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A DISCUSSION OF THE STURMIAN CONSTANTS FOR CUBIC AND QUARTIC EQUATIONS.

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FOR the cubic equation

$$(a,\ b,\ c,\ d)\,(x,\ 1)^3 = 0,$$

the Sturmian Constants (or leading coefficients of the Sturmian functions) are

$$a,\ a,\ b^2 - ac,\ -a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2.$$

If the signs of the constants, that is, of the functions for $+\infty$, are	then the signs of the functions for $-\infty$ are	
$+\ +\ +\ +$	$-\ +\ -\ +$	three real roots.
$+\ +\ -\ +$	$-\ +\ +\ +$	case cannot occur.
$+\ +\ +\ -$	$-\ +\ -\ -$	one real root.
$+\ +\ -\ -$	$-\ +\ +\ -$	

The second case would give a loss of variations of sign in passing from ∞ to $-\infty$, which is inconsistent with Sturm's theorem. To show *à posteriori* that the case cannot occur, we may form the identical equation

$$(a^2d - 3abc + 2b^3)^2 = -a^2(-a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2) + 4(b^2 - ac)^3,$$

and, this being so, then in the case in question, the right-hand side would consist of two terms, each of them negative, while the left-hand side is essentially positive.

In the particular case where the third constant vanishes, or

$$b^2 - ac = 0,$$

we have

$$\begin{aligned} -a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2 \\ = -(ad-bc)^2 + 4(b^2-ac)(c^2-bd) \\ = -(ad-bc)^2, \text{ is negative}; \end{aligned}$$

hence, regarding the evanescent term as being at pleasure positive or negative, we have in each case a combination of signs corresponding to one real root.

The general result (which is well known) is, that there are three real roots or one real root according as

$$-a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2$$

is positive or negative.

For the quartic equation

$$(a, b, c, d, e)(x, 1)^4 = 0,$$

the Sturmian constants are

$$a,\ a,\ b^2 - ac,\ 3aJ + 2(b^2 - ac)I,\ I^3 - 27J^2,$$

if, as usual,

$$\begin{aligned} I &= ae - 4bd + 3c^2, \\ J &= ace - ad^2 - b^2e + 2bcd - c^3. \end{aligned}$$

If the signs of the constants, that is, of the functions for $+\infty$, are		then the signs of the functions for $-\infty$ are		
$+ + + + +$	0	$+ - + - +$	4	4 real roots.
$+ + - + +$	2	$+ - - - +$	2	no real root.
$+ + + - +$	2	$+ - + + +$	2	no real root.
$+ + - - +$	2	$+ - - + +$	2	no real root.
$+ + + + -$	1	$+ - + - -$	3	2 real roots.
$+ + - + -$	3	$+ - - - -$	1	-2, cannot occur.
$+ + + - -$	1	$+ - + + -$	3	2 real roots.
$+ + - - -$	1	$+ - - + -$	3	2 real roots.

The non-existing combination of signs is

$$\begin{aligned} I^3 - 27J^2 &= -, \\ 3aJ + 2(b^2 - ac)I &= +, \\ b^2 - ac &= -. \end{aligned}$$

To show *à posteriori* that this case cannot occur, write

$$\begin{aligned} \vartheta &= a^2d - 3abc + 2b^3, \\ X &= 3aJ + 2(b^2 - ac)I, \end{aligned}$$

then we have identically

$$9(3a^2J^2+X^2)\vartheta^2=-4a^2X^3+36(b^2-ac)^3X^2-4a^2(b^2-ac)^3(I^3-27J^2),$$

which is impossible under the given combination of signs, since the left-hand side would be positive, and the right-hand side negative.

To prove the above identity—the relation $JU^3-IU^2H+4H^3+\Phi^2=0$, between the covariants of the quartic, gives

$$a^3J+a^2(b^2-ac)I-4(b^2-ac)^3+\vartheta^2=0,$$

or, what is the same thing,

$$\vartheta^2=-a^3J-a^2(b^2-ac)I+4(b^2-ac)^3.$$

But

$$X=3aJ+2(b^2-ac)I,$$

and thence

$$3\vartheta^2+a^2X=-a^2(b^2-ac)I+12(b^2-ac)^3,$$

or

$$3\vartheta^2=-a^2X-a^2(b^2-ac)I+12(b^2-ac)^3,$$

and the identity will be true, if

$$(3X^2+9a^2J^2)\left\{-X-(b^2-ac)I+12\frac{(b^2-ac)^3}{a^2}\right\}$$

$$=-4X^3+36\frac{(b^2-ac)^3}{a^2}X^2-4(b^2-ac)^3(I^3-27J^2).$$

This gives

$$(3X^2+9a^2J^2)\{-X-(b^2-ac)I\}=-4X^3-4(b^2-ac)^3I^3,$$

or, what is the same thing,

$$(3X^2+9a^2J^2)\{\;X+(b^2-ac)I\}=4\{\;X^3+\;(b^2-ac)^3I^3\},$$

or, dividing by $X+(b^2-ac)I$,

$$3X^2+9a^2J^2=4\{X^2-X(b^2-ac)I+(b^2-ac)^2I^2\},$$

and reducing

$$X^2-4X(b^2-ac)I-9a^2J^2+4(b^2-ac)I^2=0,$$

or finally

$$\{X-3aJ-2(b^2-ac)I\}\{X+3aJ-2(b^2-ac)I\}=0,$$

which is true in virtue of

$$X=3aJ+2(b^2-ac)I,$$

and the identity is thus proved.

The general conclusion is,

if $I^3 - 27J^2$ is positive, the four roots are all real or all imaginary, viz., all real if $b^2 - ac$ and $3aJ + 2(b^2 - ac) I$ are both positive, imaginary if otherwise. But if $I^3 - 27J^2$ is negative, the roots are two of them real, and the other two imaginary.

The following special cases may be noticed,

1°. $b^2 - ac = 0$,

here

$$9(3a^2J^2 + X^2)\vartheta^2 = -4a^2X^3, \text{ or } X = 3aJ + 2(b^2 - ac) I = 3aJ, \text{ is negative,}$$

so that,

if $I^3 - 27J^2$ is $+$, the roots are all imaginary;

if $I^3 - 27J^2$ is $-$, the roots are two real and two imaginary.

2°. $X = 3aJ + 2(b^2 - ac) I = 0$,

here

$$27a^2J^2\vartheta^2 = -4a^2(b^2 - ac)^3 (I^3 - 27J^2),$$

or $b^2 - ac$, $I^3 - 27J^2$ are of opposite signs, and if

$b^2 - ac = -$, $I^3 - 27J^2 = +$, the roots are all imaginary,

$b^2 - ac = +$, $I^3 - 27J^2 = -$, the roots are two real and two imaginary.

3°. $b^2 - ac = 0$, $X = 3aJ + 2(b^2 - ac) I = 0$,

here $J = 0$, that is,

$$2bcd - ad^2 - c^3 = 0, \text{ or } (ad - bc)^2 + c^2(ac - b^2) = 0, \text{ or } ad - bc = 0, \text{ and } I^3 - 27J^2 = I^3,$$

$$I = ae - 4bd + 3c^2 = ae - 4\frac{b^4}{a^2} + 3\frac{b^4}{a^2} = ae - \frac{b^4}{a^2} = \frac{1}{a^2}(a^3e - b^4),$$

whence $I = +$, the roots are all imaginary.

$I = -$, the roots are two real and two imaginary.

This is easily verified, in fact $ac - b^2 = 0$, $ad - bc = 0$, give $c = \dfrac{b^2}{a}$, $d = \dfrac{bc}{a} = \dfrac{b^3}{a^2}$, and the equation becomes

$$ax^4 + 4bx^3 + 6\frac{b^2}{a}x^2 + 4\frac{b^3}{a^2}x + e = 0,$$

or, which is the same thing,

$$(ax + b)^4 + (a^3e - b^4) = 0,$$

so that the roots are all imaginary, or two real and two imaginary, according to the sign of $a^3e - b^4$ as above.

It may be noticed that for a quintic equation

$$(a, b, c, d, e, f)(x, 1)^5,$$

if the Sturmian Constants are

$$a, a, C, D, E, F,$$

where as before a is positive, then the roots are real or imaginary as follows: viz.,

C, D, E, F	
$+ + + +$,	5 real roots.
$- + +$, $+ - +$, $- - +$, $+ + -$, $+ - -$, $- - -$ followed by $+$,	1 real root, 4 imaginary roots.
$+ + +$, $+ + -$, $+ - -$, $- - -$ followed by $-$,	3 real roots, 2 imaginary roots.
$- + - +$,	case which does not occur.
$- + +$, $+ - +$, $- - +$, $- + -$ followed by $-$,	cases which do not occur.

The values of C, D, E, and F are given in my "Tables of the Sturmian Functions for Equations of the Second, Third, Fourth, and Fifth Degrees," *Phil. Trans.*, t. 147 (1857), pp. 733—736, [151], but I have not further examined this case.

2, *Stone Buildings, W.C., September* 29, 1859.