2\}+2\alpha^2\Theta=0.$$

The equation with y_2z_2, y_1z_1 is in a similar manner found to be

$$\begin{aligned} \square\,\{\mathfrak{F}x^2-(\mathfrak{C}+\mathfrak{G})\,y^2-(\mathfrak{B}+\mathfrak{H})\,z^2+(\mathfrak{A}+2\mathfrak{F}+\mathfrak{G}+\mathfrak{H})\,yz+(-\mathfrak{B}-\mathfrak{H}+\mathfrak{F})\,zx+(-\mathfrak{C}-\mathfrak{G}+\mathfrak{F})\,xy\} \\ -(\beta^2+\gamma^2-\alpha^2)\,\Theta=0, \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} \square\,[(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+2\mathfrak{F}+2\mathfrak{G}+2\mathfrak{H})\,yz-\{(\mathfrak{G}+\mathfrak{F}+\mathfrak{C})\,y+(\mathfrak{H}+\mathfrak{B}+\mathfrak{F})\,z\}(x+y+z)+\mathfrak{F}\,(x+y+z)^2] \\ -(\beta^2+\gamma^2-\alpha^2)\,\Theta=0, \end{aligned}$$

or, finally,

$$\square\,\{\mathfrak{P}yz-(\mathfrak{N}y+\mathfrak{M}z)\,(x+y+z)+\mathfrak{F}\,(x+y+z)^2\}-(\beta^2+\gamma^2-\alpha^2)\,\Theta=0.$$

Hence the entire system of equations is

$$\begin{aligned} &\square\,\{\mathfrak{P}x^2 - 2\mathfrak{L}\,x\,(x+y+z)+\mathfrak{A}\,(x+y+z)^2\}+2\alpha^2\Theta=0, \\ &\square\,\{\mathfrak{P}y^2 - 2\mathfrak{M}y\,(x+y+z)+\mathfrak{B}\,(x+y+z)^2\}+2\beta^2\Theta=0, \\ &\square\,\{\mathfrak{P}z^2 - 2\mathfrak{N}\,z\,(x+y+z)+\mathfrak{C}\,(x+y+z)^2\}+2\gamma^2\Theta=0, \\ &\square\,\{\mathfrak{P}yz-(\mathfrak{N}y+\mathfrak{M}z)\,(x+y+z)+\mathfrak{F}\,(x+y+z)^2\}-(\beta^2+\gamma^2-\alpha^2)\,\Theta=0, \\ &\square\,\{\mathfrak{P}zx-(\mathfrak{L}z+\mathfrak{N}x)\,(x+y+z)+\mathfrak{G}\,(x+y+z)^2\}-(\gamma^2+\alpha^2-\beta^2)\,\Theta=0, \\ &\square\,\{\mathfrak{P}xy-(\mathfrak{M}x+\mathfrak{L}\,y)\,(x+y+z)+\mathfrak{H}\,(x+y+z)^2\}-(\alpha^2+\beta^2-\gamma^2)\,\Theta=0, \end{aligned}$$

which are the equations of six conics, each of them passing through the four foci.

From the first three of these, we have

$$\begin{aligned} &\frac{1}{\alpha^2}\,\{\mathfrak{P}x^2-2\mathfrak{L}x\,(x+y+z)+\mathfrak{A}\,(x+y+z)^2\} \\ &=\frac{1}{\beta^2}\,\{\mathfrak{P}y^2-2\mathfrak{M}y\,(x+y+z)+\mathfrak{B}\,(x+y+z)^2\} \\ &=\frac{1}{\gamma^2}\,\{\mathfrak{P}z^2-2\mathfrak{N}z\,(x+y+z)+\mathfrak{C}\,(x+y+z)^2\}, \end{aligned}$$

which, allowing for the difference of notation, are Mr Hensley's equations: it appears by his investigation that their geometrical signification is as follows; viz. if for shortness we denote the equations by

$$A = B = C,$$

then if we consider the tangents parallel to the x-side of the fundamental triangle, and the tangents parallel to the y-side of the fundamental triangle, the equation $A = B$ is the locus of a point such that the feet of the perpendiculars let fall from it on the four tangents lie in a circle. And similarly for the equations $A = C$, $B = C$.

If we multiply the six equations by 1, 1, 1, 2, 2, 2 and add, we obtain the identical equation $0 = 0$; if we multiply them by a, b, c, $2f$, $2g$, $2h$ and add, then after some easy reductions, we obtain for the equation of a new conic passing through the four foci

$$\square\,\{\mathfrak{P}U + K(x+y+z)^2\} + 2S\Theta = 0,$$

where

$$U = (a,\ b,\ c,\ f,\ g,\ h)(x,\ y\rbrack\ z)^2,$$

K is the discriminant $abc - af^2 - bg^2 - ch^2 + 2fgh$, and

$$S = a\alpha^2 + b\beta^2 + c\gamma^2 - f(\beta^2 + \gamma^2 - \alpha^2) - g(\gamma^2 + \alpha^2 - \beta^2) - h(\alpha^2 + \beta^2 - \gamma^2),$$

or, what is the same thing,

$$S = (f + a - h - g)\,\alpha^2 + (g - h + b - f)\,\beta^2 + (h - g - f + c)\,\gamma^2.$$

It would be interesting to ascertain the geometrical signification of the six conics and of the last-mentioned new conic.

2, *Stone Buildings, W.C., March* 13*th*, 1862.