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NOTE ON THE CURVATURE OF A PLANE CURVE AT A DOUBLE POINT, AND ON THE CURVATURE OF SURFACES.

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THE radius of curvature of a plane curve at a double point is most readily determined as follows: viz. if x, y be the coordinates of the double point, θ the inclination to the axis of x of the tangent to the branch under consideration, and u, v the coordinates measured from the double point as origin, parallel and perpendicular to the tangent, of the consecutive point on such branch, then the radius of curvature for the branch in question is

$$= \frac{u^2}{2v};$$

the coordinates of the consecutive point are

$$x + u\cos\theta - v\sin\theta,$$

$$y + u\sin\theta + v\cos\theta,$$

and the value of $u^2 \div 2v$ is to be found by substituting these expressions for x, y in the equation of the curve. Let $U = 0$ be the equation of the curve; then at the double point, U and the differential coefficients of the first order vanish, let those of the second order be $(a,\ b,\ c)$ and those of the third order $(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d})$, then substituting we have

$$\tfrac{1}{2}(a,\ b,\ c)(u\cos\theta - v\sin\theta,\ u\sin\theta + v\cos\theta)^2$$

$$+\tfrac{1}{6}(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d})(u\cos\theta - v\sin\theta,\ u\sin\theta + v\cos\theta)^3 = 0.$$

The coefficient of u^2 is

$$\tfrac{1}{2}(a,\ b,\ c)(\cos\theta,\ \sin\theta)^2,$$

which vanishes, since it is by putting this function equal to zero that we find the direction of the tangents at the double point. For the present purpose we are concerned with one of the two branches only, and in all that follows the ratio $\cos\theta : \sin\theta$ will denote a determinate root of the quadratic equation; viz. the root which corresponds to the branch in question.

The equation takes therefore the form

$$Buv + \tfrac{1}{2}Cv^2 + \tfrac{1}{6}Du^3 + \&\text{c.} = 0,$$

which might be satisfied by assuming $Buv + \frac{1}{2}Cv^2 = 0$, but the values so obtained belong to the branch which does not touch the tangent; the proper solution is

$$Buv + \tfrac{1}{6}Du^3 = 0,$$

and thence

$$\frac{u^2}{2v} = -\frac{3B}{D},$$

or substituting for B, D their values, the radius of curvature is

$$= \frac{3\,(a,\ b,\ c)\,(\cos\theta,\ \sin\theta)\,(-\sin\theta,\ \cos\theta)}{(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d})\,(\cos\theta,\ \sin\theta)^3},$$

where the expression in the numerator is

$$= (-a + c)\sin\theta\cos\theta + b\,(\cos^2\theta - \sin^2\theta),$$

$$= \tfrac{1}{2}\,\{(c - a)\sin 2\theta + 2b\cos 2\theta\}.$$

The curvature, at the point of contact, of the curve in which any surface is intersected by a tangent plane cannot be found by the ordinary theory of the curvature of surfaces, being (by reason that the point of contact is a double point on the curve) a case of exception from that theory. The foregoing method may be applied to the solution of the question as follows: Let $U = 0$ be the equation of the surface, and suppose that (x, y, z) refer to the point of contact of the tangent plane. Let L, M, N be the first differential coefficients of U at this point, (a, b, c, f, g, h) the second differential coefficients, (a, b, c, f, g, h, i, j, k, l) the third differential coefficients. Let λ, μ, ν be proportional to the cosines of the inclinations to the axis of the tangent to one of the two branches through the double point; the ratios $\lambda : \mu : \nu$ are determined by

$$(L,\ M,\ N)\,(\lambda,\ \mu,\ \nu) = 0,$$

$$(a,\ b,\ c,\ f,\ g,\ h)\,(\lambda,\ \mu,\ \nu)^2 = 0.$$

Let u, v be proportional to the coordinates of a consecutive point measured, from the point of contact as origin, in the direction of the tangent and in the perpendicular direction in the tangent plane. The cosines of the inclinations of u to the axes are as $\lambda : \mu : \nu$; those for the inclinations of the normal to the axes are as $L : M : N$; hence for the coordinate v which is perpendicular to the plane of the last-mentioned two lines, the cosines of the inclinations to the axes are as $N\mu - M\nu : L\nu - N\lambda : M\lambda - L\mu$, and the coordinates of the consecutive point may be taken to be

$$\begin{aligned} &x + \lambda u + (N\mu - M\nu)\, v,\\ &y + \mu u + (L\nu \ - N\lambda)\, v,\\ &z + \nu u \ + (M\lambda - L\mu)\, v. \end{aligned}$$

Substituting these values in the equation of the surface, the terms involving the first powers of u, v vanish, and the term involving u^2 also vanishes in virtue of the relation $(a, b, c, f, g, h)(\lambda, \mu, \nu)^2 = 0$. The equation consequently becomes

$$Buv + \tfrac{1}{2}Cv^2 + \tfrac{1}{6}Du^3 + \ldots = 0,$$

and we have as before for the branch in question,

$$\frac{u^2}{2v} = -\frac{3B}{D}.$$

In the present case u, v have been taken, not as before equal, but only proportional, to the coordinates of the consecutive point measured from the point of contact parallel and perpendicular to the tangent, the values of the coordinates are in fact

$$\sqrt{\lambda^2 + \mu^2 + \nu^2}\, u, \ \ \sqrt{L^2 + M^2 + N^2}\ \sqrt{\lambda^2 + \mu^2 + \nu^2}\, v,$$

and the expression for the radius of curvature is

$$= \frac{\sqrt{\lambda^2 + \mu^2 + \nu^2}}{\sqrt{L^2 + M^2 + N^2}} \frac{u^2}{2v},$$

or substituting for $\dfrac{u^2}{2v}$ the value $-\dfrac{3B}{D}$ and for B, D their values, the radius of curvature is

$$= -\frac{3\sqrt{\lambda^2 + \mu^2 + \nu^2}}{\sqrt{L^2 + M^2 + N^2}} \times \frac{(a, b, c, f, g, h)(\lambda, \mu, \nu)(N\mu - M\nu, L\nu - N\lambda, M\lambda - L\mu)}{(\mathrm{a, b, c, f, g, h, i, j, k, l})(\lambda, \mu, \nu)^3},$$

where, as already noticed, the ratios $\lambda : \mu : \nu$ are determined by the equations

$$\begin{aligned} (L, M, N)(\lambda, \mu, \nu) &= 0,\\ (a, b, c, f, g, h)(\lambda, \mu, \nu)^2 &= 0, \end{aligned}$$

the system of roots selected being that which corresponds to the branch under consideration. It may be noticed that these two equations give

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(L, M, N)^2 \cdot (a, b, c, f, g, h)(\lambda, \mu, \nu)^2 - K[(L, M, N)(\lambda, \mu, \nu)]^2 = 0,$$

where as usual

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}) = (bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch),$$

$$K = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

and the expression on the left-hand side considered as a function of (λ, μ, ν) is decomposable into a pair of factors. Selecting the proper factor and equating it to zero, we have in addition to the linear equation

$$(L, M, N)(\lambda, \mu, \nu) = 0,$$

a new linear equation; these two equations determine the ratios $\lambda : \mu : \nu$.

P.S.—I remark that the process adopted for finding the radius of curvature at a double point of a plane curve is the most simple one for the case of an ordinary point on the curve. In fact let $U = 0$ be the curve, (x, y) the coordinates of the point in question, L, M the corresponding values of the first differential coefficients, (a, b, c) those of the second differential coefficients. The coordinates of the consecutive point are

$$x + u\cos\theta - v\sin\theta,$$
$$y + u\sin\theta + v\cos\theta.$$

Substituting these in the equation of the curve, we have

$$L(u\cos\theta - v\sin\theta) + M(u\sin\theta + v\cos\theta) + \tfrac{1}{2}(a, b, c)(u\cos\theta - v\sin\theta,\ u\sin\theta + v\cos\theta)^2 = 0,$$

the coefficient of u must be zero, or we have

$$L\cos\theta + M\sin\theta = 0,$$

giving

$$\sin\theta = -\frac{L}{\sqrt{L^2+M^2}},\quad \cos\theta = \frac{M}{\sqrt{L^2+M^2}},$$

and the equation may be reduced to

$$\sqrt{L^2+M^2}\,v + \tfrac{1}{2}(a, b, c)(\cos\theta, \sin\theta)^2 u^2 = 0,$$

or what is the same thing

$$(L^2+M^2)^{\frac{3}{2}}v + \tfrac{1}{2}(a, b, c)(M, -L)^2 u^2 = 0,$$

whence the radius of curvature

$$= \frac{u^2}{2v} = -\frac{(a, b, c)(M, -L)^2}{(L^2+M^2)^{\frac{3}{2}}},$$

which is the ordinary form for the radius of curvature at any point of a curve represented by the equation $U = 0$.

2, *Stone Buildings, W.C., October 2nd*, 1859.