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NOTE ON THE EQUATION FOR THE SQUARED DIFFERENCES OF THE ROOTS OF A CUBIC EQUATION.

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THE question of finding the equation for the squared differences of the roots, presents, in the case of a cubic equation, a peculiarity which does not occur for equations of a higher order, viz. we may in the first instance form the equation for the differences of the roots taken in a given cyclical order, and thence deduce the equation for the squared differences of the roots. Let the cubic equation be

$$U = (a,\ b,\ c,\ d\between x,\ 1)^3 = a\,(x-\alpha)\,(x-\beta)\,(x-\gamma) = 0,$$

the function

$$\Pi\,\{\theta - (\beta - \gamma)\},$$

which equated to zero gives for θ the values $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$, which are the differences of the roots taken in a given circular order, has for any interchanges whatever of the roots, two values only, viz. that just written down, and the value $\Pi\,\{\theta-(\gamma-\beta)\}$, which may be deduced therefrom by changing first the sign of θ and then the sign of the entire expression (or what is the same thing, by changing the signs of the terms containing the even powers of θ); we may consequently write

$$\Pi\,\{\theta - (\beta-\gamma)\} = P - Q\,\zeta^{\frac{1}{2}}\,(\alpha,\ \beta,\ \gamma),$$

which P, Q are symmetrical functions of the roots, and $\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma)$ or $(\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)$ is a function the square of which is a symmetrical function of the roots, and such symmetrical functions of the roots can of course be expressed as functions of the coefficients. We have in fact

$$P = \theta^3 + \theta(\Sigma\beta\gamma - \Sigma\alpha^2) = a^{-2}\{a^2\theta^3 + 9(ac - b^2)\theta\},$$
$$Q = 1,$$

and

$$\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma) = a^{-2}\sqrt{-27\Box},$$

where $\Box$ is the discriminant of the cubic function,

$$= a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2.$$

Consequently

$$\Pi\{\theta - (\beta - \gamma)\} = a^{-2}\{a^2\theta^3 + 9(ac - b^2)\theta - \sqrt{-27\Box}\},$$

and forming the similar equation

$$\Pi\{\theta + (\beta - \gamma)\} = a^{-2}\{a^2\theta^3 + 9(ac - b^2)\theta + \sqrt{-27\Box}\},$$

multiplying the two equations together and writing u in the place of θ^2, we find

$$\Pi\{u - (\beta - \gamma)^2\} = a^{-4}\{[a^2u + 9(ac - b^2)]^2 u + 27\Box\},$$

and the equation for the squared differences of the roots is thus seen to be

$$[a^2u + 9(ac - b^2)]^2 u + 27\Box = 0,$$

or what is the same thing

$$a^4u^3 + 18a^2(ac - b^2)u^2 + 81(ac - b^2)^2 u + 27\Box = 0.$$

I remark that if ω is an imaginary cube root of unity (so that $(\omega - \omega^2)^2 = -3$, $\omega - \omega^2$ being thus only another form of $\sqrt{-3}$) then if in the expression for $\Pi\{\theta - (\beta - \gamma)\}$ we write $\dfrac{3\theta}{(\omega - \omega^2)a}$ in the place of θ, the equation assumes the more simple form

$$\Pi\{\theta - \tfrac{1}{3}a(\omega - \omega^2)(\beta - \gamma)\} = \theta^3 - 3(ac - b^2)\theta - a\sqrt{\Box},$$

which if U be the cubic function, H the Hessian $= (ac - b^2, ad - bc, bd - c^2)(x, y)^2$, and $\Box$ the discriminant as before, is a particular case (obtained by writing $x = 1$, $y = 0$) of the equation

$$\Pi\{\theta - \tfrac{1}{3}a(\omega - \omega^2)(x - \alpha y)\} = \theta^3 - 3H\theta - U\sqrt{\Box},$$

which equation can be at once obtained from the equation (where Φ is the cubicovariant of the cubic function)

$$\sqrt[3]{\tfrac{1}{2}(\Phi + U\sqrt{\Box})} - \sqrt[3]{\tfrac{1}{2}(\phi - U\sqrt{\Box})} = \tfrac{1}{3}a(\omega - \omega^2)(\beta - \gamma)(x - \alpha y),$$

given in my Fifth Memoir on Quantics, *Phil. Trans.*, t. CXLVIII. (1858), [156]. For writing for a moment

$$\theta = \sqrt[3]{X} - \sqrt[3]{Y},$$

we find

$$\theta^3 = X - Y - 3\sqrt[3]{XY}\theta,$$

or

$$\theta^3 + 3\sqrt[3]{XY}\theta - (X - Y) = 0,$$

where $\sqrt[3]{XY} = \sqrt[3]{\tfrac{1}{4}(\Phi^2 - U^2\square)}$, which by the equation

$$\phi^2 - U^2\square = -4H^3$$

(given in the Memoir) is $= -H$, and $(X - Y)$ is $= U\sqrt{\square}$, so that the equation in θ is, as above, $\theta^3 - 3H\theta - U\sqrt{\square} = 0$, an equation which is satisfied by $\theta = \tfrac{1}{3}a(\omega - \omega^2)(\beta - \gamma)(x - \alpha y)$; and the other two roots being of course of the like form, the cubic function in θ is equal to $\Pi\{\theta - \tfrac{1}{3}a(\omega - \omega^2)(\beta - \gamma)(x - \alpha y)\}$ which proves the theorem.

2, *Stone Buildings, W.C., Nov.* 3*rd*, 1859.