286.

NOTE ON THE VALUE OF CERTAIN DETERMINANTS, THE TERMS OF WHICH ARE THE SQUARED DISTANCES OF POINTS IN A PLANE OR IN SPACE.

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THE values of the several determinants mentioned in my paper "On a Certain Theorem in the Geometry of Position," *Cambridge Mathematical Journal*, Old Series, t. II. (1842), p. 267, [1], are as follows:

(1)

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = \Sigma\, \overline{12}^2\, \overline{21}^2 - \Sigma\, \overline{12}^2\, \overline{23}^2,$$

where the Σ, Σ contain 3 and 6 terms respectively.

(2)

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = \Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2 - \Sigma\, \overline{12}^2\, \overline{34}^2\, \overline{43}^2 - \Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{31}^2,$$

where the Σ, Σ, Σ contain 24, 12 and 8 terms respectively.

(3)

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2, & 1 \\ \overline{31}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = \begin{matrix} -\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2\, \overline{45}^2 \\ -\Sigma\, \overline{12}^2\, \overline{21}^2\, \overline{34}^2\, \overline{43}^2 \\ +\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{45}^2\, \overline{54}^2 \\ +\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2\, \overline{41}^2 \\ +\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{31}^2\, \overline{45}^2 \end{matrix}$$

where the Σ, Σ, Σ, Σ, Σ contain 120, 15, 60, 30, and 40 terms respectively.

(4)

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = \Sigma\, \overline{12}^2\, \overline{21}^2\, \overline{34}^2\, \overline{43}^2 - \Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2\, \overline{41}^2,$$

where the Σ, Σ contain 3 and 6 terms respectively.

(5)

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0 \end{vmatrix} = \begin{matrix} \Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2\, \overline{45}^2\, \overline{51}^2 \\ -\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{31}^2\, \overline{45}^2\, \overline{54}^2 \end{matrix}$$

where the Σ, Σ contain 24 and 20 terms respectively.

And it is proper to remark that it is not in the preceding formulæ (as in the memoir above referred to in which $\overline{12}$ denotes a distance between two points 1 and 2) assumed that $\overline{12}$ and $\overline{21}$ are equal.

The formulæ (1) gives, if a, b, c represent the sides, the well-known expression for the area of a triangle

$$(\text{area})^2 = \tfrac{1}{16}\,(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4).$$

Similarly the formula (2), if a, b, c, f, g, h represent the edges, viz. $\overline{23}=a$, $\overline{31}=b$, $\overline{12}=c$, $\overline{14}=f$, $\overline{24}=g$, $\overline{34}=h$, gives for the volume of a tetrahedron

$$\begin{aligned}(\text{volume})^2 = \tfrac{1}{144}\,\{\ & b^2c^2\,(g^2+h^2) + c^2a^2\,(h^2+f^2) + a^2b^2\,(f^2+g^2) \\ & + g^2h^2\,(b^2+c^2) + h^2f^2\,(c^2+a^2) + f^2g^2\,(a^2+b^2) \\ & - a^2f^2\,(a^2+f^2) - b^2g^2\,(b^2+g^2) - c^2h^2\,(c^2+h^2) \\ & - a^2g^2h^2 \qquad - b^2h^2f^2 \qquad - c^2f^2g^2 \quad - a^2b^2c^2\}, \\ = \tfrac{1}{144}\, & W \text{ suppose.}\end{aligned}$$

Now

$$\begin{aligned} 4 \times \text{surface} &\\ = &\sqrt{2h^2g^2 + 2g^2a^2 + 2a^2h^2 - a^4 - h^4 - g^4} \\ &+ \sqrt{2f^2h^2 + 2h^2b^2 + 2b^2f^2 - h^4 - b^4 - f^4} \\ &+ \sqrt{2g^2f^2 + 2f^2c^2 + 2c^2g^2 - g^4 - f^4 - c^4} \\ &+ \sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}, \\ &= x + y + z + w \text{ suppose,} \end{aligned}$$

and the norm of this is

$$(x^4 + y^4 + z^4 + w^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 - 2x^2w^2 - 2y^2w^2 - 2z^2w^2)^2 - 64x^2y^2z^2w^2,$$

which is of the sixteenth order, and must be of the form WQ where Q is a function of a, b, c, f, g, h of the tenth order. The expression of this function is given by Prof. Sylvester in his paper "On the Relation between the Volume of a Tetrahedron &c.," *Camb. and Dubl. Math. Jour.*, t. VIII. (1853), pp. 171—178, viz. the value is

$$\begin{aligned} Q = \; & a^2b^2c^2\{f^4 + g^4 + h^4 + g^2h^2 + h^2f^2 + f^2g^2 + b^2c^2 + c^2a^2 + a^2b^2 - (f^2 + g^2 + h^2)(a^2 + b^2 + c^2)\} \\ & + a^2g^2h^2\{f^4 + b^4 + c^4 + b^2c^2 + c^2f^2 + f^2b^2 + g^2h^2 + h^2a^2 + a^2g^2 - (f^2 + b^2 + c^2)(a^2 + g^2 + h^2)\} \\ & + b^2h^2f^2\{g^4 + c^4 + a^4 + c^2a^2 + a^2g^2 + g^2c^2 + h^2f^2 + f^2b^2 + b^2h^2 - (g^2 + c^2 + a^2)(b^2 + h^2 + f^2)\} \\ & + c^2f^2g^2\{h^4 + a^4 + b^4 + a^2b^2 + b^2h^2 + h^2a^2 + f^2g^2 + g^2c^2 + c^2f^2 - (h^2 + a^2 + b^2)(c^2 + f^2 + g^2)\}, \end{aligned}$$

and, as there remarked, the equation $Q = 0$ expresses the condition that the radius of the inscribed sphere may be infinite.

2, *Stone Buildings, W.C., June* 10*th*, 1859.